XX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. First day. 8 grade. Solutions
Ratmino. 2024. July 31.

1. (A.Zaslavsky) A circle ω centered at O and a point P inside it are given. Let X be an arbitrary point of ω, the line XP and the circle XOP meet ω for a second time at points X_1, X_2 respectively. Prove that all lines X_1X_2 are parallel.

   Solution. Since $XPOX_2$ is cyclic and XOX_1 is isosceles we have $\angle PX_2O = \angle PXO = \angle PX_1O$ (fig. 8.1). And since $OX_1 = OX_2$ we obtain that $\angle PX_1X_2 = \angle PX_2X_1$ and $PX_1 = PX_2$. Thus PO is the perpendicular bisector to all segments X_1X_2, i.e. all these segments are parallel.

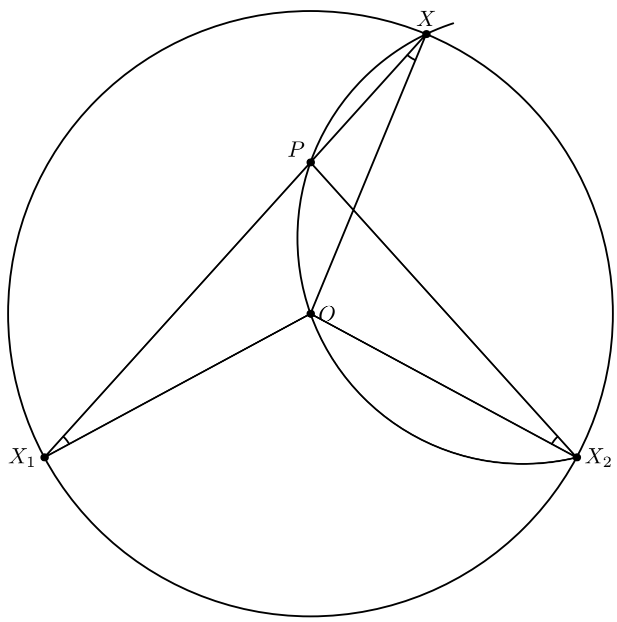

Fig. 8.1.

2. (L.Emelyanov) Let CM be the median of an acute-angled triangle ABC, and P be the projection of the orthocenter H to the bisector of angle C. Prove that MP bisects the segment CH.

   Solution. Let E be the midpoint of CH. Then $CE = EH = EP$ and $\angle PEH = 2\angle PCH = |\angle A - \angle B|$. But E and M lie on the nine-points circle, hence $\angle MEH = \angle MND = |\angle A - \angle B|$, where N is the midpoint of BC, and D is the foot of altitude from C (fig. 8.2).

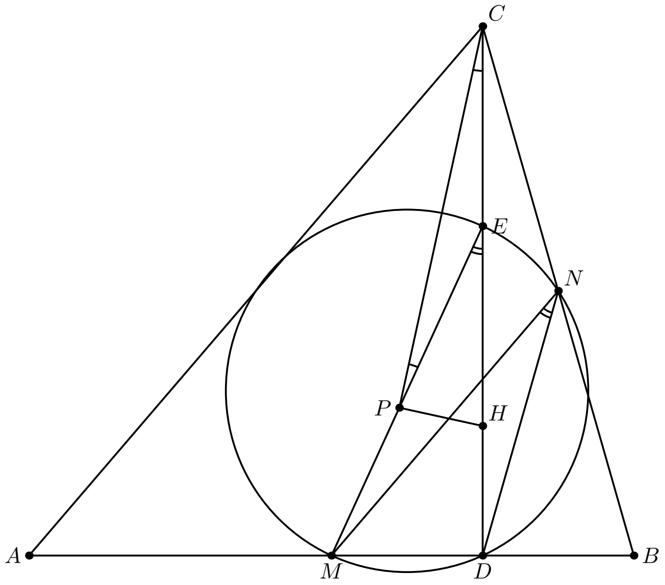

Fig. 8.2.

3. (R.Prozorov) Let AD be the altitude of an acute-angled triangle ABC, and A' be the point of its circumcircle opposite to A. A point P lies on the segment AD, and points X, Y lie on the segments AB, AC respectively in such a way that $\angle CBP = \angle ADY$, $\angle BCP = \angle ADX$. Let PA' meet BC at point T. Prove that D, X, Y, T are concyclic.

   Solution. Let the circle DXY meet BC at point T'. Let L be the second common point of AP with the circle BPC, then $\angle PCB = \angle PLB = \angle ADX$ and $XD \parallel BL$, similarly $DY \parallel CL$, therefore the triangles DXY and LBC are homothetic with center A, then their circumcircles are also homothetic. Let the line passing through L and parallel to BC meet the circle BPC for the second time at point N, then T' and N are corresponding points, and A, T', N are collinear. The projection G of A' to BC is the reflection of D about the midpoint of BC, also the projection of N is the reflection of D about the midpoint of BC, i.e. $NA' \perp BC$. Let K be the second common point of NA' with the circle ABC, then $AD \cdot A'G = A'G \cdot GK = GC \cdot GB = DB \cdot DC = DP \cdot DL = DP \cdot GN$, which yields $A'G : GN = DP : DA$, hence P and A' are corresponding points in similar triangles $DT'A$ and $GT'N$, and P, T', A' are collinear (fig. 8.3). This ends the proof.

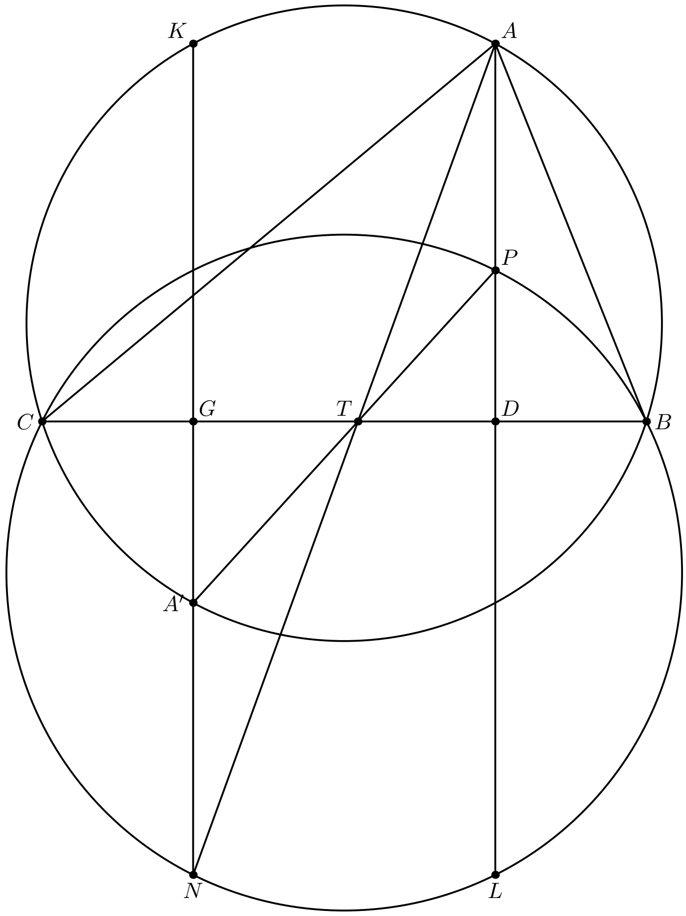

Fig. 8.3.

