XX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutons

1. (8, D.Shvetsov) Bisectors AI and CI meet the circumcircle of triangle ABC at points A_1, C_1 respectively. The circumcircle of triangle AIC_1 meets AB at point C_0; point A_0 is defined similarly. Prove that A_0, A_1, C_0, C_1 are collinear.

   Solution. Let A_1C_1 meet AB at point A'. Then

   $$\angle C_1A'A = (\smile AC_1 + \smile BA_1)/2 = (\smile AC_1 + \smile A_1C)/2 = \angle C_1IA.$$

   Therefore A, I, A', C_1 are concyclic and A' coincides with A_0 (fig. 1). Similarly we obtain that A_1C_1 passes through C_0.

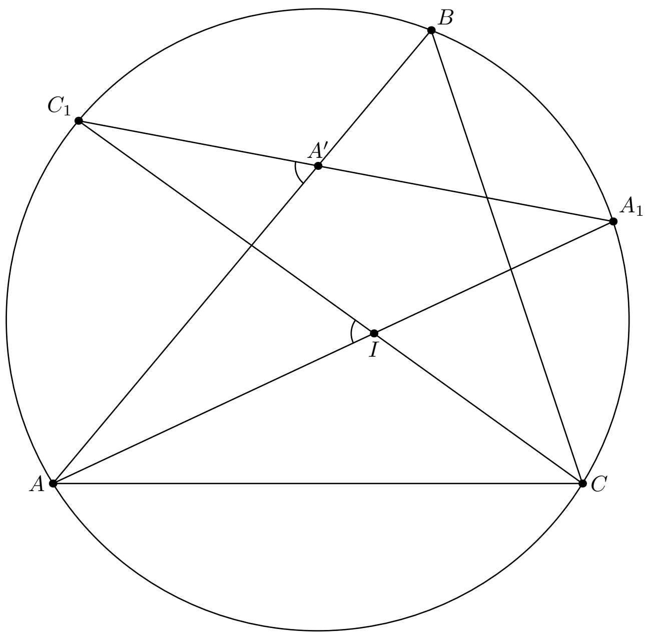

Fig. 1.

2. (8, B.Frenkin) Three different collinear points are given. What is the number of isosceles triangles such that these points are their circumcenter, incenter and excenter (in some order)?

   Answer. Two, if the medial point divide the segment between two remaining ones into two parts such that their ratio is not greater than 3, and three otherwise.

   Solution. Let ABC be an isosceles triangle with vertex C; O, I, I_c be its circumcenter, incenter and excenter (center of excircle touching AB). Then A, B, I, I_c lie on the circle with diameter II_c centered at the midpoint W of arc AB. Hence if O, I, I_c are given, then we construct the triangle ABC finding the midpoint W of segment II_c, drawing the circles centered at O, W with radii OW, WI respectively, finding the common points A and B of these circles, and the reflection C of W about O. But I has to lie inside the circumcircle. This is always correct if O is one of two extreme points (then I is the medial point). And if O is the medial point, then the inequality $OW > OI$ has to be correct, i.e. $OI_c : OI > 3$.

3. (8, K.Belskij) Let ABC be an acute-angled triangle, and M be the midpoint of the minor arc BC of its circumcircle. A circle ω touches the side AB, AC at points P, Q respectively and passes through M. Prove that $BP + CQ = PQ$.

   Solution. By the assumption $\angle PAQ = (\smile PMQ - \smile PQ)/2 = \pi - \smile PQ = \pi - 2\angle PMQ$, i.e. $\angle PMQ = \angle APQ$. Thus PM and QM bisect the angles BPQ, CQP respectively, and M is the incenter of triangle APQ. Then constructing perpendiculars MX, MY to AB, AC respectively we obtain that $PX = QY = PQ/2$. Also since $MB = MC$, we have $BX = CY$ (fig. 3). Therefore $BP + QC = PX + XB + QY - YC = PQ$.

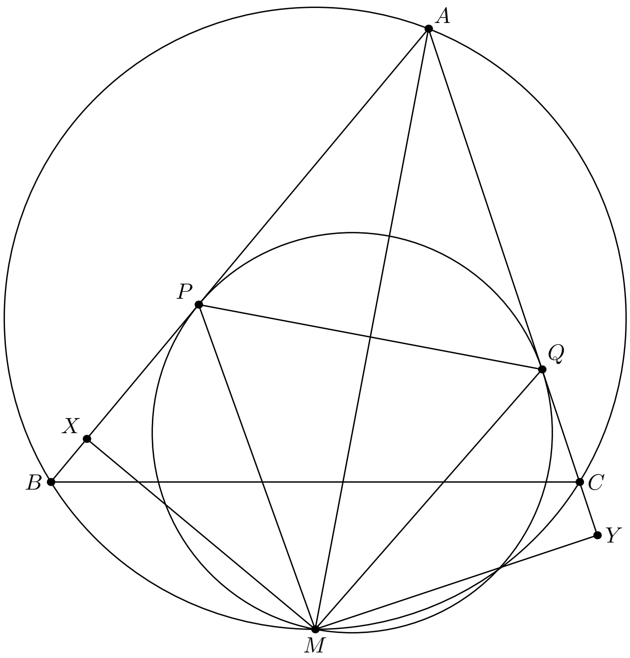

Fig. 3.

4. (8, L.Emelyanov) The incircle ω of triangle ABC touches BC, CA, AB at points A_1, B_1 and C_1 respectively, P is an arbitrary point on ω. The line AP meets the circumcircle of triangle AB_1C_1 for the second time at point A_2. Points B_2 and C_2 are defined similarly. Prove that the circumcircle of triangle $A_2B_2C_2$ touches ω.

   Solution. Points A, B_1, C_1 lie on the circle with diameter AI, where I is the incenter of ABC. Hence $\angle IA_2A = \angle IA_2P = 90^\circ$, and A_2 lies on the circle with diameter IP, touching ω at P (fig. 4). Points B_2, C_2 also lie on this circle.

