VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 8th grade. Solutions.

1. (A.Blinkov) The diagonals of a trapezoid are perpendicular, and its altitude is equal to the medial line. Prove that this trapezoid is isosceles.

   First solution. Consider the line passing through C and parallel to BD. Let E be the common point of this line and the extension of base AD. Then ACE is a right-angled triangle, thus its median from vertex C is equal to the half of the hypothenuse, i.e. to the medial line of the trapezoid. By condition, this median coincides with the corresponding altitude. Hence the diagonals of trapezoid are equal.

   Second solution. Let AD, BC be the bases of the trapezoid and O be the common point of its diagonals. Then the medians of right-angled triangles OAD, OBC are equal to halfs of their hypothenuses, i.e. the sum of these medians is equal to the medial line. On the other hand, the altitude of the trapezoid is equal to the sum of altitudes of these triangles. By the assumption the medians coincide with the altitudes, thus triangles OAD, OBC are isosceles, and this yields $AB = CD$.

2. (T.Golenishcheva-Kutuzova) Peter made a paper rectangle, put it on an identical rectangle and pasted both rectangles along their perimeters. Then he cut the upper rectangle along one of its diagonals and along the perpendiculars to this diagonal from two remaining vertices. After this he turned back the obtained triangles in such a way that they, along with the lower rectangle form a new rectangle.

   Let this new rectangle be given. Restore the original rectangle using compass and ruler.

   Solution. Let $ABCD$ be the obtained rectangle; O be its center; K, M be the midpoints of its shortest sides AB and CD; L, N be the meets of BC and AD respectively with the circle with diameter KM (fig.8.2). Then $KLMN$ is the desired rectangle. In fact, let P be the projection of M to LN. Since $\angle CLM = \angle OML = \angle MLO$, the triangles MCL and MPL are equal. Thus the bend along ML matches these triangles. Similarly the bend along MN matches triangles MDN and MPN. Finally, since the construction is symmetric wrt point O, the bend along KL and KN matches triangles BKL and AKN with triangle NKL.

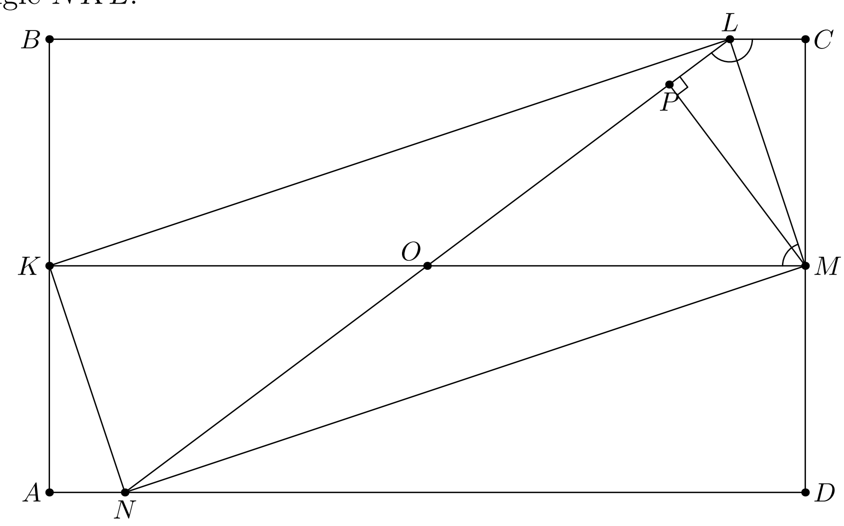

Fig.8.2

3. (A.Myakishev, D.Mavlo) The line passing through vertex A of triangle ABC and parallel to BC meets the circumcircle of ABC for the second time at point A_1. Points B_1 and C_1 are defined similarly. Prove that the perpendiculars from A_1, B_1, C_1 to BC, CA, AB respectively concur.

   First solution. Since A_1 is the reflection of A in the medial perpendicular to BC, the perpendicular from A_1 is the reflection of the altitude from A. Thus by the Thales theorem it passes through the reflection of the orthocenter in the circumcenter. The two remaining perpendiculars also pass through this point.

   Second solution. Let K, L and M be a common points of lines AA_1, BB_1 и CC_1 (fig.8.3). Prove that KC_1 is the altitude of triangle KLM. Since $KBCA$ is a parallelogram, and AC_1CB is an isosceles trapezoid, we have $KA = BC = AC_1$, $\angle KAB = \angle ABC = \angle BAC_1$. Therefore AB is the bisector and the altitude of isosceles triangle KAC_1. Thus $AB \perp KC_1$, and from this $CC_1 \perp KC_1$. Similarly LA_1 and MB_1 are also the altitudes of triangle KLM. Since the altitudes concur we obtain the assertion of the problem.

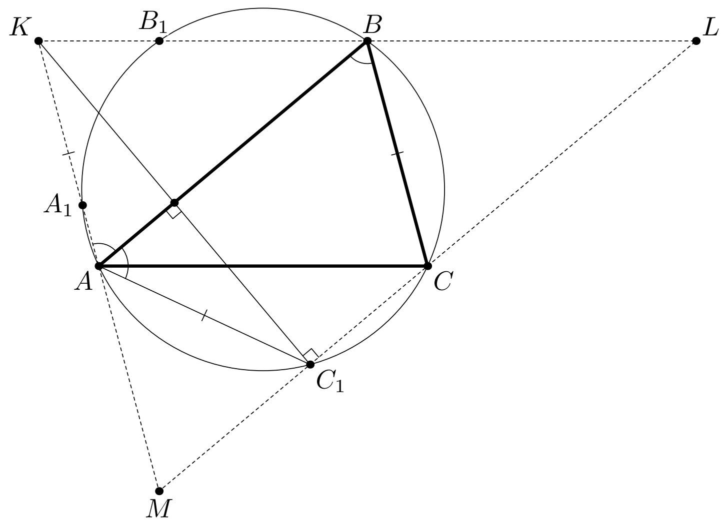

Fig.8.3.

