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On Two Remarkable Lines Related to a Quadrilateral

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Abstract. We study the Euler line of an arbitrary quadrilateral and the Nagel line of a circumscriptible quadrilateral.

1. Introduction

Among the various lines related to a triangle the most popular are Euler and Nagel lines. Recall that the Euler line contains the orthocenter H, the centroid G, the circumcenter O and the nine-point center E, so that $HE : EG : GO = 3 : 1 : 2$. On the other hand, the Nagel line contains the Nagel point N, the centroid M, the incenter I and Spieker point S (which is the centroid of the perimeter of the triangle) so that $NS : SG : GI = 3 : 1 : 2$. The aim of this paper is to find some analogies of these lines for quadrilaterals.

It is well known that in a triangle, the following two notions of centroids coincide:
(i) the barycenter of the system of unit masses at the vertices,
(ii) the center of mass of the boundary and interior of the triangle.

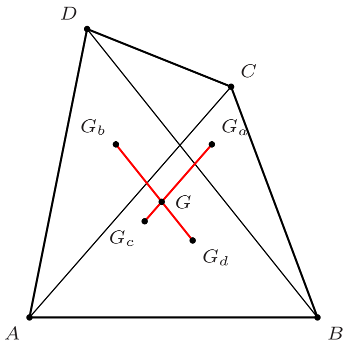

Figure 1.

But for quadrilaterals these are not necessarily the same. We shall show in this note, that to get some fruitful analogies for quadrilateral it is useful to consider the centroid G of quadrilateral as a whole figure. For a quadrilateral $ABCD$, this centroid G can be determined as follows. Let G_a, G_b, G_c, G_d be the centroids of triangles BCD, ACD, ABD, ABC respectively. The centroid G is the intersection of the lines G_aG_c and G_bG_d:

$$G = G_aG_c \cap G_bG_d.$$

See Figure 1.

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2. The Euler line of a quadrilateral

Given a quadrilateral $ABCD$, denote by O_a and H_a the circumcenter and the orthocenter respectively of triangle BCD, and similarly, O_b, H_b for triangle ACD, O_c, H_c for triangle ABD, and O_d, H_d for triangle ABC. Let