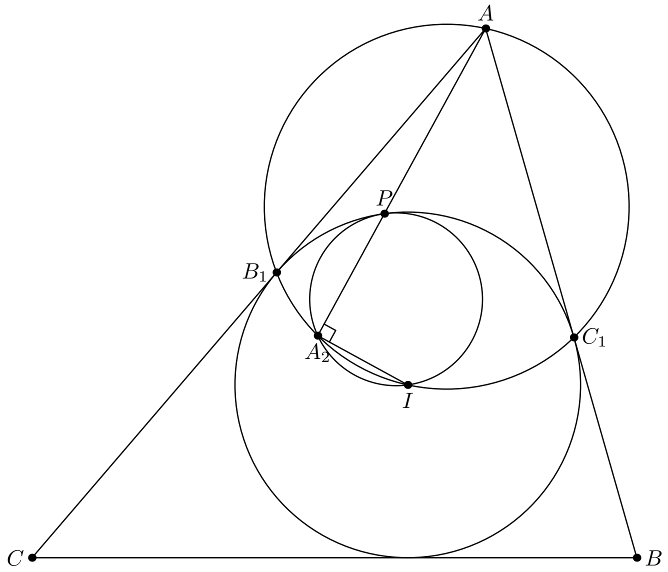

Fig. 4.

5. (8, P.Pogosjan) Points A', B', C' are the reflections of vertices A, B, C about the opposite sidelines of triangle ABC. Prove that the circles $AB'C'$, $A'BC'$, and $A'B'C$ have a common point.

   Solution. Let X be the second common point of circles $AB'C'$ and $A'BC'$. Then $\angle(XB', XC') = \angle(AB', AC') = 3\angle(AC, AB)$. Similarly $\angle(XC', XA') = 3\angle(BA, BC)$. Therefore $\angle(XB', XA') = 3\angle(AC, BC) = \angle(CB', CA')$.

6. (8–9, A.Shekera) A circle ω and two points A, B of this circle are given. Let C be an arbitrary point on one of arcs AB of ω; CL be the bisector of triangle ABC; the circle BCL meet AC at point E; and CL meet BE at point F. Find the locus of circumcenters of triangles AFC.

   Answer. A segment with the endpoint at the midpoint of arc ACB, forming the angle with AB equal to $\pi/2 - \angle ACB$

   Solution. Let O be the center of circle ACF. Then $\angle AOF = 2\angle ACF = \angle ACB$ do not depend on C. Hence all triangles AOF are similar, and O is the image of F about the spiral similarity with center A. Also $\angle ABF = \angle LBE = \angle LCE$ do not depend on C, thus F moves along a line. Therefore all points O also lie on a line. The angle between this line and BE equals $OAF = (\pi - \angle ACB)/2 = i/2 - \angle ABE$, hence it is perpendicular to the reflection of AB about BE (fig. 6). When C tends to B, O tends to the midpoint of arc ACB, and when C tends to A, F tends to the tangent to the circumcircle ABC at A. Therefore the required locus is the segment given in the answer.

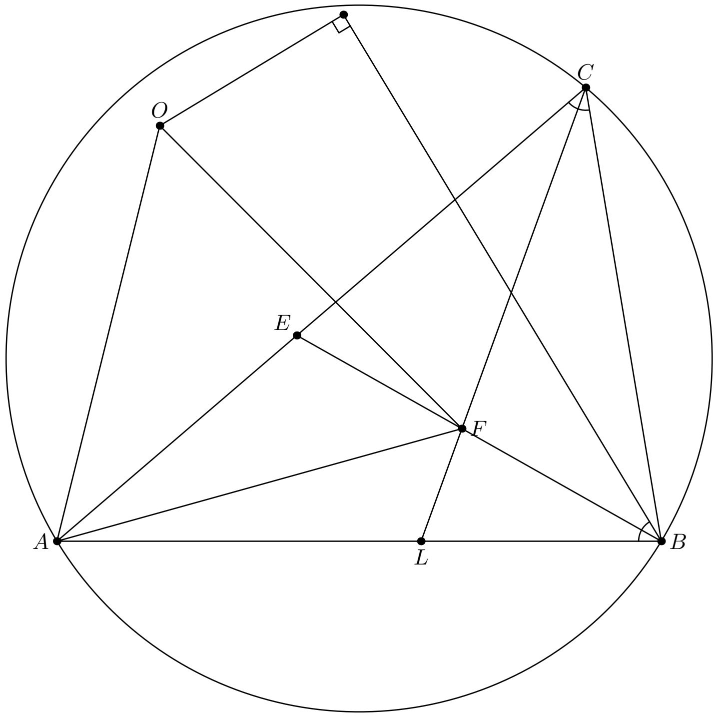

Fig. 6.

7. (8–9, B.Frenkin) Restore a bicentral quadrilateral if two opposite vertices and the incenter are given.

   Solution. Let the incenter I of bicentral quadrilateral $ABCD$ lie inside triangle ABC. Then $\angle AIC = \angle ABC + \angle IAB + \angle ICB = \angle ABC + \pi/2$. Thus if A, C, I are given, we can find the value of angle B and construct the circumcircle centered at O. Also from the Poncelet theorem we obtain that the line OI passes through the common point L of diagonals of $ABCD$, the midpoints M, N of AC, BD respectively lie on the circle with diameter OL, and the line MN passes through I. Hence we can construct the point N and the diagonal BD.

8. (8–9, K.Belskij) Let $ABCD$ be a quadrilateral with $\angle B = \angle D$ and $AD = CD$. The incircle of triangle ABC touches the sides BC and AB at points E and F respectively. Prove that the midpoints of segments AC, BD, AE, and CF are concyclic.

   Solution. Let K, L, M, N be the midpoints of BD, AC, AE, CF respectively. Since $LM \parallel BC$, and $LN \parallel AB$, we have $\angle MLN = \angle CBA$. On the other hand $\overrightarrow{KM} = (\overrightarrow{DA} + \overrightarrow{BE})/2$, $\overrightarrow{KN} = (\overrightarrow{DC} + \overrightarrow{BF})/2$, and since $DA = DC$, $BE = BF$, and the angle between $\overrightarrow{DA}$ and $\overrightarrow{DC}$ equals the angle between $\overrightarrow{BE}$ and $\overrightarrow{BF}$, we obtain that the angle MKN also is equal to these angles.

9. (8–9, A.Mardanov) Let $ABCD$ ($AD \parallel BC$) be a trapezoid circumscribed around a circle ω, which touches the sides AB, BC, CD, and AD at points P, Q, R, S respectively. The line passing through P and parallel to the bases of the trapezoid meets QR at point X. Prove that AB, QS, and DX concur.

   First solution. Let I be the center of ω. Then the sidelines of triangles PQX and AID are parallel (fig. 9). Thus these triangles are homothetic which yields the required concurrency.

