IX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN.
THE CORRESPONDENCE ROUND. SOLUTIONS.

1. (N.Moskvitin) Let ABC be an isosceles triangle with $AB = BC$. Point E lies on side AB, and ED is the perpendicular from E to BC. It is known that $AE = DE$. Find $\angle DAC$.

   Answer. 45°.

   Solution. By the external angle theorem $\angle AED = 90^\circ + \angle B = 270^\circ - 2\angle A$ (fig.1). Therefore, $\angle EAD = (180^\circ - \angle AED)/2 = \angle A - 45^\circ$.

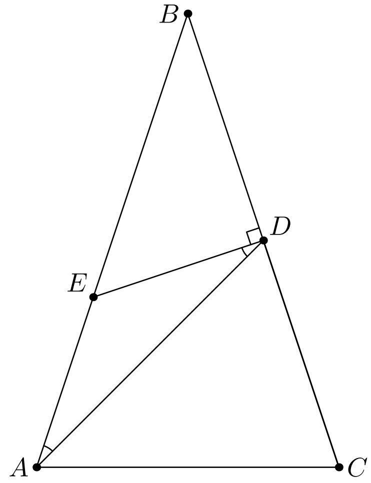

Fig.1

2. (L.Steingarts) Let ABC be an isosceles triangle ($AC = BC$) with $\angle C = 20^\circ$. The bisectors of angles A and B meet the opposite sides in points A_1 and B_1 respectively. Prove that triangle A_1OB_1 (where O is the circumcenter of ABC) is regular.

   Solution. On sides BC and AC take points A' and B' such that $AB' = B'O = OA' = A'B$. It is clear that $A'B' \parallel AB$, i.e. $\angle CA'B' = \angle CBA = 80^\circ$. Also $\angle A'OB = \angle A'BO = \angle BCO = 10^\circ$. Thus $\angle CA'O = 20^\circ$ and $\angle OA'B' = 60^\circ$, i.e triangle $OA'B'$ is regular. Then $A'B' = A'B$ and $\angle A'BB' = \angle A'B'B = \angle ABB'$ (fig.2). Therefore B' coincides with B_1. Similarly A' coincides with A_1, q.e.d.

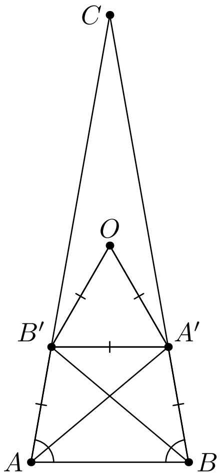

Fig.2

3. (D.Shvetsov) Let ABC be a right-angled triangle ($\angle B = 90^\circ$). The excircle inscribed into the angle A touches the extensions of the sides AB, AC at points A_1, A_2 respectively; points C_1,

C_2 are defined similarly. Prove that the perpendiculars from A, B, C to C_1C_2, A_1C_1, A_1A_2 respectively, concur.

Solution.

Let I be the incenter of ABC, and D be the fourth vertex of rectangle $ABCD$. Since $AI \perp A_1A_2$, $CI \perp C_1C_2$, the perpendiculars from A to CC_1 and from C to AA_1 meet in the incenter J of triangle ACD. Then it is sufficient to prove that $DI \perp A_1C_1$. Let X, Y, Z be the projections of I to AB, BC, CD respectively. Then $BC_1 = XC_2 = ZD$ and $A_1B = CY = IZ$, thus triangles A_1BC_1 and IZD are equal, i.e. $\angle IDZ = \angle A_1C_1B$ (fig.3), that proves the required assertion.

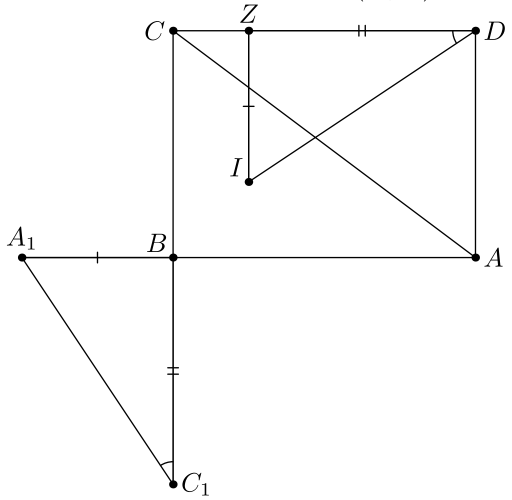

Fig.3

4. (F.Ivlev) Let ABC be a nonisosceles triangle. Point O is its circumcenter, and point K is the center of the circumcircle w of triangle BCO. The altitude of ABC from A meets w at a point P. The line PK intersects the circumcircle of ABC at points E and F. Prove that one of the segments EP and FP is equal to the segment PA.