4. (M.Evdokimov) A square with sidelength 1 is cut from the paper. Construct a segment with length $1/2024$ using at most 20 folds. No instruments are available, it is allowed only to fold the paper and to mark the common points of folding lines.

First solution. Let $ABCD$ be the given square.

1–2. Fold the square two times along the lines parallel to AD, and obtain the points U, V lying on AB, CD respectively and such that $AU = DV = 1/4$.

3–7. Fold the square five times along the lines parallel to AB, and divide the side AD and the segment UV into 32 equal parts.

8. Fold the square along the line ST, where S is the point on AD such that $SD = 23/32$, and T is the point on UV such that $TV = 1/32$. We obtain the point P on CD such that $PV : PD = 1 : 23$, i.e. $PV = 1/88$ (fig. 8.4).

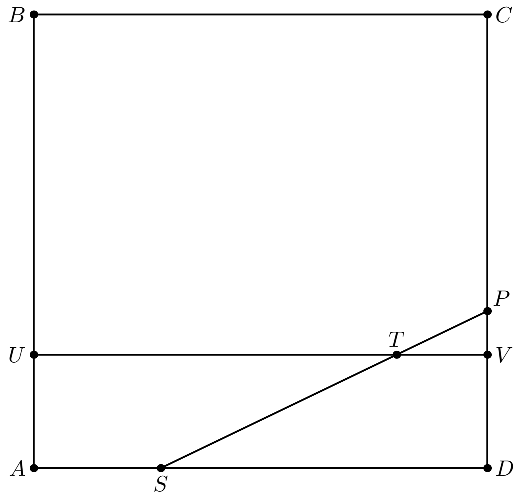

Fig. 8.4.

9. Similarly fold the square along $S'T$, where $DS' = 24/32$, and obtain the point Q on CD such that $VQ = 1/92$. Therefore $PQ = 1/2024$.

Solution. Let $ABCD$ be the given square. Denote by X_n the point on AD, such that $DX_n = AD/n$, and denote by Y_n the point on BD, such that $DY_n = BD/n$.

Lemma. For any n the line X_nY_{n+1} passes through C.

We obtain the proof applying the Menelaos theorem to the triangle AOD, where O is the center of the square, and the points X_n, Y_{n+1}, C.

Using the lemma we obtain the following construction.

1. Fold the square along the diagonal BD.

2. Folding the square along the medial line mark the point X_2.

3. Folding the square along CX_2 mark Y_3.

4–5. Divide the segment DY_3 into four equal parts and mark the point Y_{12}.

6. Fold the square along CY_{12} and mark the point X_{11}.

7. Bisect the segment DX_{11} and mark the point X_{22}.

8. Fold the square along CX_{22} and mark the point Y_{23}.

9. Folding the square along the line passing through Y_{23} and parallel to CD mark the point X_{23}.

10–11. Dividing the segments $X_{22}X_{23}$ into four equal parts we obtain the segment with length $(1/22 - 1/23)/4 = 1/2024$.

Remark. Another constructions are also possible.

XX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Second day. 8 grade. Solutions

Ratmino. 2024. August 1.

5. (M.Evdokimov, T.Kazitsyna) The vertices M, N, K of rectangle $KLMN$ lie on the sides AB, BC, CA respectively of a regular triangle ABC in such a way that $AM = 2$, $KC = 1$, the vertex L lies outside the triangle. Find the value of angle KMN.

   Answer. 30°.

   Solution. Take a point N' on BC such that $CN' = 2$. It is clear that $MN' \parallel AC$. Also, since $CN' = 2CK$ and $\angle N'CK = 60^\circ$, we obtain that the triangle CKN' is right-angled, therefore $\angle MN'K = \angle MNK = 90^\circ$. And since L does not lie on AC we obtain that $N \neq N'$.

   Since N and N' lie on the circle with diameter MK, we have $\angle MKN = \angle MN'N = 60^\circ$ (fig. 8.5). Hence $\angle KMN = 30^\circ$.

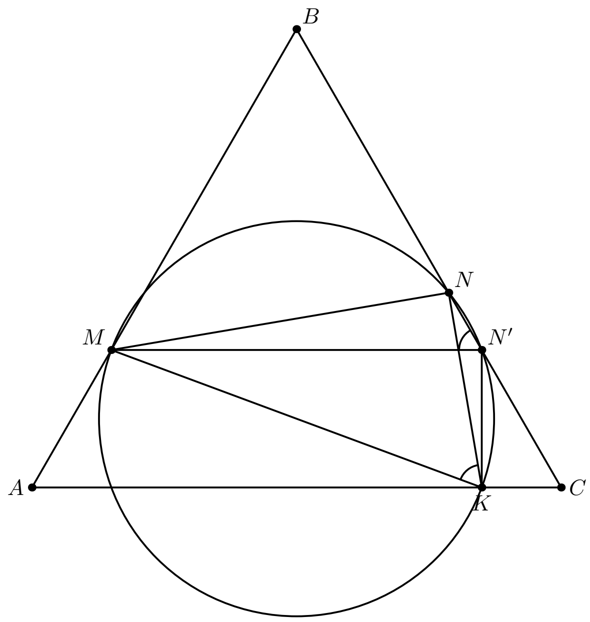

Fig. 8.5.

6. (F.Nilov) A circle ω touches lines a and b at points A and B respectively. An arbitrary tangent to the circle meets a and b at X and Y respectively. Points X' and Y' are the reflections of X and Y about A and B respectively. Find the locus of projections of the center of the circle to the lines $X'Y'$.

Answer. The reflection of circle ω about AB without two points.

Solution. Consider a case when ω is the incircle of the triangle formed by lines a, b, and XY. The reasoning for other cases is similar.

Let I be the center of ω, P be the projection of I to $X'Y'$, and T be the touching point of ω with XY. Since A and P lie on the circle with diameter IX', we have $\angle API = \angle AX'I = \angle IXA = \angle ITA$ (fig. 8.6). Similarly $\angle IPB = \angle BTI$ and therefore $\angle APB = \angle BTA$.