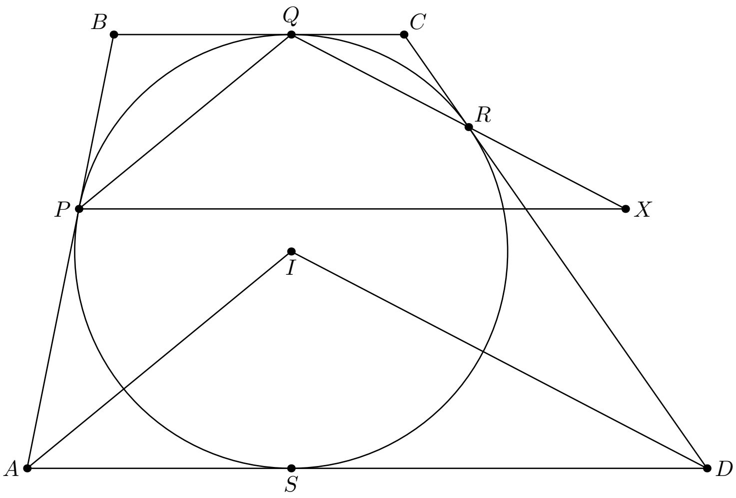

Fig. 9

Second solution. Fix the points A, B, P, Q, S and move R along ω. Then D and X move along AS and parallel line passing through P respectively. Clearly the correspondence between D and X is projective, and since they meet at infinity, this correspondence is linear, i.e. all lines DX are concurrent. Also it is clear that their common point lies on AB, hence it is sufficient to find one position such that DX, AB, and QS concur. This is correct if the trapezoid $ABCD$ is isosceles.

10. (8–9, A.Tereshin) Let ω be the circumcircle of a triangle ABC. A point T on the line BC is such that AT touches ω. The bisector of angle BAC meets BC and ω at points L and A_0 respectively. The line TA_0 meets ω at point P. The point K lies on the segment BC in such a way that $BL = CK$. Prove that $\angle BAP = \angle CAK$.

    Solution. The projection of ω to itself from T swaps B with C, P with A_0, and conserve A. This yields the equality of cross-ratios $(BCAA_0) = (CBAP)$, i.e. $\sin\angle BAP : \sin\angle CAP = PB : PC = AB^2 : AC^2$. On the other hand applying the sines law to triangles AKC and BKC we obtain that $\sin\angle CAK : \sin\angle BAK = (CK/AC) : (BK/AB) = (AB/AC)\cdot(BL/AL) = AB^2 : AC^2$, which yields the required equality.

11. (8–10, B.Butyrin) Let M, N be the midpoints of sides AB, AC respectively of a triangle ABC. The perpendicular bisector to the bisectrix AL meets the bisectrixes of angles B and C at points P and Q respectively. Prove that the common point of lines PM and QN lies on the tangent to the circumcircle of ABC at A.

    Solution. Note that the lines PQ and MN meet at the midpoint K of segment AL. Also P bisects the arc AL of circle ABL, therefore $\angle BPL = \angle CAL = \angle BIC - \pi/2$, where I is the incenter of ABC, i.e. $PL \perp CI$. Similarly $QL \perp BI$. Thus the sidelines of triangle PQL are parallel to the sidelines of triangle formed by the touching points of sides of ABC with the incircle, and the tangents to the circumcircle of PQL at the vertices are parallel to the sidelines of ABC. Since the circle APQ is the reflection of circle LPQ about PQ, the tangent to this circle at P is parallel to AB, and the tangent at A coincide with the tangent to the circumcircle of ABC. Also $\angle PAQ = \angle PLQ = \pi - \angle PIQ$, hence I lies on the circle APQ, and the tangent to this circle at I is parallel to BC. Denote the common point of tangents at A and I as T (fig. 11)

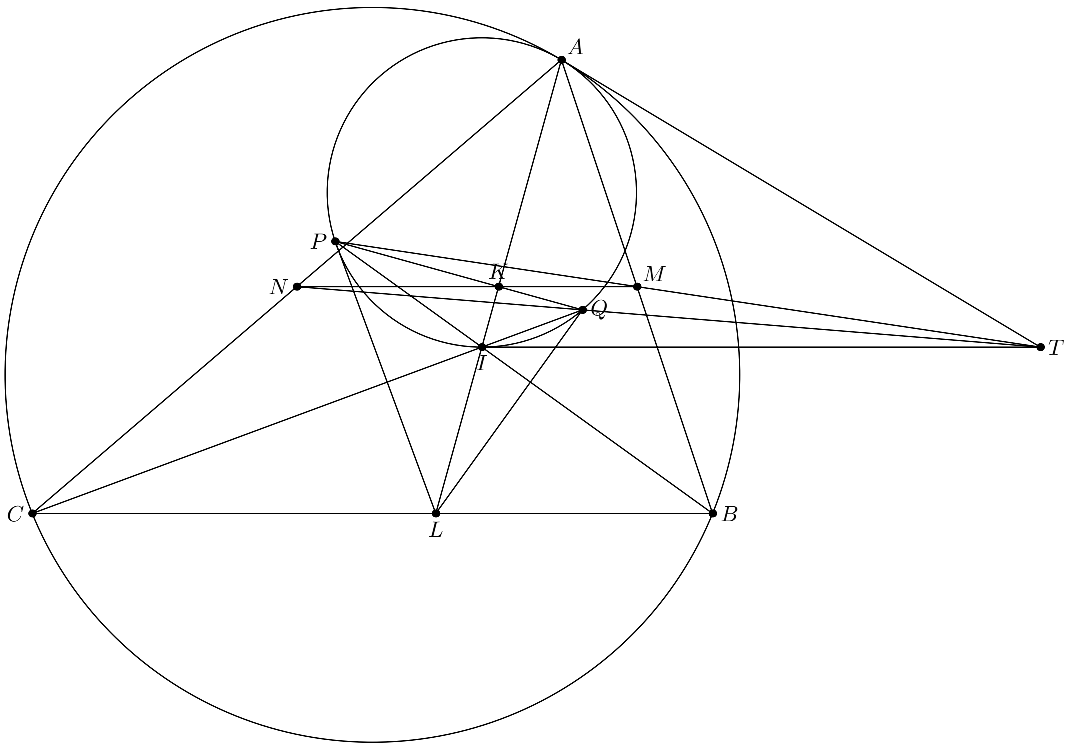

Fig. 11

We have to prove that PM passes through T, i.e. that the line passing through A and parallel to the tangent at P, and the line passing through the projection K of P to AI, parallel to the tangent at I meet AT at the same point. Let S be the common point of tangents at A and P, U be the common point of AI with the line passing through P and parallel to IT. then $IK : KU = AT : AS = \operatorname{ctg}\angle IPA : \operatorname{ctg}\angle AIP$, and we obtain the required assertion.