4. (A.Shapovalov) Given the circle of radius 1 and several its chords with the sum of lengths 1. Prove that one can be inscribe a regular hexagon into that circle so that its sides don't intersect those chords.

   Solution. Paint the smallest arcs corresponding to given chords. If we rotate the painted arcs in such a way that the corresponding chords form a polygonal line, then the distance between the ends of it is less than 1, and since a chord with length 1 corresponds to an arc equal to 1/6 of the circle, the total length of painted arcs is less than 1/6 of the circle.

   Now inscribe a regular hexagon into the circle and mark one of its vertices. Rotate the hexagon, and when the marked vertex coincides with a painted point, paint the points corresponding to all remaining vertices. The total length of painted arcs increases at most 6 times, therefore there exists an inscribed regular hexagon with non-painted vertices. Obviously its sides don't intersect the given chords.

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 8th grade. Solutions.

5. (S.Markelov) A line passing through vertex A of regular triangle ABC doesn't intersect segment BC. Points M and N lie on this line, and $AM = AN = AB$ (point B lies inside angle MAC). Prove that the quadrilateral formed by lines AB, AC, BN, CM is cyclic.

   Solution. Since triangle BAN is isosceles, $\angle ANB = \frac{\angle MAB}{2}$ (fig.8.5). Similarly $\angle AMC = \frac{\angle NAC}{2}$. Thus the sum of these angles is equal to $60°$, and the angle between lines BN and CM is equal to $120°$, which yields the assertion of the problem.

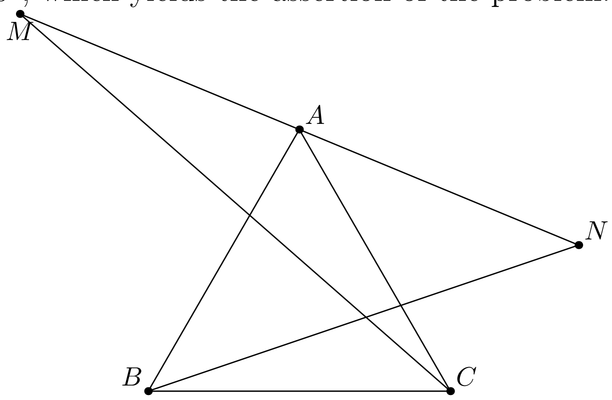

Fig.8.5

6. (D.Prokopenko) Let BB_1 and CC_1 be the altitudes of acute-angled triangle ABC, and A_0 is the midpoint of BC. Lines A_0B_1 and A_0C_1 meet the line passing through A and parallel to BC in points P and Q. Prove that the incenter of triangle PA_0Q lies on the altitude of triangle ABC.

   First solution. Since triangles BCB_1 and BCC_1 are right-angled, their medians B_1A_0, B_1C_0 are equal to the half of hypothenuse: $B_1A_0 = A_0C = A_0B = C_1A_0$. Now $\angle PB_1A = \angle CB_1A_0 = \angle B_1CA_0 = \angle PAC$, thus $PA = PB_1$ (fig.8.6.1). Similarly, $QA = QC_1$. Then the incircle of triangle A_0PQ touches its sides in points A, B_1, C_1, which yields the assertion of the problem.

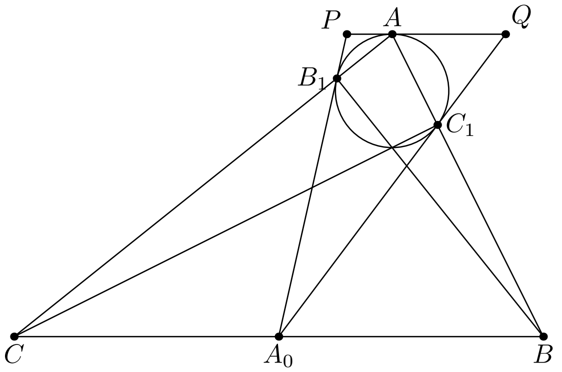

Fig.8.6

   Second solution. Let H be the orthocenter of ABC and O be the midpoint of AH. Then points A_0, B_1, C_1, O lie on the nine-points-circle of ABC, and A_0O is the diameter of this circle. On the other hand, points B_1, C_1 lie on the circle with diameter AH, thus this circle coincides with the incircle of APQ (fig.8.6.2).

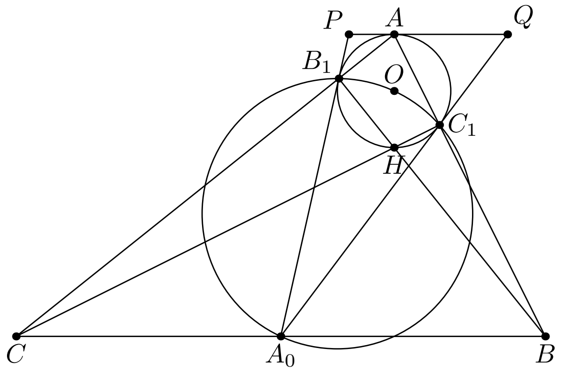

Fig.8.6.2

7. (A.Akopyan) Let a point M not lying on coordinates axes be given. Points Q and P move along Y- and X-axis respectively so that angle PMQ is always right. Find the locus of points symmetric to M wrt PQ.