   Solution. Points O and K lie on the bisector of segment BC, thus $OK \parallel AP$ and $\angle OPK = \angle POK = \angle OPA$. Therefore the reflection A' of A in OP lies on PK. Also $OA' = OA$, i.e. A' lies on the circumcircle of ABC (fig.4). Thus A' coincides with one of points E, F.

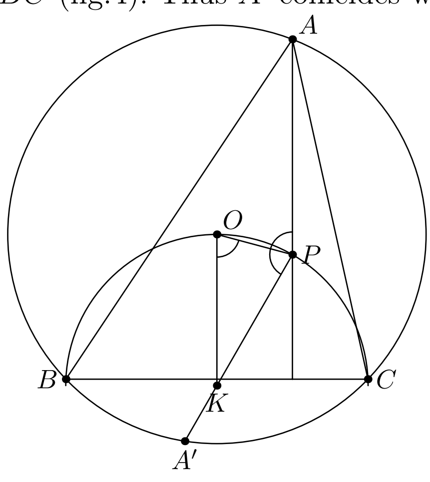

Fig.4

5. (B.Frenkin) Four segments drawn from a given point inside a convex quadrilateral to its vertices, split the quadrilateral into four equal triangles. Can we assert that this quadrilateral is a rhombus?

Answer. Yes.

Solution. Let $ABCD$ and O be the given quadrilateral and point. In equal triangles the angles opposite to equal sides are equal. Since $\triangle ABO = \triangle CBO$, angles BAO and BCO opposite to BO are equal. Similarly $\angle DAO = \angle DCO$, thus $\angle BAD = \angle BCD$. Two remaining angles of the quadrilateral are similarly equal, therefore $ABCD$ is a parallelogram.

There exist two adjacent angles with vertex O such that their sum is not less than π, suppose that these angles are $\angle AOB$ and $\angle COB$. The second angle is equal to some angle of triangle AOB. This can be only $\angle AOB$, because its sum with any of two remaining angles of AOB is less than π. The sides of equal triangles AOB and COB opposite to these angles are equal. Then $AB = BC$ and $ABCD$ is a rhombus.

6. (D.Shvetsov) Diagonals AC and BD of a trapezoid $ABCD$ meet at point P. The circumcircles of triangles ABP and CDP intersect the line AD for the second time at points X and Y respectively. Let M be the midpoint of segment XY. Prove that $BM = CM$.

Solution.

By condition, $\angle BXA = \angle BPA = \angle CPD = \angle CYD$ (fig.6). Thus $BXYC$ is an isosceles trapezoid, which proves the required assertion.

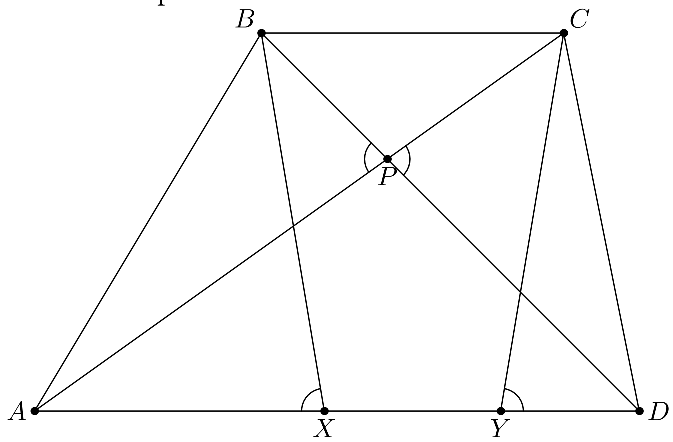

Fig.6

7. (D.Shvetsov) Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90°$.

Solution.

Lines AI_a and CI_c meet in the incenter I of ABC. By the bisectrix theorem $AI_a/I_aI = AD/ID$, $CI_c/I_cI = CD/ID$. By the Menelaos theorem $QA/QC = AD/CD = AB/BC$. Therefore BQ is the external bisectrix of angle B, q.e.d.

8. (M.Plotnikov) Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet the circumcircle for the second time at points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.

Answer. The arc of the circle passing through B, C and the circumcenter O of ABC.