12. (8–10, D.Shvetsov) The bisectors AA_1, CC_1 of a triangle ABC with $\angle B = 60^\circ$ meet at point I. The circumcircles of triangles ABC, A_1IC_1 meet at point P. Prove that the line PI bisects the side AC.

    Solution. Since $\angle A_1IC_1 = 120^\circ = 180^\circ - \angle A_1BC_1$, the circles ABC and A_1IC_1 meet at points B and P (fig. 12). Hence the triangles PA_1C and PC_1A are similar, i.e. $PB_1 : PC_1 = A_1C : AC_1$. On the other hand since $\angle AC_1I + \angle IA_1C = 180^\circ$, we obtain applying the sines law to the triangles AC_1I and CA_1I that $A_1C : AC_1 = IC : IA$. therefore $\sin\angle PIA_1 : \sin\angle PIC_1 = IC : IA$, and IP is the median of triangle IAC.

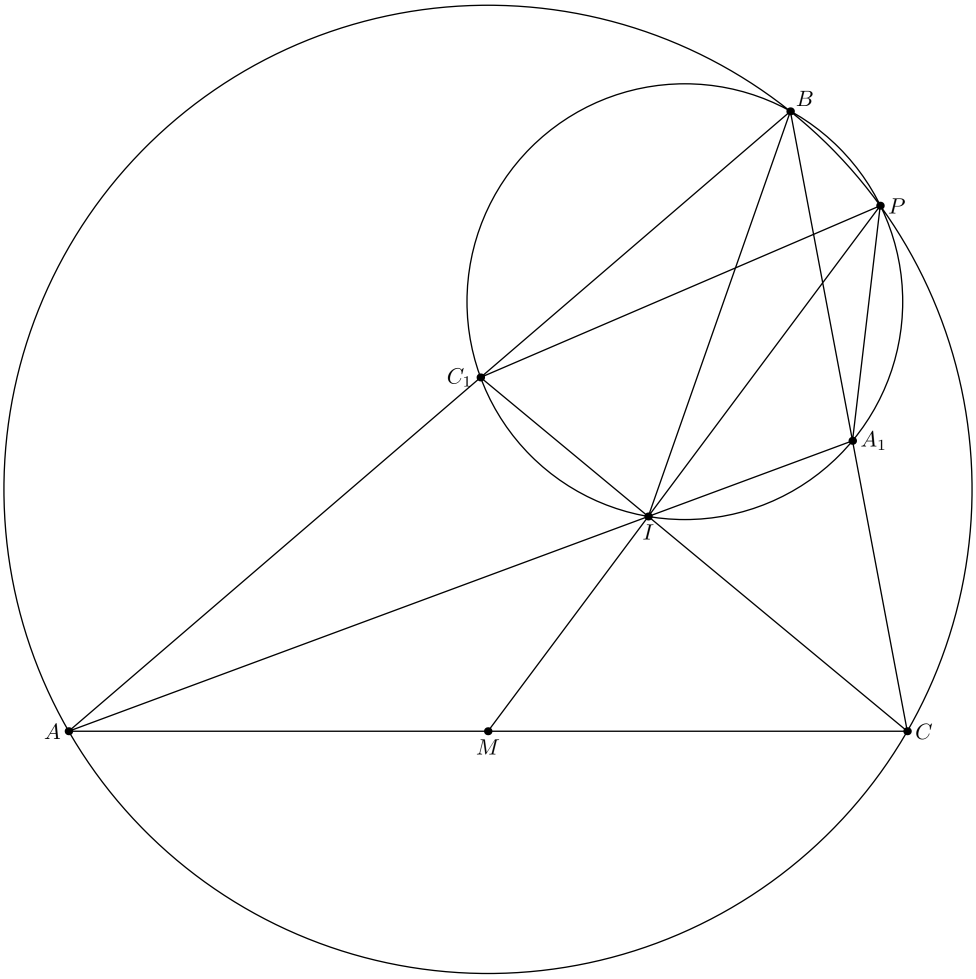

Fig. 12.

13. (8–11, A.Zaslavsky) Can an arbitrary polygon be cut into isosceles trapezoids?

    Answer. Yes.

    Solution. Since an arbitrary polygon can be cut into triangles, and an arbitrary triangle can be cut into isosceles triangles (drawing the altitude to the longer side and joining its foot with the midpoints of two remaining sides), It is sufficient to solve the problem for isosceles triangles.

    Note that we can cut an isosceles triangle into three isosceles trapezoids drawing three rays parallel to its sides from the center (fig. 13.1).

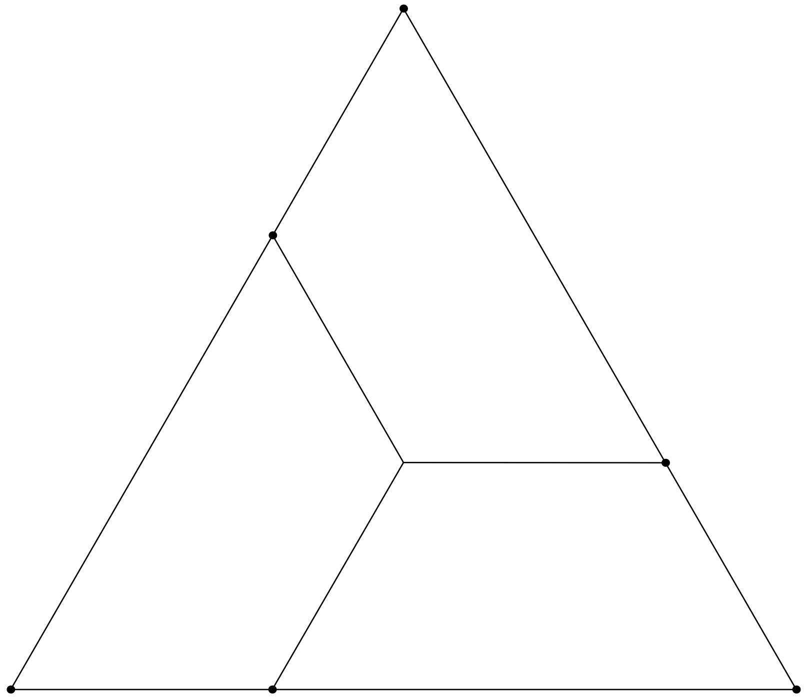

Fig. 13.1

Now if the angle at the vertex of the triangle is greater than 60° we can cut it into isosceles trapezoids applying several times the construction on fig. 13.2.

