XX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XX Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade (at the start of the correspondence round) have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. **If a problem has an explicit answer, this answer must be presented distinctly**. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The **solutions** for the problems (in Russian or in English) must be **delivered not before December 1, 2023 and not later than on March 1, 2024**. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, press «Registration» in the left upper part, and follow the instructions.

Attention:

1. The solution of each problem (and of each part of it if any) must be contained in a **separate** pdf, doc, docx or jpg file. If the solution is contained in several files then pack them to an **archive** (zip or rar) and load it.
2. We recommend to prepare the paper using computer or to scan it rather than to photograph it. **In all cases, please check readability of the file before uploading**.
3. If you upload the solution of some problem more than once then only the last version is retained in the checking system. **Thus if you need to change something in your solution then you have to upload the whole solution again.**
4. After uploading, log in to the server, open the loaded file and check its correctness.

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round will be held in July–August 2024 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before the round. The cutoff is determined after the examination of the papers of the correspondence round according to the number of participants with any given score. The graduates who are winners of the correspondence round will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** up to June 1, 2024. If you want to know your detailed results, please apply to **geomshar@yandex.ru** after publication of the list.

1. (8) Bisectors AI and CI meet the circumcircle of triangle ABC at points A_1, C_1 respectively. The circumcircle of triangle AIC_1 meets AB at point C_0; point A_0 is defined similarly. Prove that A_0, A_1, C_0, C_1 are collinear.

2. (8) Three different collinear points are given. What is the number of isosceles triangles such that these points are their circumcenter, incenter and excenter (in some order)?

3. (8) Let ABC be an acute-angled triangle, and M be the midpoint of the minor arc BC of its circumcircle. A circle ω touches the side AB, AC at points P, Q respectively and passes through M. Prove that $BP + CQ = PQ$.

4. (8) The incircle ω of triangle ABC touches BC, CA, AB at points A_1, B_1 and C_1 respectively, P is an arbitrary point on ω. The line AP meets the circumcircle of triangle AB_1C_1 for the second time at point A_2. Points B_2 and C_2 are defined similarly. Prove that the circumcircle of triangle $A_2B_2C_2$ touches ω.

5. (8) Points A', B', C' are the reflections of vertices A, B, C about the opposite sidelines of triangle ABC. Prove that the circles $AB'C'$, $A'BC'$, and $A'B'C$ have a common point.

6. (8–9) A circle ω and two points A, B of this circle are given. Let C be an arbitrary point on one of arcs AB of ω; CL be the bisector of triangle ABC; the circle BCL meet AC at point E; and CL meet BE at point F. Find the locus of circumcenters of triangles AFC.

7. (8–9) Restore a bicentral quadrilateral if two opposite vertices and the incenter are given.

8. (8–9) Let $ABCD$ be a quadrilateral with $\angle B = \angle D$ and $AD = CD$. The incircle of triangle ABC touches the sides BC and AB at points E and F respectively. Prove that the midpoints of segments AC, BD, AE, and CF are concyclic.

9. (8–9) Let $ABCD$ ($AD \parallel BC$) be a trapezoid circumscribed around a circle ω, which touches the sides AB, BC, CD, and AD at points P, Q, R, S respectively. The line passing through P and parallel to the bases of the trapezoid meets QR at point X. Prove that AB, QS, and DX concur.

10. (8–9) Let ω be the circumcircle of a triangle ABC. A point T on the line BC is such that AT touches ω. The bisector of angle BAC meets BC and ω at points L and A_0 respectively. The line TA_0 meets ω at point P. The point K lies on the segment BC in such a way that $BL = CK$. Prove that $\angle BAP = \angle CAK$.

11. (8–10) Let M, N be the midpoints of sides AB, AC respectively of a triangle ABC. The perpendicular bisector to the bisectrix AL meets the bisectrixes of angles B and C at points P and Q respectively. Prove that the common point of lines PM and QN lies on the tangent to the circumcircle of ABC at A.

12. (8–10) The bisectors AA_1, CC_1 of a triangle ABC with $\angle B = 60^\circ$ meet at point I. The circumcircles of triangles ABC, A_1IC_1 meet at point P. Prove that the line PI bisects the side AC.

13. (8–11) Can an arbitrary polygon be cut into isosceles trapezoids?

14. (9–11) The incircle ω of a right-angled triangle ABC touches the circumcircle of its medial triangle at point F. Let OE be the tangent to ω from the midpoint O of the hypothenuse AB, distinct from AB. Prove that $CE = CF$.

15. (9–11) The difference of two angles of a triangle is greater than 90°. Prove that the ratio of its circumradius and inradius is greater than 4.

16. (9–11) Let AA_1, BB_1, and CC_1 be the bisectors of a triangle ABC. The segments BB_1 and A_1C_1 meet at point D. Let E be the projection of D to AC. Points P and Q on the sides AB and BC respectively are such that $EP = PD$, $EQ = QD$. Prove that $\angle PDB_1 = \angle EDQ$.

17. (9–11) Let ABC be a non-isosceles triangle, ω be its incircle. Let D, E and F be the points at which the incircle of ABC touches the sides BC, CA and AB, respectively. Let M be the point on ray EF such that $EM = AB$. Let N be the point on ray FE such that $FN = AC$. Let the circumcircles of $\triangle BFM$ and $\triangle CEN$ intersect ω again at S and T, respectively. Prove that BS, CT and AD concur.

18. (9–11) Let AA_1, BB_1, CC_1 be the altitudes of an acute-angled triangle ABC; I_a be its excenter corresponding to A; I'_a be the reflection of I_a about the line AA_1. Points I'_b, I'_c are defined similarly. Prove that the lines $A_1I'_a$, $B_1I'_b$, $C_1I'_c$ concur.

19. (10–11) A triangle ABC, its circumcircle, and its incenter I are drawn on the plane. Construct the circumcenter of ABC using only a ruler.

20. (10–11) Lines a_1, b_1, c_1 pass through the vertices A, B, C respectively of a triangle ABC; a_2, b_2, c_2 are the reflections of a_1, b_1, c_1 about the corresponding bisectors of ABC; $A_1 = b_1 \cap c_1$, $B_1 = a_1 \cap c_1$, $C_1 = a_1 \cap b_1$, and A_2, B_2, C_2 are defined similarly. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ have the same ratios of the area and the circumradius (i.e. $\frac{S_1}{R_1} = \frac{S_2}{R_2}$, where $S_i = S(\triangle A_iB_iC_i)$, $R_i = R(\triangle A_iB_iC_i)$).

21. (10–11) A chord PQ of the circumcircle of a triangle ABC meets the sides BC, AC at points A', B' respectively. The tangents to the circumcircle at A and B meet at point X, and the tangents at points P and Q meet at point Y. The line XY meets AB at point C'. Prove that the lines AA', BB', and CC' concur.

22. (10–11) A segment AB is given. Let C be an arbitrary point of the perpendicular bisector to AB; O be the point on the circumcircle of ABC opposite to C; and an ellipse centered at O touch AB, BC, CA. Find the locus of touching points of the ellipse with the line BC.

23. (10–11) A point P moves along a circle Ω. Let A and B be fixed points of Ω, and C be an arbitrary point inside Ω. The common external tangents to the circumcircles of triangles APC and BCP meet at point Q. Prove that all points Q lie on two fixed lines.

24. (11) Let $SABC$ be a pyramid with right angles at the vertex S. Points A', B', C' lie on the edges SA, SB, SC respectively in such a way that the triangles ABC and $A'B'C'$ are similar. Does this yield that the planes ABC and $A'B'C'$ are parallel?