Solution. Let the circles touche. Then the angles between their common tangent and lines AC and AB are equal to angles ALE and AKF respectively. Since these two angles are equal to angles ABX and ACX, their sum is equal to angle A and $\angle BXC = 2\angle A = \angle BOC$. Similarly we obtain that for any point of the arc the correspondent circles touche.

9. (M.Plotnikov) Let T_1 and T_2 be the points of tangency of the excircles of a triangle ABC with its sides BC and AC respectively. It is known that the reflection of the incenter of ABC across the midpoint of AB lies on the circumcircle of triangle CT_1T_2. Find $\angle BCA$.

   Answer. $90°$.

   Solution. Let D be the fourth vertex of parallelogram $ACBD$, J be the incenter of ABD, S_1, S_2 be the points of tangency of the incircle of ABC with AD and BD. Then $S_1T_1 \parallel AC$, $S_2T_2 \parallel BC$ and $\angle T_1JT_2 = \angle S_1JS_2 = \pi - \angle C$. Also $DS_1 = DS_2$, i.e. lines S_1T_1, S_2T_2 and DJ concur. Therefore J coincides with the common point of lines S_1T_1 and S_2T_2, i.e. $\angle C = 90°$ (fig.9).

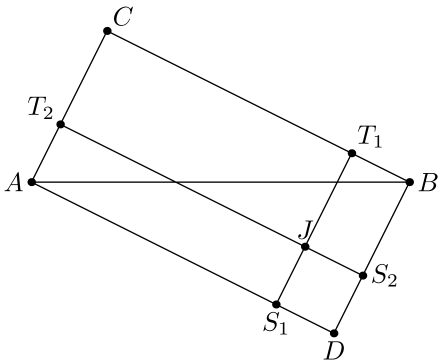

Fig.9

10. (D.Shvetsov) The incircle of triangle ABC touches the side AB at point C'; the incircle of triangle ACC' touches the sides AB and AC at points C_1, B_1; the incircle of triangle BCC' touches the sides AB and BC at points C_2, A_2. Prove that the lines B_1C_1, A_2C_2, and CC' concur.

    Solution.

    Since $AC' - BC' = AC - BC$, the incircles of triangles ACC' and BCC' touche CC' at the same point. Therefore $CB_1 = CA_2$. Also $AB_1 = AC_1$, $BA_2 = BC_2$, and if we find the angles of quadrilateral $A_2B_1C_1C_2$, we obtain that it is cyclic. Thus B_1C_1, A_2C_2 and CC' concur in the radical center of three circles: the circumcircle of $A_2B_1C_1C_2$ and the incircles of triangles ACC', BCC' (fig.10).

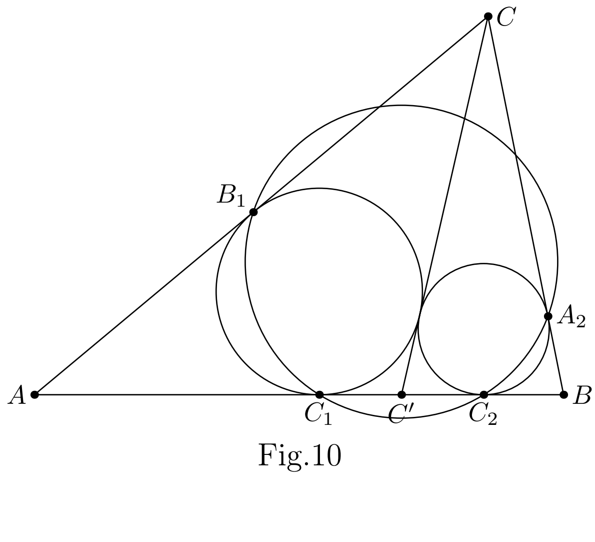

Fig.10

11. (P.Kozhevnikov) a) Let $ABCD$ be a convex quadrilateral and $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABC, BCD, CDA, DAB. Can the inequality $r_4 > 2r_3$ hold?

   b)The diagonals of a convex quadrilateral $ABCD$ meet in point E. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABE, BCE, CDE, DAE. Can the inequality $r_2 > 2r_1$ hold?

   Answer. a) No. b) No.