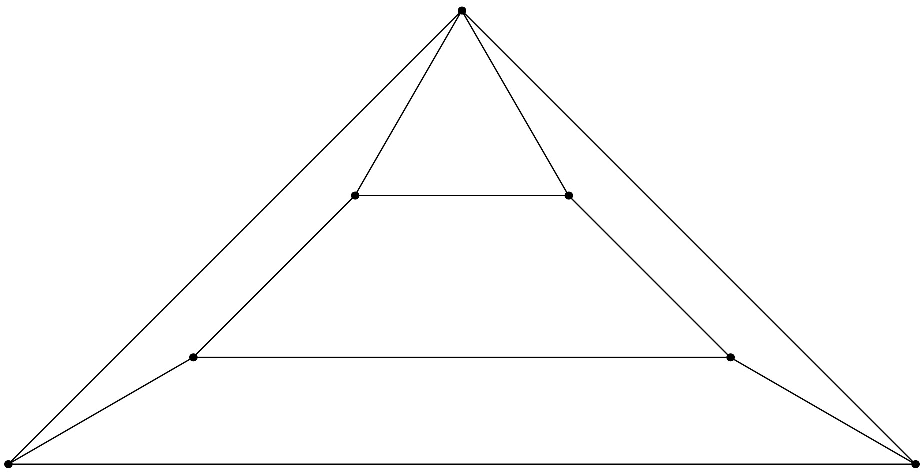

Fig. 13.2

Finally if the angel at the vertex of the triangle is less than 60° cut it into three triangles joining the vertices with the circumcenter. Two of obtained triangles are obtuse-angled, and the angle at the vertex of third one is twice greater than the angle of the original triangle. Repeating this trick several times we cut the given triangle into isosceles triangles with the angles at the vertices greater than 60°.

14. (9–11, A.Tereshin) The incircle ω of a right-angled triangle ABC touches the circumcircle of its medial triangle at point F. Let OE be the tangent to ω from the midpoint O of the hypothenuse AB, distinct from AB. Prove that $CE = CF$.

Solution. The homothety with center C and coefficient 2 maps F to the touching point of the circumcircle and the semiincircle of given triangle. Hence the line CF is the reflection about the bisector of angle C of the line joining C wit the touching point of the hypothenuse with the corresponding excircle. Let the hypothenuse touches the incircle and the excircle at points T and S respectively. Since $OE = OT = OS$ we obtain that $\angle SET = \pi/2$, i.e. the line SE passes through the point of ω opposite to T. But SC also passes through this point, therefore E lies on SC and is the reflection of F about the bisector of angle C (fig. 14).

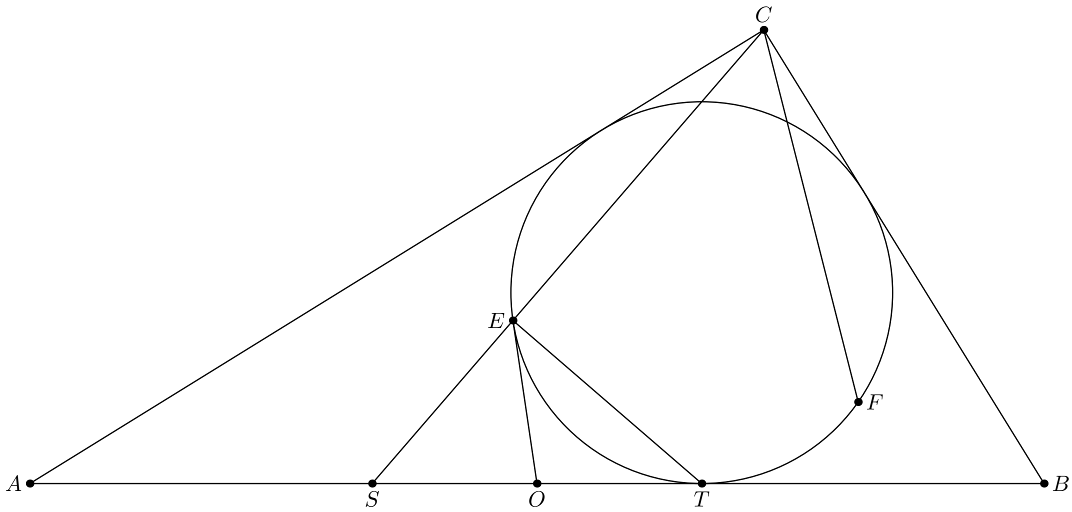

Fig. 14

15. (9–11, M.Panov) The difference of two angles of a triangle is greater than 90°. Prove that the ratio of its circumradius and inradius is greater than 4.

First solution. Let A be the smallest angle of triangle ABC, B be the greatest angle, O be the circumcenter, L be the midpoint of arc AB, CD and PQ be the chord and the diameter of the circumcircle parallel to AB (fig. 15). Since $\smile CD = \smile ADC - \smile AD = \smile ADC - \smile BC > \pi$, we obtain that A, B, C lie on the same semiplane with respect to PQ, i.e. $\angle OLC > \pi/4$, and the distance from O to the line LC passing through the incenter I is greater than $R/\sqrt{2}$. Therefore $OI^2 = R^2 - 4Rr > R^2/2$ which is equivalent to the required inequality.