$$O = O_aO_c \cap O_bO_d,$$
$$H = H_aH_c \cap H_bH_d.$$

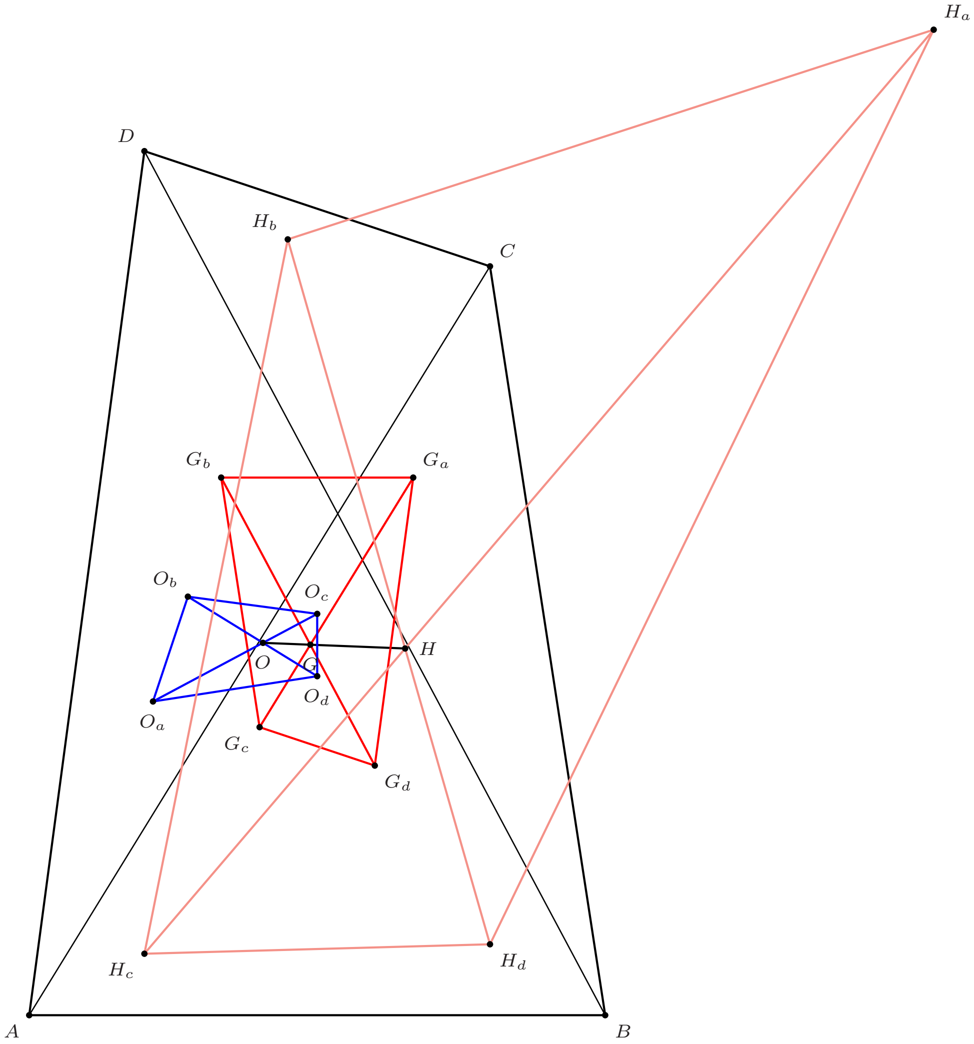

Figure 2

We shall call O the quasicircumcenter and H the quasiorthocenter of the quadrilateral $ABCD$. Clearly, the quasicircumcenter O is the intersection of perpendicular bisectors of the diagonals of $ABCD$. Therefore, if the quadrilateral is cyclic, then O is the center of its circumcircle. Figure 2 shows the three associated quadrilaterals $G_aG_bG_cG_d$, $O_aO_bO_cO_d$, and $H_aH_bH_cH_d$.

The following theorem was discovered by Jaroslav Ganin, (see [2]), and the idea of the proof was due to François Rideau [3].

Theorem 1. *In any arbitrary quadrilateral the quasiorthocenter H, the centroid G, and the quasicircumcenter O are collinear. Furthermore, $OH : HG = 3 : -2$.*

Proof. Consider three affine maps f_G, f_O and f_H transforming the triangle ABC onto triangle $G_aG_bG_c$, $O_aO_bO_c$, and $H_aH_bH_c$ respectively.

In the affine plane, write $D = xA + yB + zC$ with $x + y + z = 1$.

(i) Note that

$$\begin{aligned} f_G(D) =& f_G(xA + yB + zC) \\ =& xG_a + yG_b + zG_c \\ =& \frac{1}{3}(x(B + C + D) + y(A + C + D) + z(A + B + D)) \\ =& \frac{1}{3}((y + z)A + (z + x)B + (x + y)C + (x + y + z)D) \\ =& \frac{1}{3}((y + z)A + (z + x)B + (x + y)C + (xA + yB + zC)) \\ =& \frac{1}{3}(x + y + z)(A + B + C) \\ =& G_d. \end{aligned}$$

(ii) It is obvious that triangles ABC and $O_aO_bO_c$ are orthologic with centers D and O_d. See Figure 3. From Theorem 1 of [1], $f_O(D) = O_d$.