   Solution. By condition, points P, Q, M and the origin O lie on the circle with diameter PQ. Thus point N symmetric to M wrt PQ also lies on this circle and $\angle PON = \angle PMN = \angle PNM = \angle POM$ (fig.8.7). Then N lies on the line symmetric to OM wrt the coordinates axes. On the other hand, if N is an arbitrary point of this line and P, Q are the common points of coordinates axes with circle OMN, then $\angle PMN = \angle PON = \angle POM = \angle PNM$ and $\angle PMQ = \angle POQ = \angle PNQ = 90°$, thus M and N are symmetric wrt PQ.

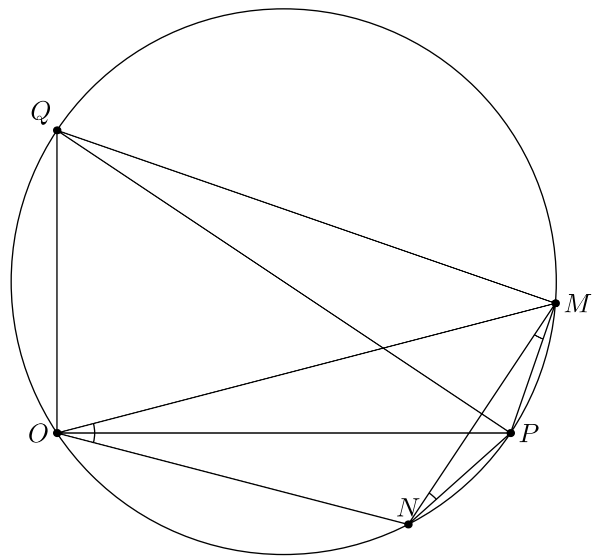

Fig.8.7

8. (A.Zaslavsky) Using only the ruler, divide the side of a square table into n equal parts. All lines drawn must lie on the surface of the table.

   Solution. Firstly bisect the side. Find center O of square $ABCD$ as a common point of its diagonals. Now let point X lie on side BC, Y be a common point of XO and AD, U be a common point of AX and BY, V be a common point of UC and XY (fig.8.8.1). Then line BV bisects the bases of trapezoid $CYUX$. The line passing through O and the midpoint of CY bisects sides AB and CD.

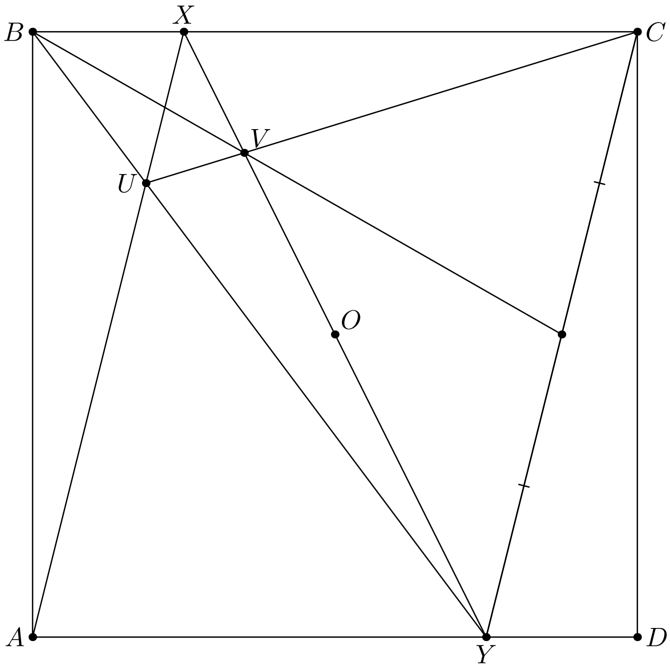

Fig.8.8.1

Now suppose that two opposite sides are divided into k equal parts. Let us demonstrate how to divide it into $k+1$ equal parts. Let $AX_1 = X_1X_2 = \cdots = X_{k-1}B$, $DY_1 = Y_1Y_2 = \cdots Y_{k-1}C$. Then by Thales theorem, lines $AY_1, X_1Y_2, \ldots, X_{k-1}C$ divide diagonal BD into $k+1$ equal parts (fig.8.8.2). Dividing similarly the second diagonal and joining the corresponding points by the lines parallel to BC we divide side AB into $k+1$ equal parts.

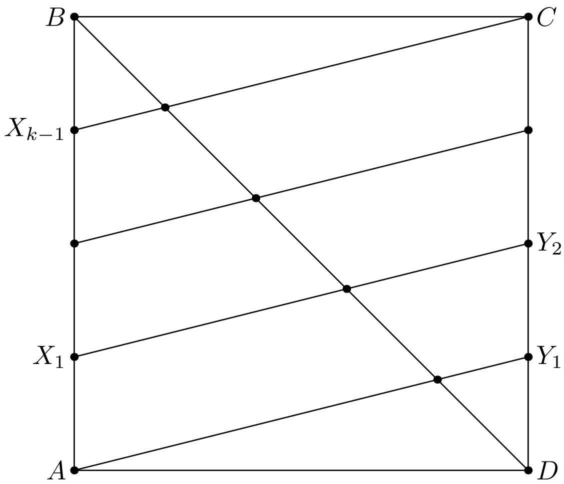

Fig.8.8.2

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 9th grade. Solutions.

1. (M.Kungozhin) Altitudes AA_1 and BB_1 of triangle ABC meet in point H. Line CH meets the semicircle with diameter AB, passing through A_1, B_1, in point D. Segments AD and BB_1 meet in point M, segments BD and AA_1 meet in point N. Prove that the circumcircles of triangles B_1DM and A_1DN touch.

   Solution. The angle between the tangent to circle B_1DM in point D and line AD is equal to angle MB_1D, which in its turn is equal to angle BAD (fig.9.1). Similarly the angle between the tangent to circle A_1DN and line BD is equal to angle ABD. Since $\angle BAD + \angle ADB = 90^\circ = \angle ADB$, these tangents coincide.