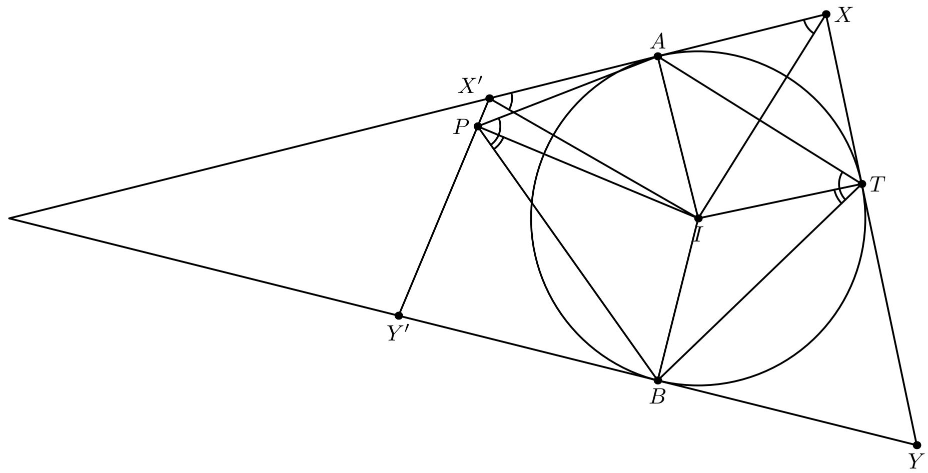

Fig. 8.6.

Similarly constructing for any point P of the obtained circle the points X, Y we obtain that XY touches ω. We can not do this only for two points such that the line IP is perpendicular to one of sidelines of the angle, because in these cases one of points X', Y' does not exist.

7. (L.Shatunov) A convex quadrilateral $ABCD$ is given. A line $l \parallel AC$ meets the lines AD, BC, AB, CD at points X, Y, Z, T respectively. The circumcircles of triangles XYB and ZTB meet for the second time at point R. Prove that R lies on BD.

Solution. Let BD meet XT at point U (fig. 8.7). Applying the Menelaos theorem to the triangle BUZ and the points X, A, D we obtain

$$\frac{XZ}{XU} \cdot \frac{UD}{DB} \cdot \frac{AB}{AZ} = 1.$$

Similarly

$$\frac{TY}{TU} \cdot \frac{UD}{DB} \cdot \frac{BC}{CY} = 1.$$

Since $AB : AZ = BC : CY$, this yields that $UX : UZ = UT : UY$, i.e. the degrees of U with respect to both circles are equal. Hence U and D lie on BR.

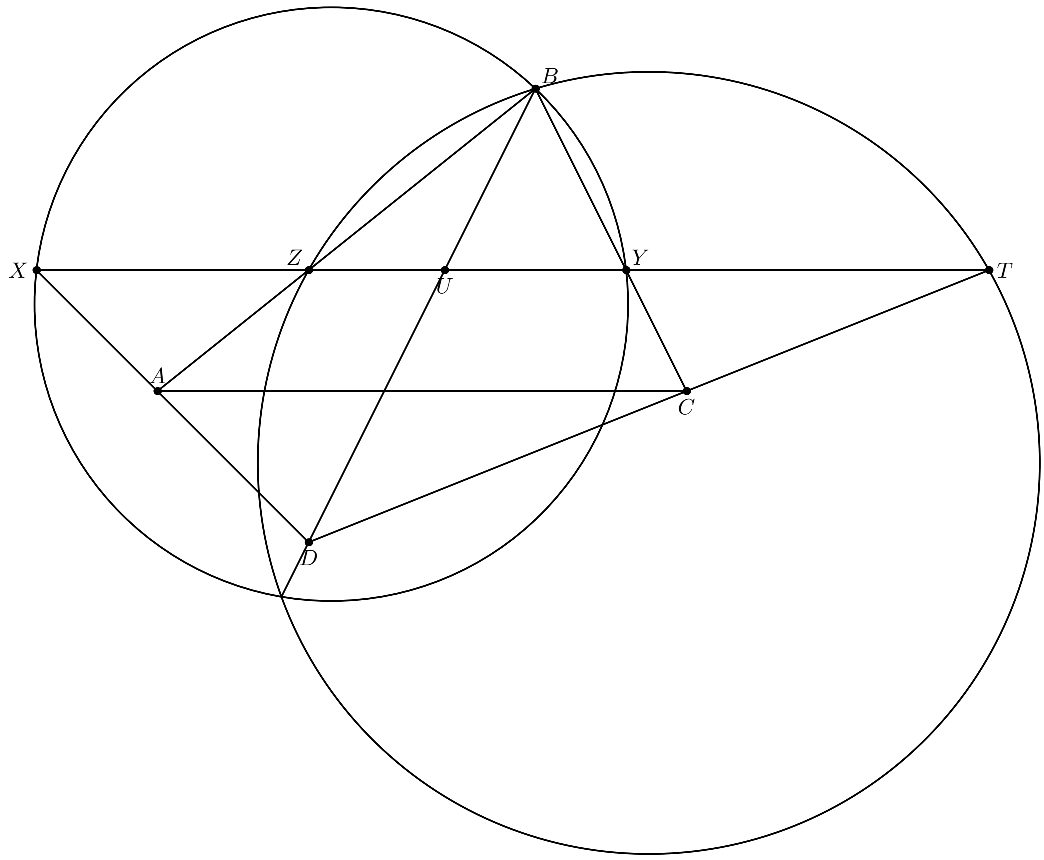

Рис. 8.7.

8. (S.Shmarin) Two polygons are cut from the cartboard. Is it possible that for any disposition of these polygons on the plane they have common inner point or have only finite number of common points?

   Answer. Yes.

   Solution. Let one polygon be an octagon $A_1 \ldots A_8$ such that $A_2A_4A_6A_8$ is a square, all sidelengths are equal, and $\angle A_2A_1A_3 > 40^\circ$ (fig.8.8), and the second polygon be a regular nonagon with sidelength greater than A_1A_3. If these polygons have a common boundary segment, it has to contain one of vertices A_1, A_3, A_5, A_7 of the octagon. Let this segment be A_1B, where B lies on A_1A_2. Since the external angle of the nonagon equals 40°, its side intersects the segment A_2A_3. Therefore the polygons have common inner points.

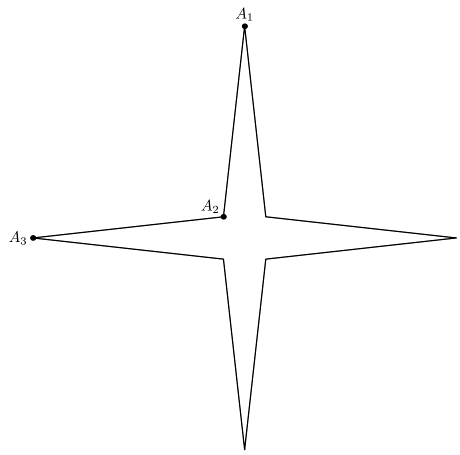

Fig. 8.8.

Remark. It is possible to prove that sum of numbers of the sides of the polygons is minimal in the above example.

XX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. First day. 9 grade. Solutions
Ratmino. 2024. July 31.

1. (L.Emelyanov) Let H be the orthocenter of an acute-angled triangle ABC; A_1, B_1, C_1 be the touching points of the incircle with BC, CA, AB respectively; E_A, E_B, E_C be the midpoints of AH, BH, CH respectively. The circle centered at E_A and passing through A meets for the second time the bisector of angle A at A_2; points B_2, C_2 are defined similarly. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar.