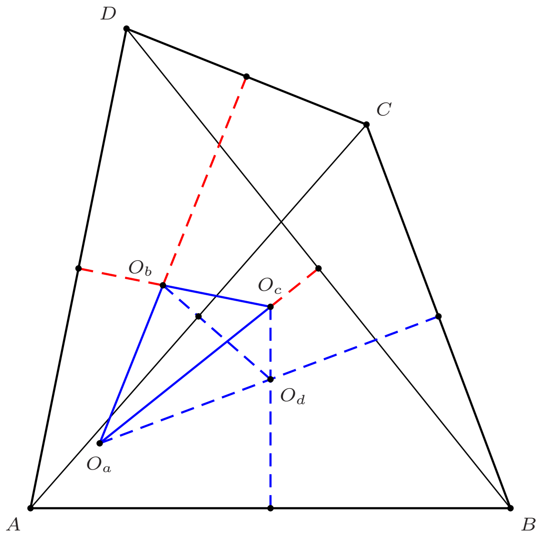

Figure 3

(iii) Since H_a divides O_aG_a in the ratio $O_aH_a : H_aG_a = 3 : -2$, and similarly for H_b and H_c, for $Q = A,\ B,\ C$, the point $f_H(Q)$ divides the segment $f_O(Q)f_G(Q)$ into the ratio $3 : -2$. It follows that for *every* point Q in the plane of ABC, $f_H(Q)$ divides $f_O(Q)f_G(Q)$ in the same ratio. In particular, $f_H(D)$ divides $f_O(D)f_G(D)$, namely, O_dG_d, in the ratio $3 : -2$. This is clearly H_d. We have shown that $f_H(D) = H_d$.

(iv) Let $Q = AC \cap BD$. Applying the affine maps we have

$$\begin{aligned} f_G(Q) =& G_aG_c \cap G_bG_d = G, \\ f_O(Q) =& O_aO_c \cap O_bO_d = O, \\ f_H(Q) =& H_aH_c \cap H_bH_d = H. \end{aligned}$$

From this we conclude that H divides OG in the ratio $3 : -2$. □

Theorem 1 enables one to define the *Euler line* of a quadrilateral $ABCD$ as the line containing the centroid, the quasicircumcenter, and the quasiorthocenter. This line contains also the quasininepoint center E defined as follows. Let E_a, E_b, E_c, E_d be the nine-point centers of the triangles BCD, ACD, ABD, ABC respectively. We define the quasininepoint center to be the point $E = E_aE_c \cap E_bE_d$. The following theorem can be proved in a way similar to Theorem 1 above.

Theorem 2. *E is the midpoint of OH.*

3. The Nagel line of a circumscriptible quadrilateral

A quadrilateral is circumscriptible if it has an incircle. Let $ABCD$ be a circumscriptible quadrilateral with incenter I. Let T_1, T_2, T_3, T_4 be the points of tangency of the incircle with the sides AB, BC, CD and DA respectively. Let N_1 be the isotomic conjugate of T_1 with respect to the segment AB. Similarly define N_2, N_3, N_4 in the same way. We shall refer to the point $N := N_1N_3 \cap N_2N_4$ as the Nagel point of the circumscriptible quadrilateral. Note that both lines divide the perimeter of the quadrilateral into two equal parts.

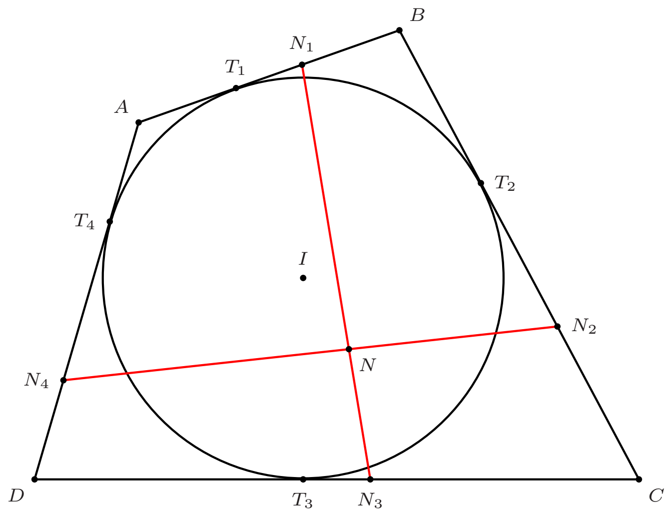

Figure 4.

In Theorem 6 below we shall show that N lies on the line joining I and G. In what follows we shall write