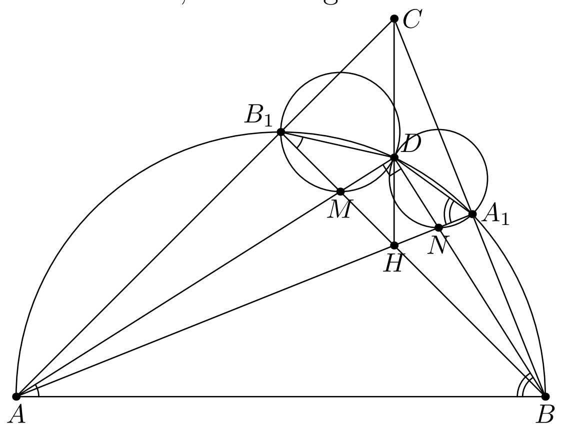

Fig.9.1

2. (D.Cheian) In triangle ABC, $\angle B = 2\angle C$. Points P and Q on the medial perpendicular to CB are such that $\angle CAP = \angle PAQ = \angle QAB = \frac{\angle A}{3}$. Prove that Q is the circumcenter of triangle CPB.

   Solution. Let D be the reflection of A in the medial perpendicular to BC. Then $ABCD$ is the isosceles trapezoid and its diagonal BD is the bisector of angle B. Thus $CD = DA = AB$. Now $\angle DAP = \angle C + \angle A/3 = (\angle A + \angle B + \angle C)/3 = 60^\circ$. Thus triangle ADP is equilateral and $AP = AB$. Since AQ is the bisector of angle PAB, $QP = QB = QC$ (fig.9.2).

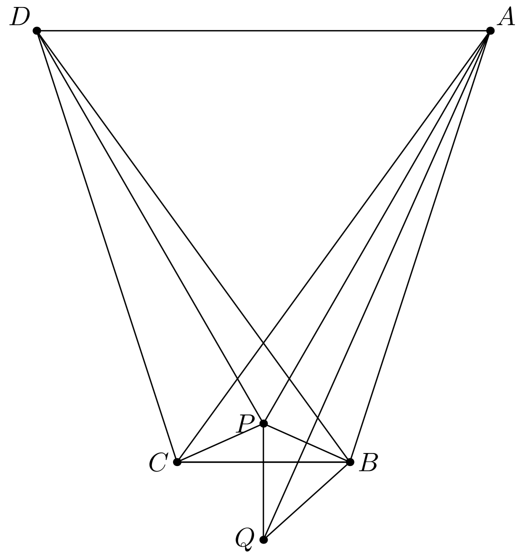

Fig.9.2

3. (A.Karluchenko) Restore the isosceles triangle ABC ($AB = AC$) if the common points I, M, H of bisectors, medians and altitudes respectively are given.

   Solution. Circumcenter O of the triangle lies on the extension of HM beyond point M, and $MO = HM/2$. Now BI, CI are the bisectors of angles OBH, OCH ($\angle CBH = \angle ABO = \pi/2 - \angle C$). Thus $BO/BH = CO/CH = IO/IH$, i.e. points B, C lie on the Apollonius circle of points O and H passing through I. But the circumcenter of triangle BIC lies on the circumcircle of ABC. So we obtain the following construction.

   Construct point O and the Apollonius circle. Now construct the circle with center O passing through the center of constructed circle. These two circles meet in points B, C, and line OH again meets the circle with center O in point A.

4. (A.Zaslavsky) Quadrilateral $ABCD$ is inscribed into a circle with center O. The bisectors of its angles form a cyclic quadrilateral with circumcenter I, and its external bisectors form a cyclic quadrilateral with circumcenter J. Prove that O is the midpoint of IJ.

   Solution. Let the bisectors of angles A and B, B and C, C and D, D and A meet in points K, L, M, N respectively (fig.9.4). Then line KM bisects the angle formed by lines AD and BC. If this angle is equal to ϕ, then by external angle theorem we obtain that $\angle LKM = \angle B/2 - \phi/2 = (\pi - \angle A)/2 = \angle C/2$ and thus $\angle LIM = \angle C$. On the other hand, the perpendiculars from L and M to BC and CD respectively form the angles with ML equal to $(\pi - \angle C)/2$, i.e. the triangle formed by these perpendiculars and ML is isosceles and the angle at its vertex is equal to C. Thus the vertex of this triangle coincides with I. So the perpendiculars from the vertices of $KLMN$ to the corresponding sidelines of $ABCD$ pass through I. Similarly the perpendiculars from the vertices of triangle formed by external bisectors pass through J.

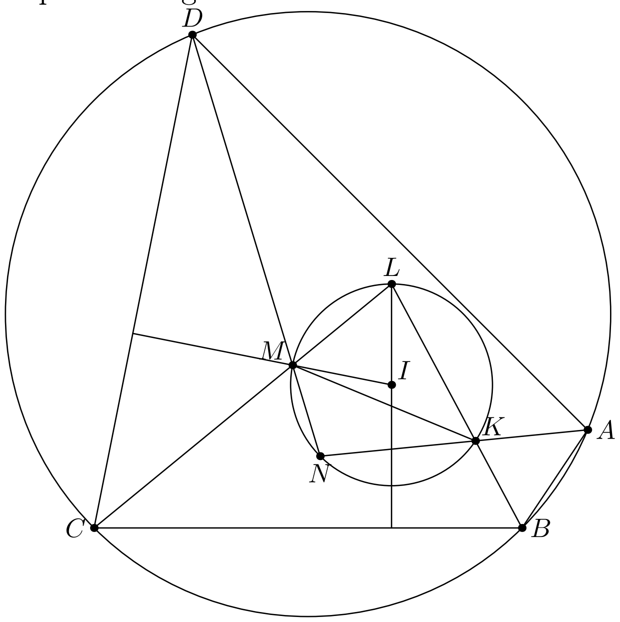

Fig.9.4

   Now let K' be the common point of external bisectors of angles A and B. Since quadrilateral $AKBK'$ is inscribed into the circle with diameter KK', the projections of K and K' to AB are symmetric wrt the midpoint of AB. From this and the above assertion, the projections of I and J to each side of $ABCD$ are symmetric wrt the midpoint of this side, and this is equivalent to the sought assertion.