   Solution. a) Suppose that $r_4 = r(ABC)$. It is sufficient to prove that $r(ABC)/2 < \max\{r(ABD), r(CB$ The midpoint K of AC lies inside one of triangles ABD, CBD, for example inside ABD. Then triangle AKL, where L is the midpoint of AB, lies inside triangle ABD, therefore $r(ABC)/2 = r(AKL) < r(ABD)$.

   b) Let $r = r_1$ be the inradius of triangle ABE. The diameters of the incircles of triangles BCE, ADE, are less than the altitudes of these triangles coinciding with altitudes h_a, h_b of ABE. Thus it is sufficient to prove that one of these altitudes is less than $4r$. Suppose that $AE \ge BE$. Then the semiperimeter $p < AE + BE \le 2AE$ and $h_b = 2S/AE = 2pr/AE < 4r$.

   Comment. Note that the answer to both questions will be positive if we replace 2 to any smaller number.

12. (B.Frenkin) (8–11) On each side of triangle ABC, two distinct points are marked. It is known that these points are the feet of the altitudes and the bisectors.

   a) Using only a ruler determine which points are the feet of the altitudes and which points are the feet of the bisectors.

   b) Solve p.a) drawing only three lines.

   Solution. Preliminary hints. Since all points are distinct the triangle isn't isosceles. For each side, the foot of the altitude lies between the foot of the bisector and the smaller of two remaining sides. Thus it is sufficient to define the smallest and the greatest of the sides. We will denote the feet of the bisector and the altitude from vertex X as L_X and H_X respectively.

   Lemma. If $|AC| > |BC|$ then lines L_BL_A and H_BH_A meet the extension of side AB beyond B.

   Proof. Let L_BD be the perpendicular from L_B to AB, and CH be the altitude. By the bisector theorem $|L_BD| : |CH| = |AB| : (|BC| + |AB|)$. Similarly if L_AE is the perpendicular from L_A to AB, then $|L_AE| : |CH| = |AB| : (|AC| + |AB|)$. Furthermore $|AC| > |BC|$, $|L_BD > |L_AE|$, thus L_BL_A meets the extension of AB beyond B.

   Points H_B, H_A lie on the semicircle with diameter AB. Since $\angle H_AAB < \angle H_BBA$, the distance from H_A to AB is less than the distance from H_B to AB. The lemma is proved.

   Simple solution of p.a). Joining the given points with the opposite vertices we obtain two families of concurrent lines. Take two points of the same family on two sides and draw the line through them. By the lemma this line meets the extension of the third side beyond the vertex lying on the smaller of two sides. Therefore we can define the smaller of any two sides.

   Solution of p.b). Take for each vertex the nearest marked points on two adjacent sides and join these points. We will prove that *these lines meet the prolongation of the greatest side beyond the vertex of the medial angle and the extensions of two remaining sides beyond the vertex of the greatest angle.* From this we can define the greatest and the smallest side.

Let us prove the above assertion. Suppose that $|AB| > |AC| > |BC|$. The marked points nearest to the vertex of the smallest angle are the feet of the bisectors, and the points nearest to the vertex of the greatest angle are the feet of the altitudes. By the lemma, the lines joining these points meet the extension of BC beyond C and the extension of AB beyond B. The marked points nearest to B are H_C and L_A. By the lemma, line L_CL_A meets the extension of AC beyond C in some point P. Ray H_CL_A passes inside triangle H_CCP and thus intersects segment CP, q.e.d.

13. (F.Ivlev) Let A_1 and C_1 be the tangency points of the incircle of triangle ABC with BC and AB respectively, A' and C' be the tangency points of the excircle inscribed into the angle B with the extensions of BC and AB respectively. Prove that the orthocenter H of triangle ABC lies on A_1C_1 if and only if the lines $A'C_1$ and BA are orthogonal.

    Solution. Suppose that $A'C_1 \perp BA$. Then by Thales theorem the altitude from C divides segment A_1C_1 in ratio $A_1C : CA' = p-c : p-a$. The altitude from A passes through the same point. The inverse assertion is obtained similarly.

14. (D.Shvetsov) Let M, N be the midpoints of diagonals AC, BD of right-angled trapezoid $ABCD$ ($\angle A = \angle D = 90°$). The circumcircles of triangles ABN, CDM meet line BC in points Q, R. Prove that the distances from Q, R to the midpoint of MN are equal.

    Solution. Let X, Y be the projections of N and M to BC. Than we have to prove that $RY = XQ$. Since $\angle NQX = \angle NAB = \angle DBA$, triangles XQN and ABD are similar (fig.14). Thus $XQ = AB \cdot NX/AD$. But $NX = CD \sin \angle BCD/2 = CD \cdot AD/2BC$, therefore $XQ = AB \cdot CD/2BC = RY$.