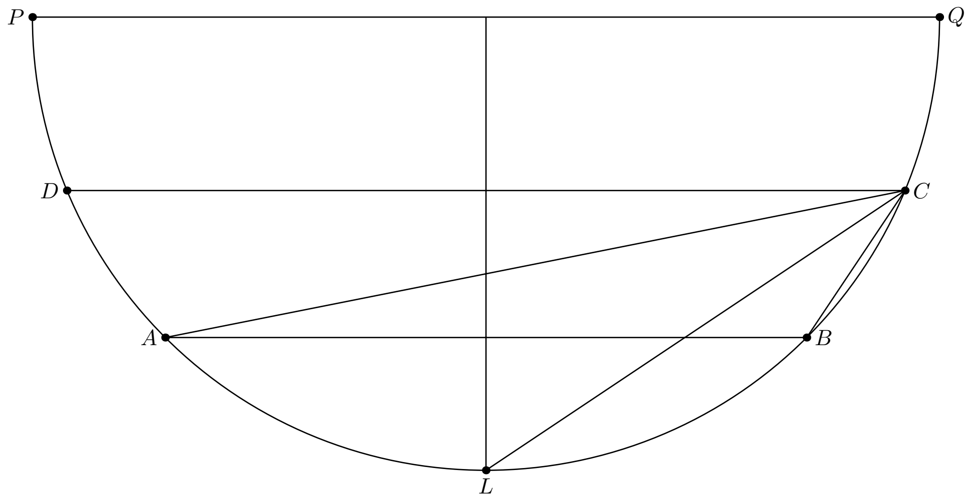

Fig. 15

Second solution. From the formula $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$ we have

$$\frac{r}{R} = 2\sin\frac{C}{2}\left(\cos\frac{B-A}{2} - \cos\frac{A+B}{2}\right) < 2\sin\frac{C}{2}\left(\frac{1}{\sqrt{2}} - \sin\frac{C}{2}\right) \leq 2\left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{4}.$$

Remark. the obtained estimation is exact. If $\cos C = 3/4$, $A = \pi/4 - C/2$, $B = 3\pi/4 - C/2$ all inequalities transform to equalities and $R = 4r$.

16. (9–11, A.Mardanov) Let AA_1, BB_1, and CC_1 be the bisectors of a triangle ABC. The segments BB_1 and A_1C_1 meet at point D. Let E be the projection of D to AC. Points P and Q on the sides AB and BC respectively are such that $EP = PD$, $EQ = QD$. Prove that $\angle PDB_1 = \angle EDQ$.

    Solution. The sum of distances from any point of segment A_1C_1 to AB and BC equals to the distance from this point to AC (because this is correct for the endpoints of the segment). Since the distances from D to AB and BC are equal each of these distances equals to a half of DE, i.e. D is the incenter of triangle BPQ (fig. 16). Thus $\angle EPQ = \angle DPB$, and $\angle EQP = \angle DQB$. Therefore B and E are isogonally conjugated with respect to triangle DPQ and we obtain the required equality.

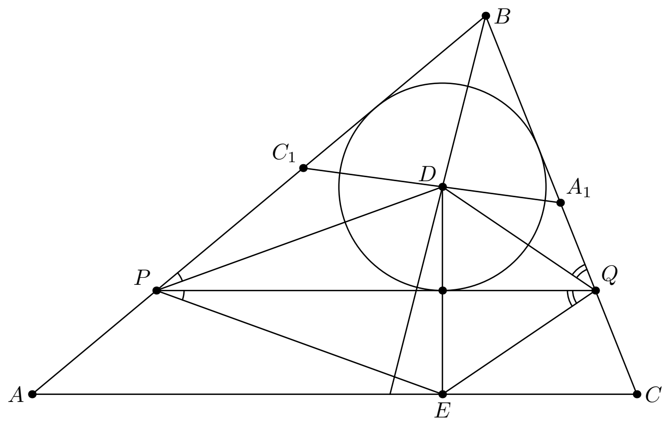

Fig. 16

17. (9–11, L.Dong) Let ABC be a non-isosceles triangle, ω be its incircle. Let D, E and F be the points at which the incircle of ABC touches the sides BC, CA and AB, respectively. Let M be the point on ray EF such that $EM = AB$. Let N be the point on ray FE such that $FN = AC$. Let the circumcircles of $\triangle BFM$ and $\triangle CEN$ intersect ω again at S and T, respectively. Prove that BS, CT and AD concur.

Solution.

Lemma. Let X, Y, Z be points on ω such that DX, EY, FZ concur. Then AX, BY, CZ concur.

Proof. Clearly, $\angle AEX = \angle EDX$ и $\angle XFA = \angle XDF$. Also by the sines law

$$\frac{\sin\angle FAX}{\sin\angle XFA} = \frac{XF}{AX}, \quad \frac{\sin\angle XAE}{\sin\angle AEX} = \frac{XE}{AX}.$$

Therefore

$$\frac{\sin\angle BAX}{\sin\angle XAC} = \frac{\sin\angle FAX}{\sin\angle XAE} = \frac{XF}{XE}\cdot\frac{\sin\angle XFA}{\sin\angle AEX} = \frac{XF}{XE}\cdot\frac{\sin\angle XDF}{\sin\angle EDX} = \left(\frac{\sin\angle XDF}{\sin\angle EDX}\right)^2.$$

Similarly

$$\frac{\sin\angle CBY}{\sin\angle YBA} = \left(\frac{\sin\angle YED}{\sin\angle FEY}\right)^2; \frac{\sin\angle ACZ}{\sin\angle ZCB} = \left(\frac{\sin\angle ZFE}{\sin\angle DFZ}\right)^2.$$

Hence

$$\frac{\sin\angle BAX}{\sin\angle XAC}\cdot\frac{\sin\angle CBY}{\sin\angle YBA}\cdot\frac{\sin\angle ACZ}{\sin\angle ZCB} = 1.$$

and the assertion of the lemma follows from the Ceva theorem.

Return to the problem.

Denote by $O(ABCD)$ the cross ratio of lines OA, OB, OC, OD. Let J be the common point of FT and ES, G be the second common point of AD and ω, K be the common point of EF and BC (fig. 17).