Note. The similar property of a triangle is well-known: the circumcenter is the midpoint of the segment between the incenter and the circumcenter of the triangle formed by external bisectors.

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9th grade. Solutions.

5. (B.Frenkin) It is possible to compose a triangle from the altitudes of a given triangle. Can we conclude that it is possible to compose a triangle from its bisectors?

   Solution. No. Consider a triangle with two sides equal to 2 and 3 and increase the angle between these sides. When the angle approaches to 180° the ratio of altitudes of triangle approaches to $1/2 : 1/3 : 1/5$, thus for any value of the angle it is possible to compose the triangle from the altitudes. On the other hand, the smallest bisector approaches to zero, and two remaining bisectors approach to different values. Thus for great values of the angle it is impossible to compose the triangle from the bisectors.

   Let us present the exact estimates. Firstly note that if a and b are two sides of a triangle, C is the angle between these sides, and l_c is the bisector of this angle, then the area of the triangle is equal to $S = ab\sin C/2 = (a+b)l_c \sin\frac{C}{2}/2$, thus $l_c = 2ab\cos\frac{C}{2}/(a+b)$. The lengths of bisectors l_a, l_b are determined similarly.

   Now let $a = 2$, $b = 3$. Then $\cos\frac{A}{2} > \cos\frac{B}{2}$. Thus

   $$l_a - l_b > 2c\cos\frac{A}{2}\left(\frac{b}{b+c} - \frac{a}{a+c}\right) = \frac{2c^2\cos\frac{A}{2}}{(c+2)(c+3)}.$$

   Take angle C sufficiently great such that $c > 4$, $\cos\frac{A}{2} > 0,9$, $\cos\frac{C}{2} < 0,1$. Then $l_a - l_b > l_c$ and it is impossible to compose a triangle from the bisectors. On the other hand, we have $h_b/h_a = 2/3$, $2/5 < h_c/h_a < 1/2$. Thus $h_b + h_c > h_a > h_b > h_c$ and it is possible to compose a triangle from the altitudes.

6. (P.Dolgirev) In triangle ABC AA_0 and BB_0 are medians, AA_1 and BB_1 are altitudes. The circumcircles of triangles CA_0B_0 and CA_1B_1 meet again in point M_c. Points M_a, M_b are defined similarly. Prove that points M_a, M_b, M_c are collinear and lines AM_a, BM_b, CM_c are parallel.

   Solution. Let O and H be the circumcenter and the orthocenter of triangle ABC. Since $\angle CA_0O = \angle CB_0O = \angle CA_1H = \angle CB_1H = 90°$, CO and CH are the diameters of circles CA_0B_0 and CA_1B_1 respectively. Thus the projection of point C to line OH lies on both circles, i.e. coincides with M_c (fig.9.6). This yields the assertion of the problem

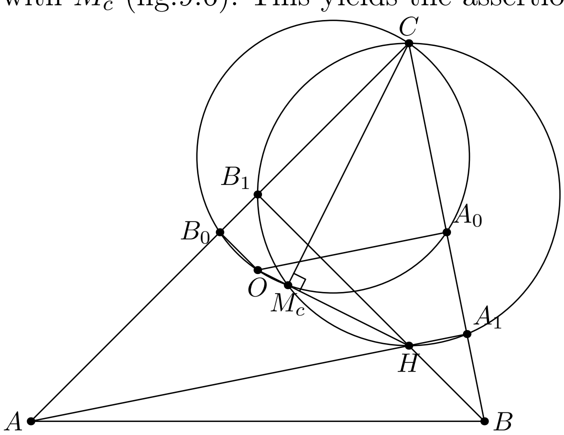

Fig.9.6

7. (I.Bogdanov) Circles ω and Ω are inscribed into the same angle. Line ℓ meets the sides of angles, ω and Ω in points A and F, B and C, D and E respectively (the order of points on the line is A, B, C, D, E, F). It is known that $BC = DE$. Prove that $AB = EF$.

First solution. Let one side of the angle touch ω and Ω in points X_1, Y_1, and the second side touch them in points X_2, Y_2; U, V are the common points of X_1X_2 and Y_1Y_2 with AF. The midpoint of CD lies on the radical axis of the circles, i.e. the medial line of trapezoid $X_1Y_1Y_2X_2$, thus $BU = EV$ and $CU = DV$ (fig.9.7). This yields that $X_1U \cdot X_2U = Y_1V \cdot Y_2V$. Hence $FY_2/FX_2 = Y_2V/X_2U = X_1U/Y_1V = AX_1/AY_1$, i.e. $AX_1 = FY_2$. Now from $AB \cdot AC = AX_1^2 = FY_2^2 = FE \cdot FD$ we obtain the assertion of the problem.