   Solution. The points A_2, B_2, C_2 are the projections of the orthocenter to the bisectors, thus they lie on the circle with diameter HI, where I is the incenter. Hence, for example, $\angle A_2C_2B_2 = \angle A_2IB_2 = (\angle A + \angle B)/2 = \angle A_1C_1B_1$ (fig. 9.1.). Similarly we obtain that the remaining corresponding angles of triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

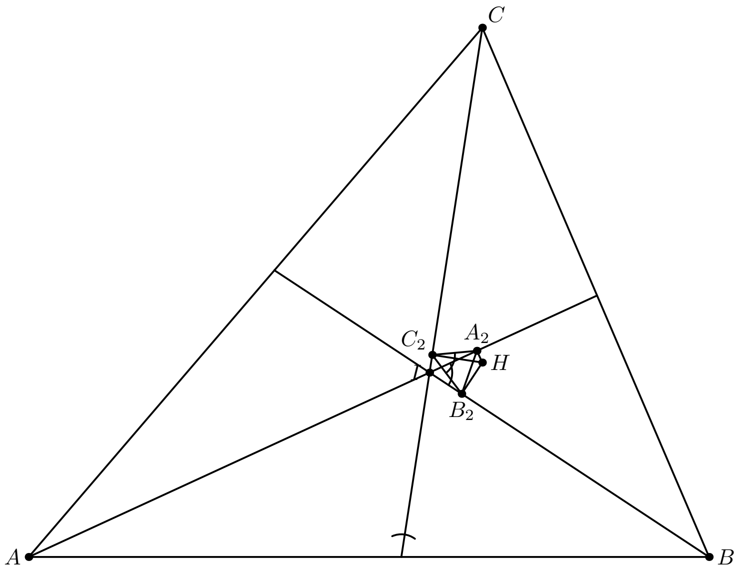

Fig. 9.1.

   Remark. The assertion of the problem is also correct if we replace the orthocenter with an arbitrary point of the plane.

2. (A.Shekera) Points A, B, C, D on the plane do not form a rectangle. Let the sidelengths of triangle T equal $AB + CD$, $AC + BD$, $AD + BC$. Prove that the triangle T is acute-angled.

Solution. Note that

$$\begin{aligned}(AB+CD)^2+(AD+BC)^2-(AC+BD)^2 &= \\ =(AB^2+BC^2+CD^2+DA^2-AC^2-BD^2) &+ \\ +2(AB\cdot CD+AD\cdot BC-AC\cdot BD).\end{aligned}$$

The second parenthesis is not negative by the Ptolemy inequality. Denote $\vec{a}=\vec{OA}$, $\vec{b}=\vec{OB}$, $\vec{c}=\vec{OC}$, $\vec{d}=\vec{OD}$, where O is an arbitrary point. Then the first parenthesis equals

$$(\vec{a}-\vec{b})^2+(\vec{b}-\vec{c})^2+(\vec{c}-\vec{d})^2+(\vec{d}-\vec{a})^2-(\vec{a}-\vec{c})^2-(\vec{b}-\vec{d})^2=(\vec{a}-\vec{b}+\vec{c}-\vec{d})^2\geq 0.$$

The first parenthesis equals zero if and only if $ABCD$ is a parallelogram, and the second one if and only if it is a cyclic quadrilateral. Since $ABCD$ is not a rectangle, both conditions cannot be realized. Thus $(AC+BD)^2<(AB+CD)^2+(AD+BC)^2$. This and two similar inequalities yield the assertion of the problem. This reasoning work also for four collinear points: the first parenthesis equals zero, when one of points A, C lies inside the segment BD, and the remaining one lies outside it; the second parenthesis equals zero when the midpoints of AC and BD coincide. These both conditions cannot be correct simultaneously.

3. (L.Shatunov, V.Shelomovsky) Let (P,P') and (Q,Q') be two pairs of points isogonally conjugated with respect to a triangle ABC, and R be the common point of lines PQ and $P'Q'$. Prove that the pedal circles of points P, Q, and R are coaxial.

First solution. Denote by X_a, X_b, X_c the projections of an arbitrary point X to BC, CA, AB respectively. Let p, q, r be the pedal circles of points P, Q, R respectively, and M, N, K be their centers. Then M, N, K lie on the Gauss line of $PQP'Q'$. Applying the Menelaos theorem to the triangles PQR' and $P'Q'R'$ we obtain (R' is isogonally conjugated to R).

$$\frac{P'Q}{P'R'}\frac{Q'R'}{Q'P}\frac{PR}{RQ}=\frac{P'Q}{QR'}\frac{R'P}{PQ'}\frac{Q'R}{RP'}=1.$$

Therefore

$$\frac{RP\cdot RP'}{RQ\cdot RQ'}=\frac{R'P\cdot R'P'}{R'Q\cdot R'Q'}.$$

By the Thales theorem

$$\frac{R_aP_a\cdot R_aP'_a}{R_aQ_a\cdot R_aQ'_a}=\frac{R'_aP_a\cdot R'_aP'_a}{R'_aQ_a\cdot R'_aQ'_a},$$

i.e. the ratios of degrees of R_a, R_a' with respect to p and q are equal. Therefore these points lie on some circle coaxial with p and q. Since the center of this circle lies on MN, it coincides with r.

Second solution. Applying two times the Thales theorem we obtain $\frac{R_aP_a\cdot R_aP_a'}{R_aQ_a\cdot R_aQ_a'} = \frac{RP\cdot RP'}{RQ\cdot RQ'}$. Similarly $\frac{RP\cdot RP'}{RQ\cdot RQ'} = \frac{R_bP_b\cdot R_bP_b'}{R_bQ_b\cdot R_bQ_b'} = \frac{R_cP_c\cdot R_cP_c'}{R_cQ_c\cdot R_cQ_c'} = \frac{R_aP_a\cdot R_aP_a'}{R_aQ_a\cdot R_aQ_a'}$. By the property of coaxial circles R_a, R_b, R_c lie on a circle coaxial with the pedal circles of P and Q.

4. (P.Puchkov) For which $n > 0$ it is possible to mark several different points and several different circles on the plane in such a way that:
- exactly n marked circles pass through each marked point;
- exactly n marked points lie on each marked circle;
- the center of each marked circle is marked?

Answer. For all n.

Solution. Construct the required configuration by induction. If $n = 1$ it contains two circles with radii 1, such that each of them passes through the center of the other, and the centers of these circles. Let the configuration for n contain 2^n unit circles and their centers. Translate it to a unit vector distinct from all vectors between the marked points. Then we obtain one new point on each of old circles — the image of its center, and one new circle passing through each of old points — the image of the circle centered at this point. Similarly n new circles and one old circle pass through each of new points, and n new points and one old point lie on each of the circles.

Remark. Joining each point of the configuration with the centers of all circles passing through it, we obtain the projection of the n-dimensional cube such that the lengths of all edges are equal.