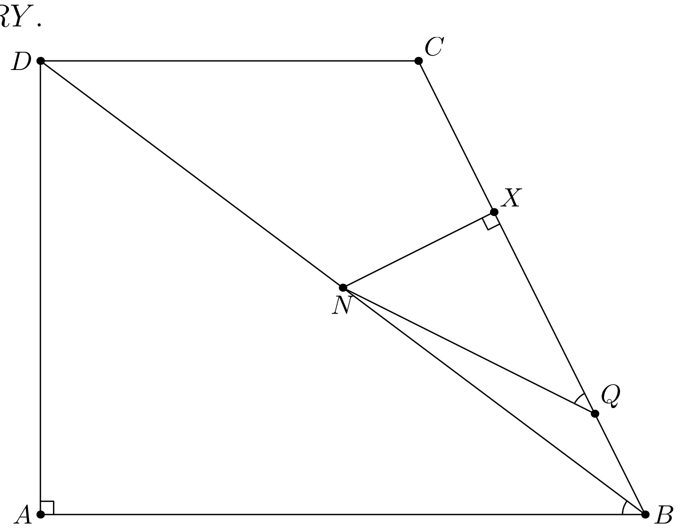

Fig.14

15. a) (V.Rastorguev) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are equal.

    b) (P.Kozhevnikov) Points $A_1, B_1, C_1, A_2, B_2, C_2$ lie inside triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2 and angles $BAA_1, CBB_1, ACC_1, CAA_2, ABB_2, BCC_2$ are equal. Prove that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

    Solution. a) Inscribe triangle $A_2B_2C_2$ into triangle $A'B'C'$ in such a way that $C_2A_2 \perp B'C'$, $A_2B_2 \perp C'A'$, $B_2C_2 \perp A'B'$. It is clear that the corresponding sidelines of triangles ABC

and $B'C'A'$ are symmetric wrt the circumcenter of $A_2B_2C_2$. This symmetry maps $A_2B_2C_2$ to $B_1C_1A_1$. Therefore these triangles are equal and their circumcenters coincide.

b) Consider the chords AA', BB', CC', AA'', BB'', CC'' of the circumcircle of ABC lying on the lines A_1B_1, B_1C_1, C_1A_1, A_2C_2, B_2A_2, C_2B_2. By condition, arcs AC', BA', CB', AB'', CA'', BC'' are equal. Let their size be φ. The rotation around the circumcenter to φ maps AA', BB', CC' to BB'', CC'', AA'' respectively, thus it maps $A_1B_1C_1$ to $A_2B_2C_2$ (fig.15).

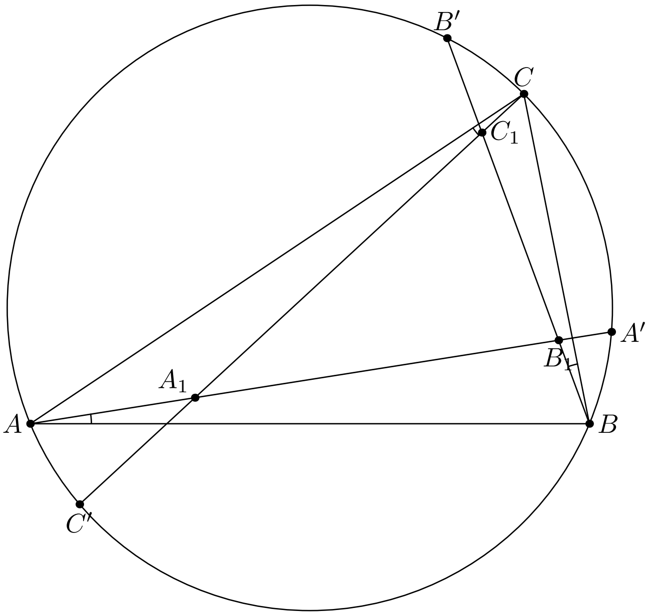

Fig.15

Comment. In a special case when triangle $A_1B_1C_1$ degenerates to a point, $A_2B_2C_2$ also degenerates to a point, and the distances from these two points to the circumcenter are equal. These points are *the Brockard points* of the triangle.

16. (F.Ivlev) The incircle of triangle ABC touches BC, CA, AB at points A', B', C' respectively. The perpendicular from the incenter I to the median from vertex C meets the line $A'B'$ in point K. Prove that $CK \parallel AB$.