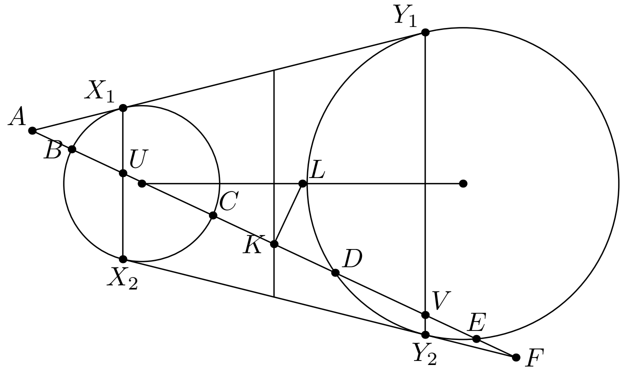

Fig.9.7

Second solution. Prove that there exists exactly one line passing through a fixed point A on a side of the angle which satisfies the condition of the problem. In fact, the distances from the midpoint K of segment CD to the projections of the centers of the circles to the sought line are equal, thus K coincides with the projection of the midpoint L of the segment between the centers. Hence K is the common point of the circle with diameter AL and the radical axis, distinct from the midpoint of segment X_1Y_1. On the other hand, if F is a point such that $AX_1 = Y_2F$ then $AB \cdot AC = FE \cdot FD$ and $AD \cdot AE = FC \cdot FB$, thus AF is the sought line.

8. (B.Frenkin) A convex n-gon P, where $n > 3$, is dissected into equal triangles by diagonals non-intersecting inside it. Which values of n are possible, if P is circumscribed?

Solution. Let us prove that $n = 4$.

Lemma. Let a convex polygon be dissected into equal triangles by non-intersecting diagonals. Then each triangle of the dissection has at least one side which is a side (not a diagonal) of the polygon

Proof. Let a triangle of the dissection have angles $\alpha \le \beta \le \gamma$; A, B, C be the corresponding vertices, AC and BC be the diagonals of the polygon. There are at least two another angles of dividing triangles adjacent to C. If one of them is greater than α, then the sum of angles adjacent to C is not less than $\gamma + \beta + \alpha = \pi$, but this sum can't be greater than an angle of convex polygon, a contradiction. Thus all angles in vertex C, except $\angle ACB$ are equal to α and $\alpha < \beta$.

Consider the second triangle adjacent to BC. Since it is equal to $\triangle ABC$, its angle opposite to BC is equal to α. But angle C also is equal to α although it must be β or γ, a contradiction. Lemma is proved.

Now we turn to the solution of the problem.

Since the sum of angles of P is equal to $\pi(n-2)$ the number of dividing triangles is equal to $n-2$. By lemma, each of these triangles has at least one side coinciding with a side of P. Hence there are two triangles having two sides coinciding with sides of P.

Let KLM be one of these triangles, KL and LM be the sides of P. KM is the side of another dissection triangle KMN. One of its sides (for example KN) is a side of P. Since the triangles are equal, $\angle NKM$ is equal to $\angle LKM$, or to $\angle KML$. In the first case KM bisects the angle P and so passes through the incenter I. In the second case $KN \parallel LM$. Then I lies on the common perpendicular to these two segments, thus it lies inside parallelogram $KLMN$, i.e. belongs to at least one of triangles $KLM,\ KMN$.

Let $K'L'M'$ be the second triangle with two sides coinciding with sides of P. Similarly we obtain that I lies inside this or adjacent dissection triangle. If I lies inside one of triangles $KLM,\ K'L'M'$, then they are adjacent and $n = 4$. In the opposite case I lies inside triangle KMN which is adjacent to both these triangles. Then MN is a side of $\triangle K'L'M'$; let $M = M',\ N = K'$. As above we obtain that $LM \parallel KN \parallel L'M$. But then sides LM and $L'M$ of the convex polygon lie on the same line, a contradiction.

$>$From the above argument we see that a convex quadrilateral satisfies the condition of the problem iff it is symmetric wrt one of its diagonals.

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 10th grade. Solutions.

1. (M.Rozhkova) In triangle ABC the midpoints of sides AC, BC, vertex C and the centroid lie on the same circle. Prove that this circle touches the circle passing through A, B and the orthocenter of triangle ABC.

   Solution. Let point C' be the reflection of C in the midpoint of AB. Then points A, B, C' and the orthocenter of ABC lie on the same circle. On the other hand, if A_0, B_0 are the midpoints of BC, AC, then triangle A_0B_0C is homothetic to triangle ABC' wrt cenroid M of ABC with coefficient $-1/2$. Thus the circumcircles of these triangles touch in M (fig.10.1).

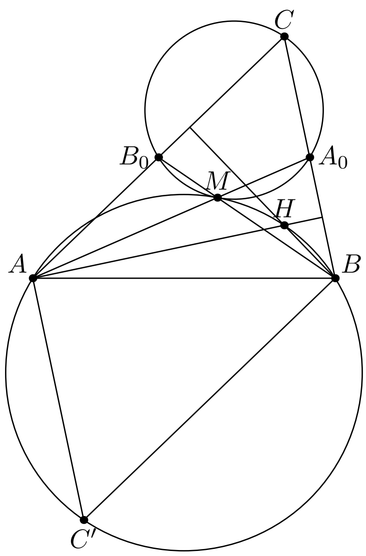

Fig.10.1

2. (L.Emelyanov) Quadrilateral $ABCD$ is circumscribed. Its incircle touches sides AB, BC, CD, DA in points K, L, M, N respectively. Points A', B', C', D' are the midpoints of segments LM, MN, NK, KL. Prove that the quadrilateral formed by lines AA', BB', CC', DD' is cyclic.

   Solution. Let us begin with the assertion which follows from a simple calculation of angles.

   Lemma. Points A, B, C, D lie on the same circle iff the bisectors of angles formed by lines AB and CD are parallel to the bisectors of angles formed by lines AD and BC.

   In fact, consider the case when $ABCD$ is a convex quadrilateral, rays BA and DC meet in point E, rays DA and BC meet in point F. Then the angles between the bisectors of angles BED and BFD are equal to half-sums of opposite angles of the quadrilateral. This clearly yields the assertion of lemma. Another cases can be considered similarly.