XX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Second day. 9 grade. Solutions

Ratmino. 2024. August 1.

5. (A.Zaslavsky) Let ABC be an isosceles triangle ($AC = BC$), O be its circumcenter, H be the orthocenter, and P be a point inside the triangle such that $\angle APH = \angle BPO = \pi/2$. Prove that $\angle PAC = \angle PBA = \angle PCB$.

   Solution. Let M be the midpoint of AB. Then B, O, P, M lie on the circle with diameter OB, and A, H, P, M lie on the circle with diameter AH. Hence $\angle PAH = \angle PMH = \angle PMO = \angle PBO$ (fig. 9.5). We obtain that $PAH \sim PBO$, i.e., P is the center of spiral similarity mapping $\vec{AH}$ to $\vec{BO}$. The calculation of angles ($\angle OBC = \angle OCB = 90^\circ - \angle ABC = \angle HAB = \angle HBA$) yields $AHB \sim BOC$. Therefore the above spiral similarity maps B to C. Thus $PAB \sim PBC$, which yields $\angle PCB = \angle PBA$ and $\angle PBC = \angle PAB$. Then $\angle PAC = \angle BAC - \angle PAB = \angle CBA - \angle PBC = \angle PBA$, q.e.d.

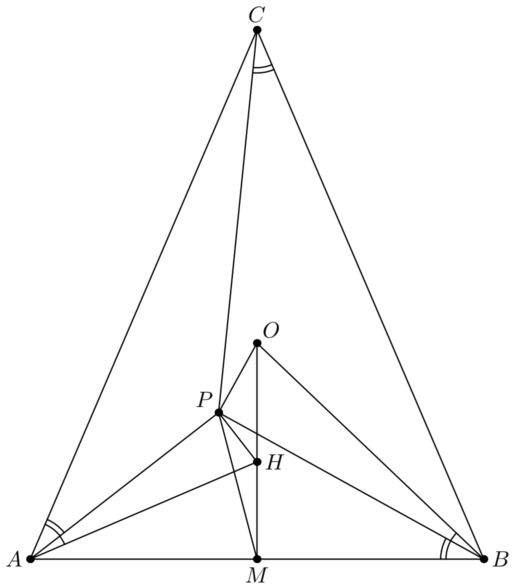

Fig. 9.5.

   Remark. A point P such that $\angle PAC = \angle PBA = \angle PCB$, and a point Q such that $\angle QAB = \angle QBC = \angle QCA$, are called *the Brocard points* of triangle ABC. From the solution we see that for an isosceles triangle P satisfies the conditions $\angle PAC = \angle PCB$, $\angle PAB = \angle PBC$. Such (and two similar) points are called *the Humpty points* and coincide with the projections of the orthocenter to the medians of triangle. Also we have $\angle PBA = \angle PCB$, $\angle PBC = \angle PAB$. Such (and similar) conditions define *the Dumpty points* coinciding with the projections of the circumcenter to the symedians. Thus we can reformulate the assertion of the problem: The Brocard points of an isosceles triangle are also the Humpty and Dumpty points corresponding to the base vertices.

6. (A.Mardanov, K.Mardanova) The incircle of a triangle ABC centered at I touches the sides BC, CA, and AB at points A_1, B_1, and C_1 respectively. The excircle centered at J touches the side AC at point B_2 and touches the extensions of AB, BC at points C_2, A_2 respectively. Let the lines IB_2 and JB_1 meet at point X, the lines IC_2 and JC_1 meet at point Y, the lines IA_2 and JA_1 meet at point Z. Prove that if one of points X, Y, Z lies on the incircle then two remaining points also lie on it.

   Solution. Since Y is the common point of diagonals of trapezoid IC_1C_2J, we have $IY : IC_2 = IC_1 : (IC_1 + JC_2) = r : (r + r_b)$, where r is the inradius, and r_b is an exradius. Therefore Y lies on the incircle if and only if $IC_2 = r + r_b$, and since $C_1C_2 = AC = b$, this is equivalent to $b^2 = r_b^2 + 2rr_b$. The same condition we obtain for Z.

   Now consider the point X. Let BH be the altitude and BL be the bisector of ABC. Since the quadruple B, L, I, J is harmonic, their projections H, L, B_1, B_2 also form a harmonic quadruple. Therefore the common point X of lateral sidelines of trapezoid IB_1JB_2 lies on BH (fig. 9.6). Also $BX : B_2J = BI : IJ = r : (r_b - r)$, i.e. $1/BX = 1/r - 1/r_b = p/S - (p-b)/S = 2/BH$ and X is the midpoint of BH. The condition $IX = r$ is equal to $r : BX = IB_2 : r_b$, and since $B_1B_2 = |AB - AC| = |a - c|$ we can rewrite this as $r_b^2 - 2rr_b = (a-c)^2$.

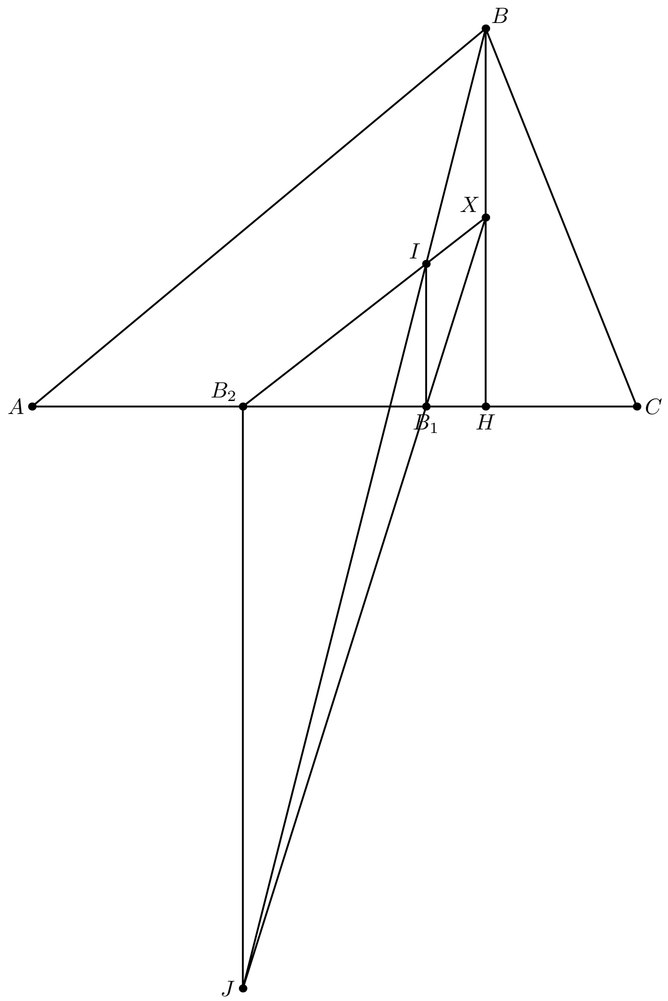

Fig. 9.6.