Solution. The polar transformation wrt the incircle maps the perpendicular from I to the median into the infinite point of this median, the image of line $A'B'$ is point C, and the image of the line passing through C and parallel to AB is the common point P of $A'B'$ and IC'. Thus we have to prove that P lies on the median.

Since $IA' = IB'$, $\angle PIB' = \angle A$, $\angle PIA' = \angle B$, we have $B'P : A'P = BC : AC$. Since $CA' = CB'$, we have $\sin\angle ACP : \sin\angle BCP = BC : AC$, i.e. CP bisects AB (fig.16).

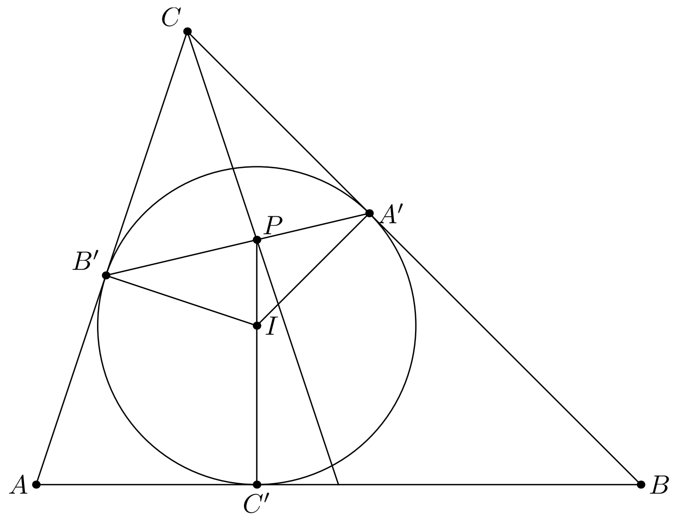

Fig.16

17. (A.Zaslavsky) An acute angle between the diagonals of a cyclic quadrilateral is equal to ϕ. Prove that an acute angle between the diagonals of any other quadrilateral having the same sidelengths is smaller than ϕ.

    Solution. Let the diagonals of quadrilateral $ABCD$ meet in point P. Put $PA = a$, $PB = b$, $PC = c$, $PD = d$ and express the sidelengths of $ABCD$ through a, b, c, d and $\cos\phi$. Then

    $$|AB^2 - BC^2 + CD^2 - CA^2| = 2\cos\phi(ab + bc + cd + da) = 2AC \cdot BD\cos\phi.$$

    By Phtolomeos theorem $AC \cdot BD \leq AB \cdot CD + BC \cdot AD$, and the equality holds only for a cyclic quadrilateral.

18. (A.Ivanov) Let AD be a bisector of triangle ABC. Points M and N are the projections of B and C to AD. The circle with diameter MN intersects BC in points X and Y. Prove that $\angle BAX = \angle CAY$.

    Solution. Let B', C', X', Y' be the reflections of B, C, X, Y in MN. Then the diagonals of isosceles trapezoid $BB'CC'$ meet at point L, which is the reflection of A in the circle with diameter MN. The diagonals of isosceles trapezoid $XX'YY'$ inscribed into this circle also meet at L. The lateral sidelines of this trapezoid meet on the polar of L, passing through A and parallel to the bases of the trapezoid. By symmetry A is the common point of the sidelines, which implies the assertion of the problem (fig.18).

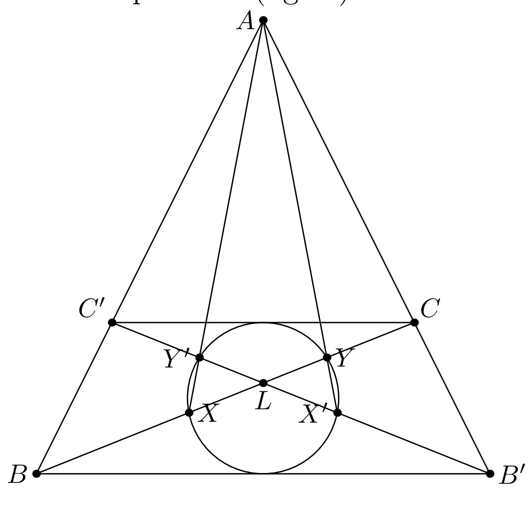

Fig.18

19. (D.Prokopenko) a) The incircle of a triangle ABC touches AC and AB at points B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to the bisector AL in points Q and P respectively. Prove that the lines PC_0, QB_0, and BC concur.

b) Let AL be the bisector of a triangle ABC. Points O_1 and O_2 are the circumcenters of triangles ABL and ACL respectively. Points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. Prove that the lines O_1C_1, O_1B_1, and BC concur.

c) Prove that two points obtained in pp. a) and b) coincide.