$$P = (x \cdot A,\ y \cdot B,\ z \cdot C,\ w \cdot D)$$

to mean that P is the barycenter of a system of masses x at A, y at B, z at C, and w at D. Clearly, x, y, z, w can be replaced by kx, ky, kz, kw for nonzero k without changing the point P. In Figure 4, assume that $AT_1 = AT_4 = p$, $BT_2 = BT_1 = q$, $CT_3 = CT_2 = r$, and $DT_4 = DT_3 = t$. Then by putting masses p at A, q at B, r at C, and t at D, we see that
(i) $N_1 = (p \cdot A,\ q \cdot B,\ 0 \cdot C,\ 0 \cdot D)$,
(ii) $N_3 = (0{\cdot}A,\ 0{\cdot}B,\ r{\cdot}C,\ t{\cdot}D)$, so that the barycenter $N = (p{\cdot}A,\ q{\cdot}B,\ r{\cdot}C,\ t{\cdot}D)$ is on the line N_1N_3. Similarly, it is also on the line N_2N_4 since
(iii) $N_2 = (0 \cdot A,\ q \cdot B,\ r \cdot C,\ 0 \cdot D)$, and
(iv) $N_4 = (p \cdot A,\ 0 \cdot B,\ 0 \cdot C,\ t \cdot D)$.

Therefore, we have established the first of the following three lemmas.

Lemma 3. $N = (p \cdot A,\ q \cdot B,\ r \cdot C,\ t \cdot D)$.

Lemma 4. $I = ((q+t)A,\ (p+r)B,\ (q+t)C,\ (p+r)D)$.

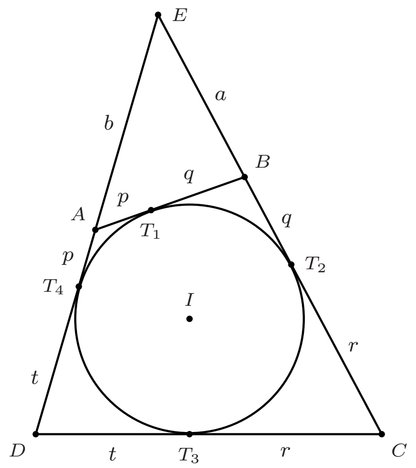

Figure 5.

Proof. Suppose the circumscriptible quadrilateral $ABCD$ has a pair of non-parallel sides AD and BC, which intersect at E. (If not, then $ABCD$ is a rhombus, $p = q = r = s$, and $I = G$; the result is trivial). Let $a = EB$ and $b = EA$.
(i) As the incenter of triangle EDC, $I = ((t+r)E,\ (a+q+r)D,\ (b+p+t)C)$.
(ii) As an excenter of triangle ABE, $I = ((p+q)E,\ -a \cdot A,\ -b \cdot B)$.

Note that $\frac{EC}{EB} = \frac{a+q+r}{a}$ and $\frac{ED}{EA} = \frac{b+p+t}{b}$, so that the system $(p+q+r+t)E$ is equivalent to the system $((a+q+r)B,\ -a \cdot C,\ (b+p+t)A,\ -b \cdot D)$. Therefore,

$$I = ((-a+b+p+t)A,\ (-b+a+q+r)B,\ (-a+b+p+t)C,\ (-b+a+q+r)D).$$

Since $b + p = a + q$, the result follows. $\square$

Lemma 5. $G = ((p+q+t)A,\ (p+q+r)B,\ (q+r+t)C,\ (p+r+t)D)$.

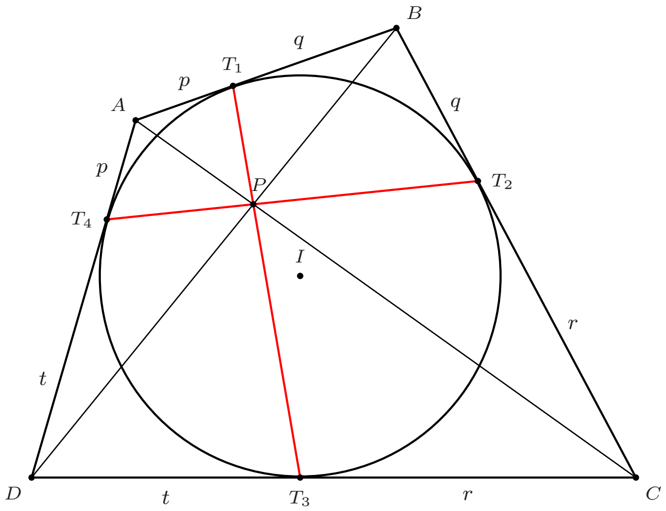

Figure 6.