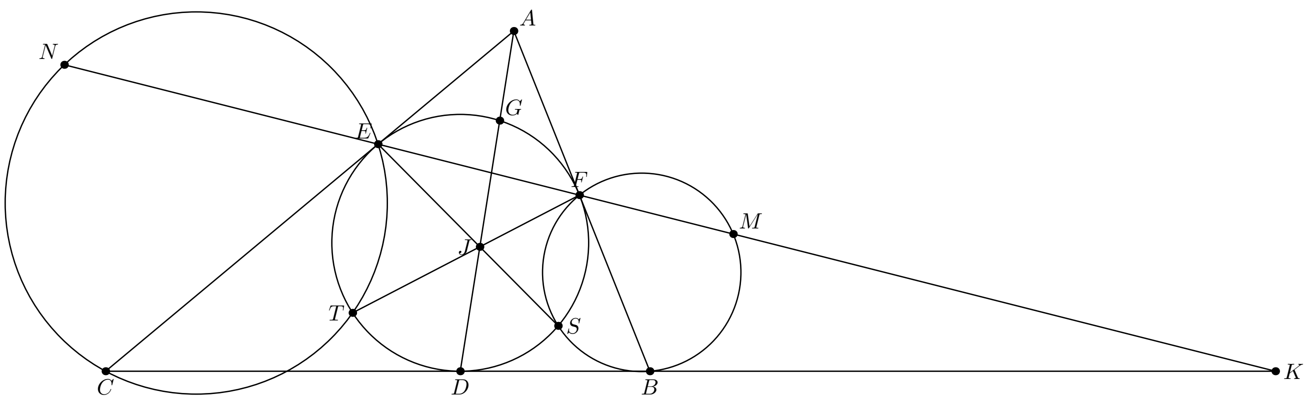

Fig. 17

Since $\angle FMS = \angle FBS$, $\angle MES = \angle BFS$, we obtain that the triangles SBF and SME are similar. Thus $SE : SF = ME : BF = AB : BF$. Then

$$E(AFJD) = E(EFSD) = (EFSD) = \frac{SE}{SF} : \frac{DE}{DF} = \frac{AB}{BF} \cdot \frac{DF}{DE}.$$

Similarly

$$F(AEJD) = \frac{AC}{CE} \cdot \frac{DE}{DF}.$$

And so

$$E(AFJD) : F(AEJD) = \frac{AB}{AC} \cdot \frac{CE}{BF} \cdot \frac{DF^2}{DE^2} = \frac{AB}{AC} \cdot \frac{CE}{BF} \cdot \frac{KF}{KE}.$$

Applying the Menelaos theorem to the triangle AEF and points K, B, C, we obtain

$$\frac{AB}{BF} \cdot \frac{FK}{KE} \cdot \frac{EC}{CA} = 1.$$

Therefore $E(AFJD) = F(AEJD)$, i.e. A, J, D are collinear. Hence DG, ES, and FT concur at J, and by the lemma BS, CT, and AG concur.

18. (9–11, D.Shvetsov) Let AA_1, BB_1, CC_1 be the altitudes of an acute-angled triangle ABC; I_a be its excenter corresponding to A; I'_a be the reflection of I_a about the line AA_1. Points I'_b, I'_c are defined similarly. Prove that the lines $A_1I'_a$, $B_1I'_b$, $C_1I'_c$ concur.

    First solution. The lines A_1A, A_1B, A_1A', and A_1I_a form a harmonic quadruple. Therefore their meeting points with AL also form a harmonic quadruple. Since the fourth harmonic point for A, L, I_a coincide with the incenter I of triangle ABC, we obtain that A_1A' passes through I. Similarly B_1B' and C_1C' pass through I.

    this reasoning can be modified. Since A_1A, A_1B, A_1I, and A_1I_a form a harmonic quadruple, and $A_1A \perp A_1B$, the lines A_1A and A_1B bisect the angles between A_1I and A_1I_a. Therefore A_1A' passes through I.

    Second solution. Prove that A_1I_a, B_1I_b, C_1I_c concur. then their reflections A_1A', B_1B', C_1C' about the bisectors of triangle $A_1B_1C_1$ also concur. Note that for example $\sin\angle I_bI_aA_1 : \sin\angle I_cI_aA_1 = (BA_1 : CA_1) \cdot (I_aC : I_aB)$. Applying the Ceva theorem to the lines I_aA, I_bB, I_cC and AA_1, BB_1, CC_1 we obtain that the product of this ratio and two similar ones equals 1.

    Remark. The assertion of the problem is a partial case of the following fact. If points A_1, B_1, C_1 lie n the sidelines BC, CA, AB of triangle ABC, and points A_2, B_2, C_2 lie on the sidelines B_1C_1, C_1A_1, A_1B_1 of triangle $A_1B_1C_1$, in such a way that AA_1, BB_1, CC_1 concur, and A_1A_2, B_1B_2, C_1C_2 concur, then AA_2, BB_2, CC_2 also concur.

19. (10–11, M.Evdokimov) A triangle ABC, its circumcircle, and its incenter I are drawn on the plane. Construct the circumcenter of ABC using only a ruler.

    Solution. Construct the common point C_1 of tangents to the circle at points A, B and the second common point C_2 of the circle and the line CI. The line C_1C_2 is the perpendicular bisector to the segment AB therefore it passes through the circumcenter. Constructing similarly the perpendicular bisector to AC find the circumcenter.

20. (10–11, L.Shatunov) Lines a_1, b_1, c_1 pass through the vertices A, B, C respectively of a triangle ABC; a_2, b_2, c_2 are the reflections of a_1, b_1, c_1 about the corresponding bisectors of ABC; $A_1 = b_1 \cap c_1$, $B_1 = a_1 \cap c_1$, $C_1 = a_1 \cap b_1$, and A_2, B_2, C_2 are defined similarly. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ have the same ratios of the area and the circumradius (i.e. $\frac{S_1}{R_1} = \frac{S_2}{R_2}$, where $S_i = S(\triangle A_iB_iC_i)$, $R_i = R(\triangle A_iB_iC_i)$).

Solution.

Lemma. Let X', Y', Z' lie on the sides YZ, ZX, XY respectively of triangle XYZ. Then

$$S_{X'Y'Z'} = \frac{XY' \cdot YZ' \cdot ZX' + X'Y \cdot Y'Z \cdot Z'X}{4R_{XYZ}}.$$

(the points X', Y', Z' may also lie on the extensions of the sides. In this case we have to consider the segments in the formula as oriented.)

Proof. Let $XY' = \alpha XZ$, $YZ' = \beta YX$, $ZX' = \gamma ZY$. Then $S_{X'Y'Z'} : S_{XYZ} = 1 - \alpha(1 - \beta) - \beta(1 - \gamma) - \gamma(1 - \alpha) = \alpha\beta\gamma + (1 - \alpha)(1 - \beta)(1 - \gamma)$ and the assertion of the lemma follows from the formula $S_{XYZ} = (XY \cdot YZ \cdot ZX)/4R_{XYZ}$.