Solution. a) It is clear that $PQ \parallel B_0C_0$. Also P lies on the circumcircle of ACL. Thus $\angle PLA = \angle C/2$ and $\angle PLB = 90^\circ - \angle B/2 = \angle C_0A_0B$, where A_0 is the touching point of the incircle with BC. Therefore the corresponding sidelines of triangles PQL and $C_0B_0A_0$ are parallel i.e., these triangles are homothetic (fig.19a). The homothety center S lies on line LA_0. Thus lines P_0 and QB_0 meet in S, i.e. on line BC.

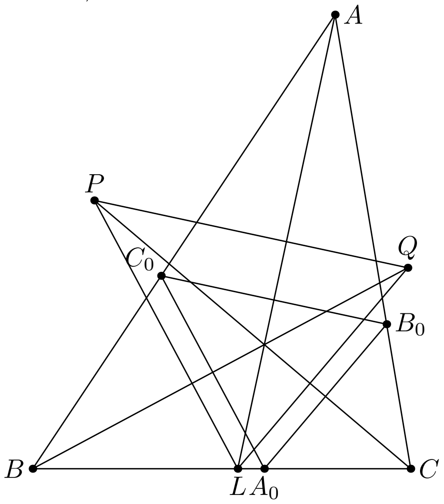

Fig.19a

b) First prove that points C_0, B_0, C_1 and B_1 are collinear. In fact, since the reflection of B in the bisector of angle C lies on AC, point C_1 lies on the medial line $A'C'$. Also we have $A'C_1 = BC/2$, and therefore $C'C_1 = |AC - BC|/2 = C'B_0$. This property defines the common point of $A'C'$ and B_0C_0. Thus lines O_1O_2 and C_1B_1 are parallel. Now quadrilateral BC_1IA_0 is cyclic, therefore $\angle C_1A_0B = 90^\circ - \angle A/2 = \angle O_1LB$ and $A_0C_1 \parallel LO_1$. Similarly $A_0B_1 \parallel LO_2$ (fig.19b). Thus triangles O_1O_2L and $C_1B_1A_0$ are homothetic.

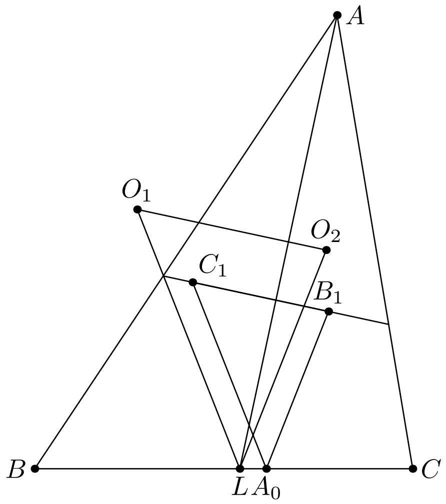

Fig.19b

c) Both homotheties of pp. a) and b) transform A_0 to L, and line B_0C_0 to the medial perpendicular to AL. Therefore their centers coincide.

20. (V.Yassinsky) Let C_1 be an arbitrary point on the side AB of triangle ABC. Points A_1 and B_1 on the rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

    Solution. By condition, quadrilaterals ACA_1C_1 and BCB_1C_1 are cyclic. Thus $\angle B_1BC_1 = \angle ACC_1$, $\angle A_1AC_1 = \angle BCC_1$, and therefore $\angle AC_2B = \pi - \angle C$, i.e. C_2 lies on the circle passing through A, B and the reflection C' of C wrt AB. Also $\angle BC'C_1 = \angle BAC_2$, thus $C'C_1$ passes through C_2 (fig.20).