   Now let us turn to the solution of the problem. Let I be the incenter of $ABCD$, r be the radius of its incircle. Then $IC' \cdot IA = r^2 = IA' \cdot IC$, i.e. points A, C, A', C' lie on the circle. By lemma, the bisectors of angles between AA' and CC' are parallel to the bisectors of angles between IA and IC, and hence to the bisectors of the angles between perpendicular lines KN and LM. Similarly the bisectors of the angles between BB' and

DD' are parallel to the bisectors of the angles between KL and MN. Using again the lemma we obtain the assertion of the problem.

3. (A.Akopyan) Given two tetrahedrons $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Consider six pairs of edges A_iA_j and B_kB_l, where (i, j, k, l) is a transposition of numbers $(1, 2, 3, 4)$ (for example A_1A_2 and B_3B_4). It is known that for all but one such pairs the edges are perpendicular. Prove that the edges in the remaining pair also are perpendicular.

   Solution. Let us prove firstly the following lemma.

   Lemma. Edges A_1A_2 and B_3B_4 are perpendicular iff the perpendiculars from points A_1, A_2 to planes $B_2B_3B_4$ and $B_1B_3B_4$ respectively intersect.

   Proof of Lemma. Let $A_1A_2 \perp B_3B_4$. Then there exists a plane passing through A_1A_2 and perpendicular to B_3B_4. The perpendiculars from the condition of Lemma lie on this plane and hence intersect. Conversely, if the perpendiculars intersect then the plane containing them is perpendicular to B_3B_4 and passes through A_1A_2.

   Now let $A_1A_2 \perp B_3B_4$, $A_1A_3 \perp B_2B_4$, $A_2A_3 \perp B_1B_4$. Then any two of three perpendiculars from A_1, A_2, A_3 to the corresponding faces of $B_1B_2B_3B_4$ intersect. Since these three perpendicular aren't complanar, this yields that they have a common point. Thus if the condition of the problem is true then all four perpendiculars from the vertices of one tetrahedron to the corresponding faces of the other have a common point and the edges in the sixth pair are perpendicular.

4. (V.Mokin) Point D lies on the side AB of triangle ABC. The circle inscribed in angle ADC touches internally the circumcircle of triangle ACD. Another circle inscribed in angle BDC touches internally the circumcircle of triangle BCD. These two circles touch segment CD in the same point X. Prove that the perpendicular from X to AB passes through the incenter of triangle ABC.

   Solution. Firstly prove next lemma.

   Lemma. Let a circle touch sides AC, BC of triangle ABC in points U, V and touch internally its circumcircle in point T. Then line UV passes through the incenter I of triangle ABC.

   Proof of Lemma. Let lines TU, TV intersect the circumcircle again in points X, Y. Since circles ABC and TUV are homothetic with center T, points X, Y are the midpoints of arcs AC, BC, i.e. lines AY and BX meet in point I (fig.10.4.1). Thus the assertion of the lemma follows from Pascal theorem applied to hexagon $AYTXBC$.

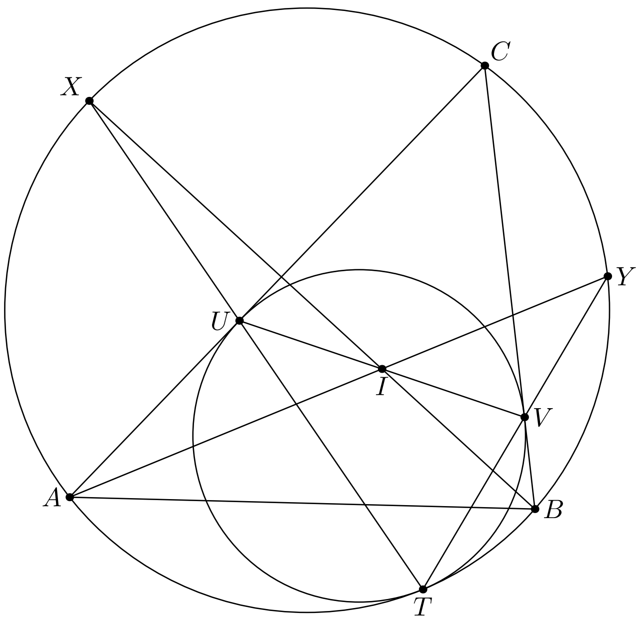

Fig.10.4.1

>From lemma and the condition of the problem we obtain that DI_1XI_2, where I_1, I_2 are the incenters of triangles ACD, BCD, is the rectangle (fig.10.4.2.). Let Y, C_1, C_2 be the projections of points X, I_1, I_2 to AB. Then $BY - AY = BC_2 + C_2Y - AC_1 - C_1Y = (BC_2 - DC_2) - (AC_1 - DC_1) = (BC - CD) - (AC - CD) = BC - AC$. Thus Y is the touching point of AB with the incircle.

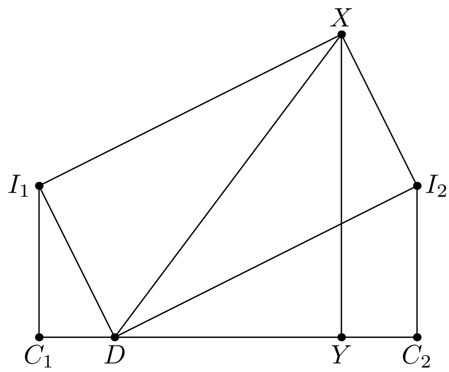

Fig.10.4.2

VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10th grade. Solutions.

5. (A.Blinkov) The touching point of the excircle with the side of a triangle and the base of the altitude to this side are symmetric wrt the base of the corresponding bisector. Prove that this side is equal to one third of the perimeter.