Now apply the lemma to the triangle $A_1B_1C_1$ and A, B, C on its sidelines. Denoting $\angle B_1AC = \alpha$, $\angle C_1BA = \beta$, $\angle A_1CB = \gamma$ we obtain (using the sines law for the triangles AB_1C_1, BC_1A_1, CA_1B_1)

$$S_{ABC} = \frac{AB \cdot BC \cdot CA(\sin\alpha \sin\beta \sin\gamma + \sin(A + \alpha)\sin(B + \beta)\sin(C + \gamma))}{4R_1 \sin\angle A_1B_1C_1 \sin\angle B_1C_1A_1 \sin\angle C_1A_1B_1}.$$

Applying the sines law to the triangle $A_1B_1C_1$ we obtain that the denominator equals $2S_1/R_1$. Finally note that if we replace the triangle $A_1B_1C_1$ to $A_2B_2C_2$ the numerator do not change.

21. (10–11, A.Zaslavsky) A chord PQ of the circumcircle of a triangle ABC meets the sides BC, AC at points A', B' respectively. The tangents to the circumcircle at A and B meet at point X, and the tangents at points P and Q meet at point Y. The line XY meets AB at point C'. Prove that the lines AA', BB', and CC' concur.

Solution. Let PQ and AB meet at point U, AA' and BB' meet at point V. Then the line XY is the polar of U with respect to the circumcircle, therefore A, B, U, C' form a harmonic quadruple. Thus CV passes through C'.

22. (10–11, D.Reznik, A.Akopyan) A segment AB is given. Let C be an arbitrary point of the perpendicular bisector to AB; O be the point on the circumcircle of ABC opposite to C; and an ellipse centered at O touch AB, BC, CA. Find the locus of touching points of the ellipse with the line BC.

Answer. The circle with diameter BQ, where Q is a point on the ray AB such that $AQ = 3AB/2$, without B and Q.

Solution. Let M be the midpoint of AB, N be the reflection of M about O, Pbe the common point of PN and BC, U, V be the common points of the line passing through N and parallel to AB with BC, AC respectively (fig. 22). Since the triangles BPQ and UPN are similar we have $BP : PN = BQ : UN = AB : UV$, i.e. the line passing through P and parallel to AB passes through the common point of diagonals of trapezoid $ABUV$. Therefore the ellipse inscribed into the trapezoid touches BC at point P. Since

$PN \parallel OB \perp BC$ this point lies on the circle with diameter BQ. It is clear that all points of this circle distinct from B and Q belong to the required locus.

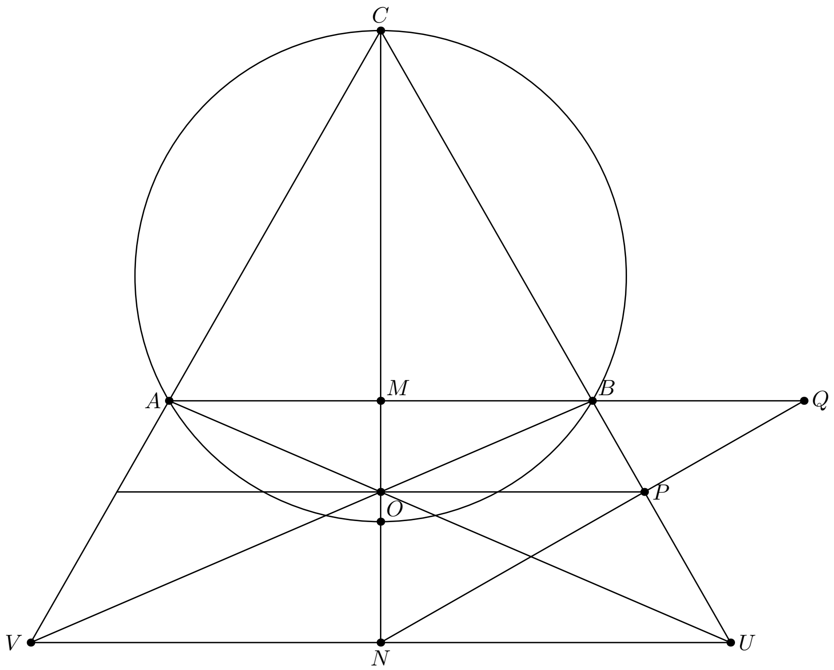

Fig. 22

23. (10–11, I.Kukharchuk) A point P moves along a circle Ω. Let A and B be fixed points of Ω, and C be an arbitrary point inside Ω. The common external tangents to the circumcircles of triangles APC and BCP meet at point Q. Prove that all points Q lie on two fixed lines.

    Solution. The common point of external tangents is the center of circle ω such that the inversion with respect to it swaps the circles APC and BPC. Consider an inversion centered at C transforming A, B, P to A', B', C' respectively. It maps the circles APC, BPC to the lines $A'P'$, $B'P'$ respectively, the image of ω is the bisector of some angle between these lines, and the image of Q is the reflection Q' of C about this bisector. Since the bisectors of angles between $P'A'$ and $P'B'$ pass through two fixed points — the midpoints of arcs $A'B'$ of circle $A'B'P'$, all points Q' lie on two circles centered at these points and passing through C. The considered inversion maps these circles to two fixed lines.

    Remark. The point Q jumps from one line to he second one when P intersect one of lines AC, BC.

24. (11, Tran Quang Hung, N.Dergiados) Let $SABC$ be a pyramid with right angles at the vertex S. Points A', B', C' lie on the edges SA, SB, SC respectively in such a way that the triangles ABC and $A'B'C'$ are similar. Does this yield that the planes ABC and $A'B'C'$ are parallel?

    Answer. Yes.

    First solution. Suppose that the planes ABC and $A'B'C'$ are not parallel. Then apply the homothety centered at S mapping $A'B'C'$ to a triangle congruent to ABC, and transform the obtained triangle to ABC by an isometry of the space. The image of S is a point S' distinct from S and its reflection about the plane ABC. On the other hand both points S, S' lie on three spheres with diameters AB, BC, CA. Since the centers of these spheres are not collinear they have only two common points symmetric with respect to the plane ABC — contradiction.

Second solution. Let $A'B' = tAB$. Then $B'C' = tBC$, $C'A' = tAC$. Since the angles at S are right we have $SA'^2 + SB'^2 = t^2AB^2$, $SA'^2 + SC'^2 = t^2AC^2$, $SB'^2 + SC'^2 = t^2BC^2$. From this we obtain that $SA'^2 = t^2(AB^2 + AC^2 - BC^2)/2 = t^2SA^2$, i.e. $SA' = tSA$. Similarly $SB' = tSB$, $SC' = tSC$ and therefore the planes ABC and $A'B'C'$ are parallel.