Finally note that $b^2 - (a-c)^2 = 4(p-a)(p-c) = 4S^2/(p(p-b)) = 4rr_b$. Hence the conditions for all three points are the same.

7. (D.Brodsky) Let P and Q be arbitrary points on the side BC of triangle ABC such that $BP = CQ$. The common points of segments AP and AQ with the incircle form a quadrilateral $XYZT$. Prove the locus of common points of diagonals of such quadrilaterals.

   Answer. The common point of the median from A and the segment joining the touching points of AB and AC with the incircle.

   Solution. Let AM be a median of the triangle, B', C' be the touching points of the incircle with AC, AB respectively. The common point M' of the diagonals of cyclic quadrilateral $XYZT$ lies on the polar of the common point of its sidelines XY and ZT, i.e. on $B'C'$. Let AP, AQ meet $B'C'$ at points P', Q' respectively. By the generalized butterfly theorem $(B'C'M'P') =$

$(C'B'M'Q')$. Since $(CBMP) = (BCMQ)$, this yields that A, M, M' are collinear (fig. 9.7).

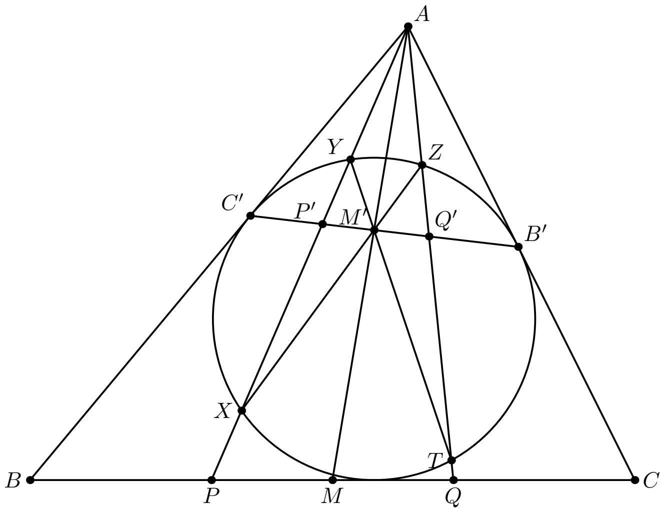

Fig. 9.7.

8. (G.Zabaznov) Let points P and Q be isogonally conjugated with respect to a triangle ABC. The line PQ meets the circumcircle of ABC at point X. The reflection of BC about PQ meets AX at point E. Prove that A, P, Q, E are concyclic.

Solution.

Lemma. Let a circle S, a point A on it, a point P and a line t passing through P be fixed. An arbitrary line q passing through P meets S at points X, Y. The lines AX, AY meet t at points E, F. Then the Miquel point of $FEXY$ is fixed.

Proof. Let W be the point on S such that $AW \parallel t$; PW meet S for the second time at M; U, V be the common points of t and S (fig. 9.8.1). Then $\angle PMY = \smile WY/2 = (\smile UY + \smile AV)/2 = \angle UFY$, i.e. M lies on the circle PFY. Similarly M lies on the circle PEX.

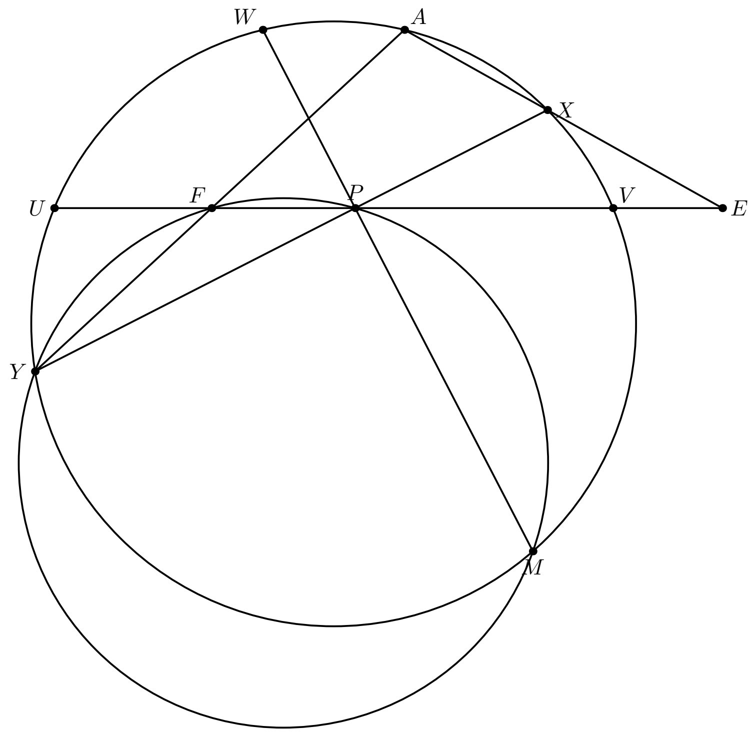

Fig. 9.8.1.

Return to the problem. Let PQ meet BC at point R, and meet the circle ABC for the second time at point Y. Let K, L, F be the common points of RE with AB, AC, AY respectively. By the lemma the Miquel points of quadrilaterals $BCLK$ and $XYFE$ coincide, denote this point by M. Note that the compositions of inversion and symmetry with center M corresponding to both quadrilaterals swap R and A. Therefore these compositions coincide.

Since P and Q lie on the bisector of angle FRC, they are isogonally conjugated with respect to the quadrilateral $CBKL$, hence the above composition of inversion and symmetry swaps them. Finally note that this composition maps the line PQ to the circle $AMEF$ (fig. 9.8.2).

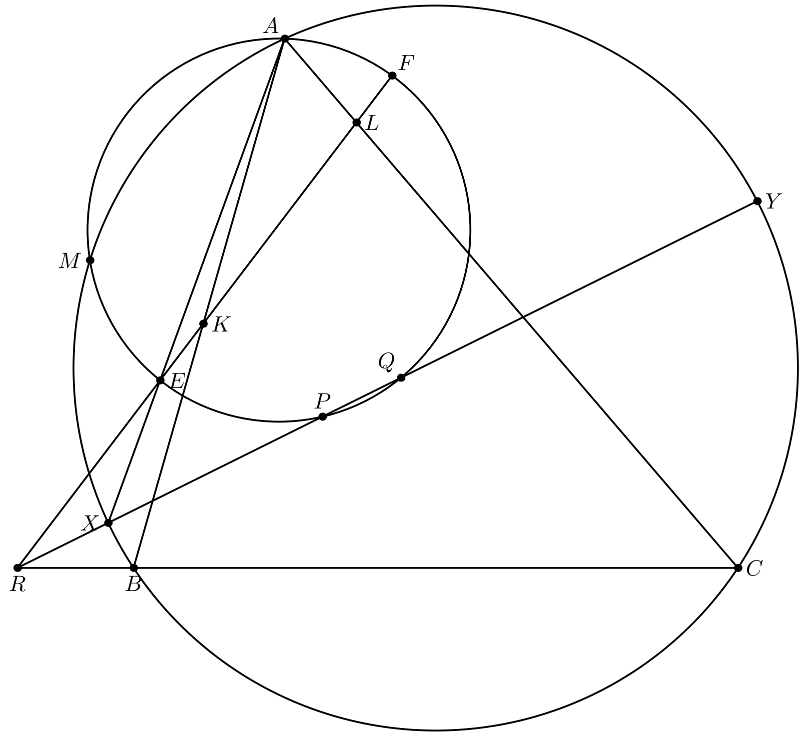

Fig. 9.8.2.