Proof. Denote the point of intersection of the diagonals by P. Note that $\frac{AP}{CP} = \frac{p}{r}$ and $\frac{BP}{DP} = \frac{q}{t}$. Actually, according to one corollary of Brianchon's theorem, the lines T_1T_3 and T_2T_4 also pass through P. For another proof, see [4, pp.156–157]. Hence,

$$P = \left(\frac{1}{p}\cdot A,\ \frac{1}{q}\cdot B,\ \frac{1}{r}\cdot C,\ \frac{1}{t}\cdot D\right).$$

Consequently, $P = \left(\frac{1}{q}\cdot B,\ \frac{1}{t}\cdot D\right)$ and also $P = \left(\frac{1}{p}\cdot A,\ \frac{1}{r}\cdot C\right)$.

The quadrilateral $G_aG_bG_cG_d$ is homothetic to $ABCD$, with homothetic center $M = (1\cdot A,\ 1\cdot B,\ 1\cdot C,\ 1\cdot D)$ and ratio $-\frac{1}{3}$. Thus, $\frac{G_aG}{G_cG} = \frac{AP}{CP} = \frac{p}{r}$ and $\frac{G_bG}{G_dG} = \frac{BP}{DP} = \frac{q}{t}$. It follows that $G = (r\cdot G_a,\ p\cdot G_c) = (p\cdot A,\ (r+p)B, r\cdot C,\ (r+p)D)$ and $G = (t\cdot G_b,\ q\cdot G_d) = ((q+t)A,\ q\cdot B,\ (q+t)C,\ t\cdot D)$. To conclude the proof, it is enough to add up the corresponding masses. $\square$

The following theorem follows easily from Lemmas 3, 4, 5.

Theorem 6. *For a circumscriptible quadrilateral, the Nagel point N, centroid G and incenter I are collinear. Furthermore, $NG : GI = 2 : 1$.*

See Figure 7.

Theorem 6 enables us to define the Nagel line of a circumscriptible quadrilateral. This line also contains the Spieker point of the quadrilateral, by which we mean the center of mass S of the perimeter of the quadrilateral, without assuming an incircle.

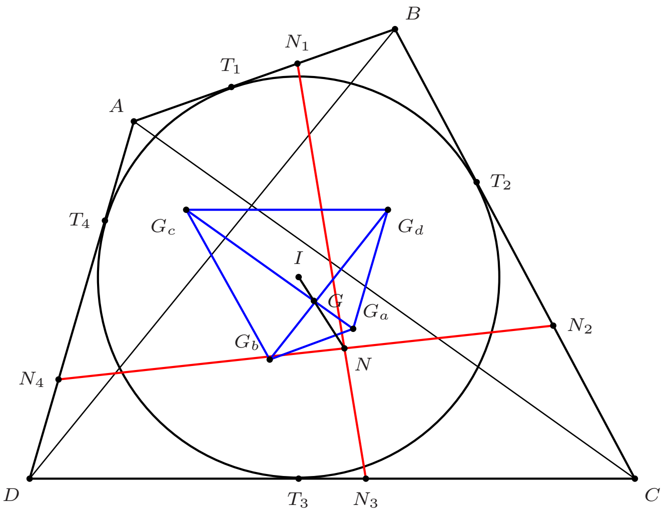

Figure 7.

Theorem 7. *For a circumscriptible quadrilateral, the Spieker point is the midpoint of the incenter and the Nagel point.*

Proof. With reference to Figure 6, each side of the circumscriptible quadrilateral is equivalent to a mass equal to its length located at each of its two vertices. Thus,

$$S = ((2p + q + t)A,\ (p + 2q + r)B,\ (q + 2r + t)C,\ (p + r + 2t)D).$$

Splitting into two systems of equal total masses, we have

$$N =(2pA,\ 2qB,\ 2rC,\ 2tD),$$
$$I =((q + t)A,\ (p + r)B,\ (q + t)C,\ ((p + r)D).$$

From this the result is clear. □

References

[1] E. Danneels and N. Dergiades, A theorem on orthology centers, *Forum Geom.*, 4 (2004) 135–141.

[2] A. Myakishev, Hyacinthos message 12400, March 16, 2006.

[3] F. Rideau, Hyacinthos message 12402, March 16, 2006.

[4] P. Yiu, *Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998.

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