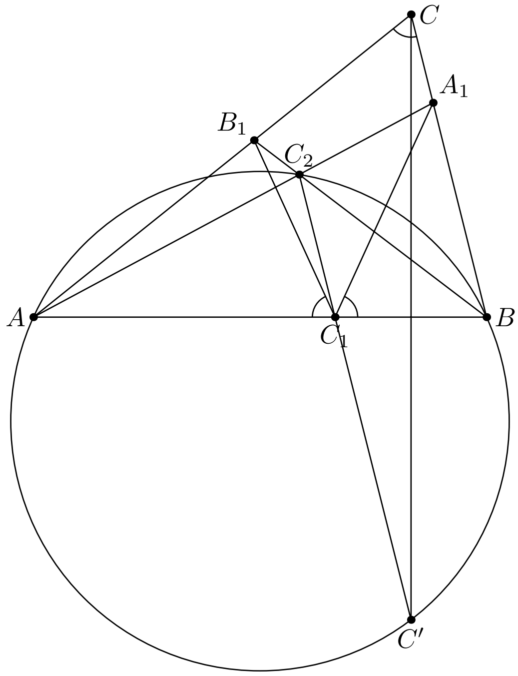

Fig.20

21. (D.Yassinsky) Let A be a point inside a circle ω. One of two lines drawn through A intersects ω at points B and C, the second one intersects it at points D and E (D lies between A and E). The line passing through D and parallel to BC meets ω for the second time at point F, and

the line AF meets ω at point T. Let M be the common point of the lines ET and BC, and N be the reflection of A across M. Prove that the circumcircle of triangle DEN passes through the midpoint of segment BC.

Solution. Firstly, project line AB to the circle from point D, and then project the circle to AB from point T. As a result we obtain that the image of A is M, the image of infinite point is A, and points B and C are fixed. From the equality of cross-ratios we obtain that $MB/MC = (AB/AC)^2$. Hence $AM = AB \cdot AC/(AB + AC)$. Now let K be the midpoint of BC. Then $AN \cdot AK = 2AM(AB + AC)/2 = AB \cdot AC = AD \cdot AE$, i.e. points D, E, K, N are concyclic.

22. (A.Zaslavsky) The common perpendiculars to the opposite sidelines of a nonplanar quadrilateral are mutually orthogonal. Prove that they intersect.

    Solution. Let K, L, M, N be the feet of common perpendiculars lying on the sides AB, BC, CD, DA of quadrilateral $ABCD$. The projection to the plane parallel to KM and LN transforms these lines to perpendicular lines $K'M'$ and $L'N'$. By three perpendiculars theorem the projections of AB and CD are perpendicular to $K'M'$, and the projections of BC and AD are perpendicular to $L'N'$. Therefore the projection of $ABCD$ is a rectangle $A'B'C'D'$, and $A'K' = D'M'$, $B'L' = A'N'$. Thus $AK/KB = DM/MC$, $BL/LC = AN/ND$ and by Menelaos theorem K, L, M, N are complanar.

23. (B.Frenkin) Two convex polytopes A and B do not intersect. The polytope A has exactly 2012 planes of symmetry. What is the maximal number of symmetry planes of the union of A and B, if B has a) 2012, b) 2013 symmetry planes?

    c) What is the answer to the question of p.b), if the symmetry planes are replaced by the symmetry axes?

    Answer. a) 2013. b) 2012. c) 1.

    Solution. a) *Estimation.* The symmetry transposes polyhedrons A and B or fixes each of them. In the first case it transposes the centroids of polyhedrons, thus the symmetry plane is the perpendicular bisector of the segment between the centroids. In the second case this plane is a symmetry plane of both polyhedrons A and B. Thus we have at most 1+2012=2013 planes. *Example.* Let A be regular 2012-gonal pyramid. Take a point outside A on its axis and construct a plane P passing through this point and perpendicular to the axis. Let B the reflection of A in P. Then all conditions are valid, and P and 2012 symmetry planes of A are the symmetry planes of the union.

    b) *Estimation.* Since A and B have a distinct number of symmetry planes, they aren't equal and can't be transposed by a symmetry. Thus each symmetry is a symmetry of polyhedron A, which has only 2012 symmetry planes. *Example.* Let A be a regular 2012-gonal pyramid. Take a point outside A on its axis, a plane passing through this point and perpendicular to the axis, and construct the reflection of the pyramid's base in this plane. Let B be a prism with this reflection as the base, disjoint from A. It is clear that B has 2013 symmetry planes: one of them is parallel to the bases of the prism and equidistant from them and 2012 remaining planes coincide with the symmetry planes of A.

    c) *Estimation.* Since A and B have a distinct number of symmetry axes they can't be transposed. Thus the sought symmetry fixes the centroid of each polyhedron. These centroids don't coincide because the polyhedrons are convex. Therefore the symmetry axis coincides with the line joining two centroids. *Example.* Let A be a regular 2011-gonal prism with horizontal bases. Then A has one vertical and 2011 horizontal symmetry axes. Now let B be a regular 2012-gonal prism with the same axis, disjoint from A. Then B has 2013 symmetry axes and the union of A and B has a vertical symmetry axis.