XX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The final round. First day. 10 grade. Solution
Ratmino. 2024. July 31.

1. (D.Shvetsov) The diagonals of a cyclic quadrilateral $ABCD$ meet at point P. The bisector of angle ABD meets AC at point E, and the bisector of angle ACD meets BD at point F. Prove that the lines AF and DE meet on the median of triangle APD.

   Solution. Since the triangles APB and DPC are similar, we have $AE : EP = AB : BP = CD : CP = DF : FP$ and the required assertion follows from the Ceva theorem (fig.10.1).

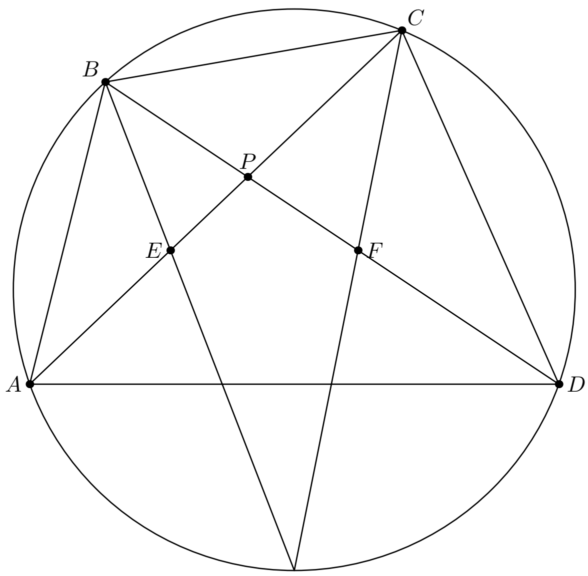

Fig. 10.1.

2. (T.Kazitsyna) For which greatest n there exists a convex polyhedron with n faces having the following property: for each face there exists a point outside the polyhedron such that the remaining $n-1$ faces are seen from this point?

   Answer. For $n = 4$.

   First solution. It is clear that a tetrahedron satisfies the condition. Suppose that $n > 4$. Then there exist three faces which have no common vertex. The polyhedron lies inside one of trihedral angles formed by the planes of these faces. An arbitrary ray from the vertex of this angle intersects the polyhedron by a segment. Deleting the faces containing the endpoints of these segments closest to the vertex we obtain a polyhedron containing the original one with smaller number of faces. All faces of this new polyhedron have to be seen from some point of space — contradiction.

Second solution. Let a point O be the origin of n vectors perpendicular to the faces of the polyhedron and lying outside it. The assumption of the problem means that for any of these vectors there exists a plane passing through O and such that this vector lies on one side, and the remaining $n-1$ vectors lie on the other side with respect to this plane. But we can choose four of n vectors such that O lies inside the tetrahedron formed by the endpoints of these four vectors. Any vector distinct from them lies inside a trihedral angle defined by some three vectors, therefore such plane can not exist for this vector.

Third solution. Each face is seen from some semispace. If $n > 4$ then each four of these semispaces have a common point. By the Helly theorem all semispaces have a common point which is impossible.

3. (N.Shteinberg, L.Finarevsky) Let BE and CF be the bisectors of a triangle ABC. Prove that $2EF \leq BF + CE$.

Solution. If $AB = AC$ then $BF = FE = EC$ and the assertion is correct. Let $AB < AC$. Then $BF : FA < CE : EA$, hence EF meets the extension of BC beyond B and $\angle BEF < \angle CBE$. On the other hand, C lies outside the circle BFE, thus $\angle BEF > \angle BCF$. Since $\angle BEF + \angle CFE = \angle BCF + \angle CBE$, we obtain that $|\angle BEF - \angle CFE| < \angle CBE - \angle BCF$ (fig. 10.3).

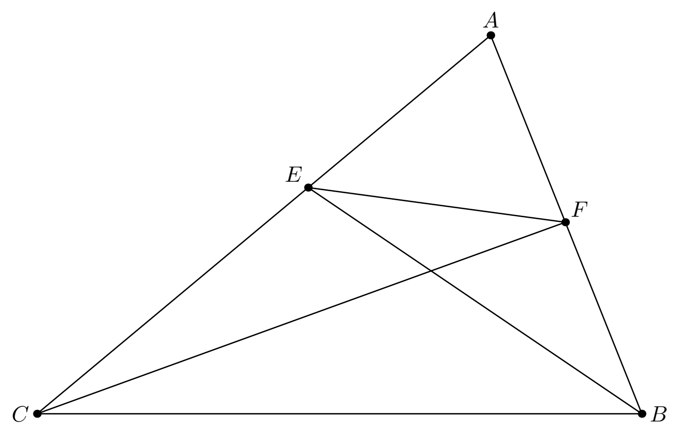

Fig. 10.3.

Applying the sine law to the triangles BFE and CEF we have

$$\frac{BF + CE}{EF} = \frac{\sin \angle BEF}{\sin \angle FBE} + \frac{\sin \angle CFE}{\sin \angle ECF}.$$

Since the product of fractions in the right part

$$\frac{\sin\angle BEF\sin\angle CFE}{\sin\angle FBE\sin\angle ECF} = \frac{\cos(\angle BEF-\angle CFE)-\cos(\angle BEF+\angle CFE)}{\cos(\angle FBE-\angle ECF)-\cos(\angle FBE+\angle ECF)} > 1,$$

their sum is greater than 2.

4. (S.Kuznetsov, M.Vekshin) Let I be the incenter of a triangle ABC. The lines passing through A and parallel to BI, CI meet the perpendicular bisector to AI at points S, T respectively. Let Y be the common point of BT and CS, and A^* be a point such that $BICA^*$ is a parallelogram. Prove that the midpoint of segment YA^* lies on the excircle of the triangle touching the side BC.

   First solution. It is clear that $\angle IST = \angle AST = \angle ICB$, $\angle ITS = \angle ATS = \angle IBC$. Hence the triangles BIC and TIS are similar, i.e. I is the Miquel point of $BCST$, and Y lies on the circles ICB, ITS. Let J be the excenter. Then A^* is the orthocenter of triangle JBC, thus the midpoint of A^*Y lies on the nine-points circle of this triangle.