   Solution. By condition the radius r_c of the excircle touching side AB of triangle ABC is equal to altitude h_c. Since the square of the triangle equals $S = (p - c)r_c = ch_c/2$, we have $c = 2(p - c) = 2p/3$.

6. (M.Rozhkova) Prove that for any nonisosceles triangle $l_1^2 > \sqrt{3}S > l_2^2$, where l_1, l_2 are the greatest and the smallest bisectors of the triangle and S is its area.

   Solution. Let $a > b > c$ be the sidelengths of the triangle. Then l_2 is the bisector of angle A and $S = bc \sin A/2 = (b + c)l_2 \sin \frac{A}{2}/2$. Thus we can write down the right inequality as $\sqrt{3}(b + c) \sin \frac{A}{2}/2 > 2bc \cos \frac{A}{2}/(b + c)$ or $\sqrt{3} \operatorname{tg} \frac{A}{2} > 4bc/(b + c)^2$. But $\pi/6 < A/2 < \pi/2$, thus the left part is greater than 1, and the right part is less than 1 by Cauchy inequality.

   Since $C < \pi/3$, we have $\sqrt{3}S < 3ab/4$. On the other hand $l_1^2 = 4a^2b^2 \cos^2 \frac{C}{2}/(a + b)^2 = 2a^2b^2(1 + \cos C)/(a + b)^2$. Since $b > c$, we have $\cos C > a/2b$ and $l_1^2 > a^2b(a + 2b)/(a + b)^2$. Thus the left inequality follows from $a(a + 2b)/(a + b)^2 = 1 - b^2/(a + b)^2 > 3/4$.

7. (G.Feldman) Point O is the circumcenter of acute-angled triangle ABC, points A_1, B_1, C_1 are the bases of its altitudes. Points A', B', C' lying on lines OA_1, OB_1, OC_1 respectively are such that quadrilaterals $AOBC'$, $BOCA'$, $COAB'$ are cyclic. Prove that the circumcircles of triangles AA_1A', BB_1B', CC_1C' have a common point.

   Solution. Let H be the orthocenter of ABC. Then $AH \cdot HA_1 = BH \cdot HB_1 = CH \cdot CH_1$, i.e the degrees of point H wrt circles AA_1A', BB_1B', CC_1C' are equal and H lies inside these circles. On the other hand $\angle BC'O = \angle BAO = \angle OBC_1$, i.e. triangles $OC'B$ and OBC_1 are similar and $OC_1 \cdot OC' = OB_0^2$ (fig.10.7). Thus the degrees of point O wrt all three circles are also equal, so these circles meet in two points lying on line OH.

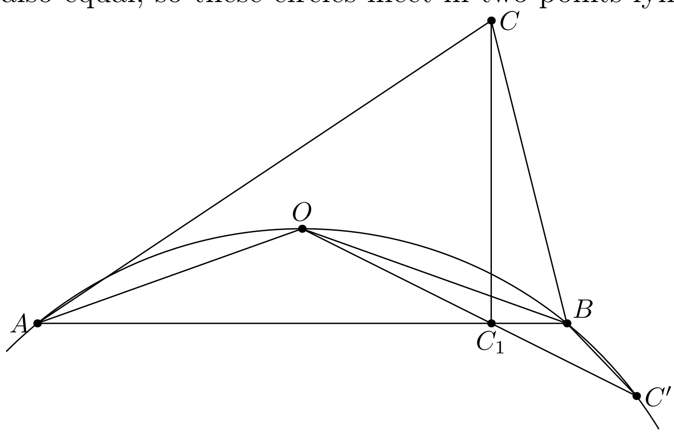

Fig.10.7

8. (S.Tokarev) Given a sheet of tin 6×6. It is allowed to bend it and to cut it but in such a way that it doesn't fall to pieces. How to make a cube with edge 2, divided by partitions into unit cubes?

Solution. The sought development is presented on Fig.10.8. Bold lines describe the cuts, thin and dotted lines describe the bends up and down. The central 2×2 square corresponds to the horizontal partition of the cube.

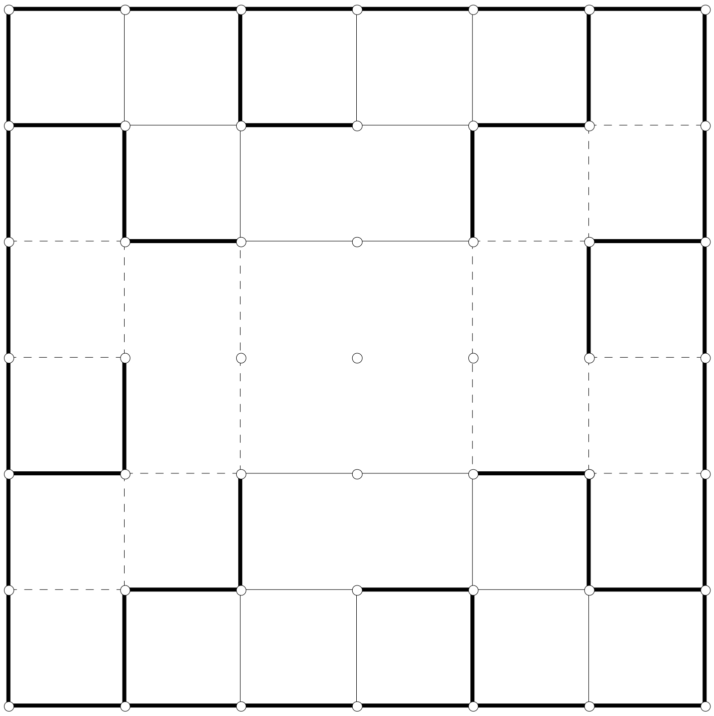

Fig.10.8.