   Let B', C' be the second common points of AS, AT with the circle ITS. Then $\angle B'C'T = \angle TC'I = \angle B'ST = \angle TSI$, therefore, $C'A$ bisects the angle $B'C'I$ equal to ACB. Similarly $B'A$ bisects angle $C'B'I$ equal to ABC. Also the corresponding sidelines of triangles ABC and $IB'C'$ are parallel, because their bisectors are parallel, thus these triangles are symmetric with respect to the midpoint of AI. Then $AB'CA^*$ and $AC'BA^*$ are parallelograms, and the homothety with center A^* and coefficient $1/2$ maps the circle ITS to the nine-points circle of triangle ABC.

   Thus the midpoint of A^*Y is the common point of nine-points circles of triangles ABC and JBC. This point lies also on the pedal circle of J with respect to triangle ABC — an excircle (fig.10.4).

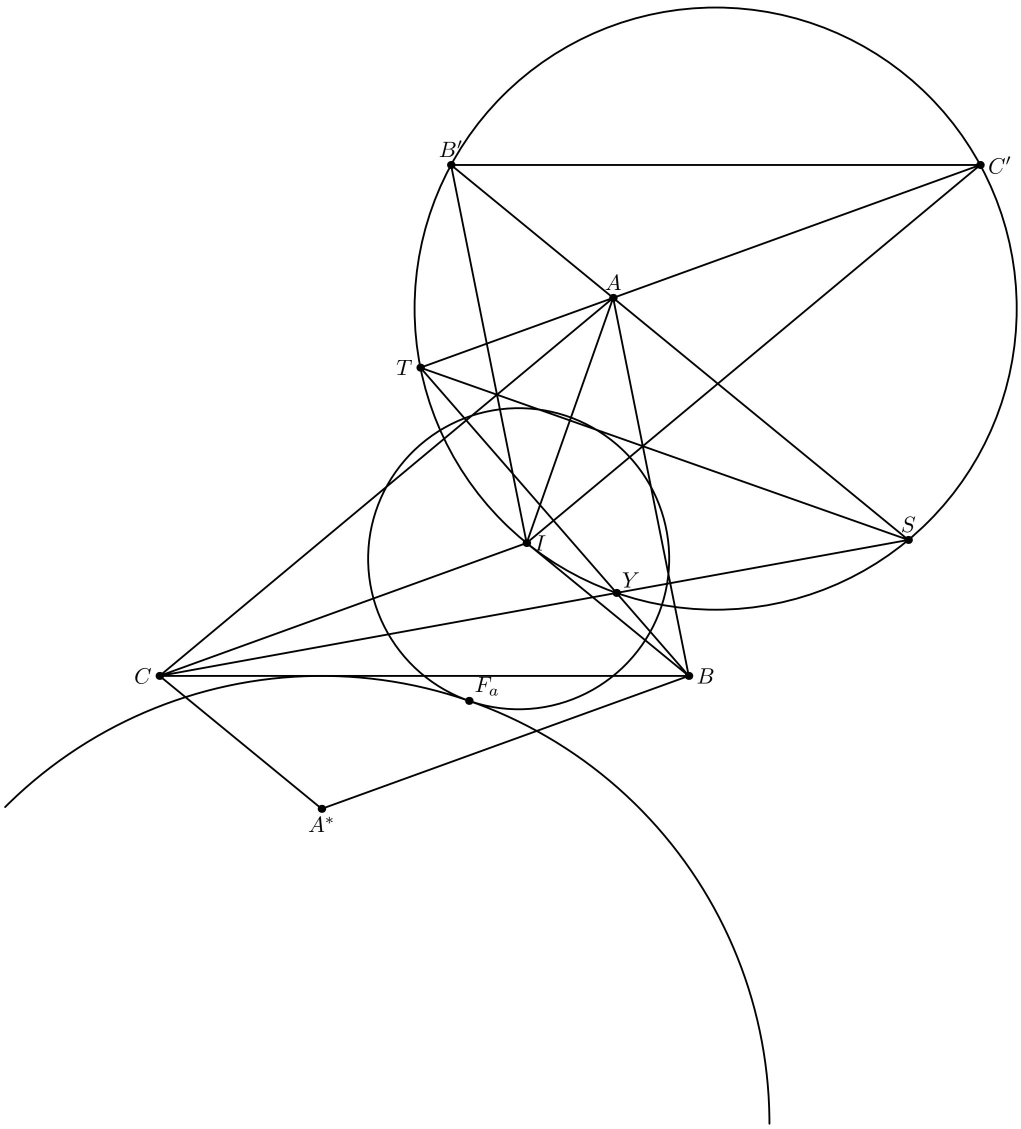

Fig. 10.4.

Remark. From the solution we see that the midpoint of A^*Y is the Feuerbach point F_a.

Second solution. Let ST meet BC at point P. The composition of inversion and symmetry with center A swapping B and C maps I to J, and maps P to a point P' on the circumcircle such that isosceles triangles API and AI_aP' are similar. Let the excircle touch BC at point D. Since the triangle formed by the external bisectors is similar to BCJ, and A, D are the corresponding points of these triangles, there exists a point P'' on the nine-points circle of triangle BCJ such that isosceles triangles AI_aP' and DI_aP'' are similar. Let M be the midpoint of BC and $\alpha = \angle API = \angle DJP''$. It is easy to see that the arc MP'' of the nine-points circle of triangle BCJ and the arc IY of the circle BCJ equal α, hence the homothety with center A^* and coefficient 2 maps M to I, and maps P'' to Y.

Third solution. Rename A^* to Z. Denote the reflection of a point about AI by prime. Let M be the midpoint of BC; the incircle and the excircle touch BC at P and Q respectively. Let us prove that the midpoint of $Y'Z'$ is the second common point of $M'Q$ with the excircle. Note that S' and T' are the midpoints of arcs AC and AB of the circumcircle. The triangles $S'IC'$ and $T'IB'$ are similar — the angles at I are equal and $IS'/IC' = IS'/IC = IT'/IB = IT'/IB'$. Thus Y' is the second common point of circles BIC and $S'IT'$. Let K be the reflection of I about the perpendicular bisector to BC (it also lies on the circle $BB'CC'IY'$), and F be the radical center of circles ABC, BIC, $S'IT'$. Then, since $MI \parallel AQ$, we have $MM' : MQ = 2IP : IA = BC : S'T' = \sin BFI : \sin S'FI = \sin Y'IK : \sin Y'KI = Y'K : Y'I$. Since $\angle QMM' = \angle KY'I$, we obtain that the triangles QMM' and $IY'K$ are similar. In particular, the lines IY' and $M'Q$ are parallel, and the homothety with center Z and the coefficient $1/2$ maps the first line to the second one. Let this homothety map Q to X. Then $\angle KXI = \angle MM'Q = \angle Y'KI$, hence $IY' \cdot IX = IK^2 = 4M'Q'^2$. Thus $M'Q'^2 = IX \cdot IY'/4 = M'Q \cdot (IY'/2)$. Therefore $M'Q$ meets the excircle for the second time at the point lying on the distance $IY'/2$ from M', i.e. at the midpoint of ZY'.

XX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Second day. 10 grade. Solutions

Ratmino. 2024. August 1.

5. (M.Vasilyev) The incircle of a right-angled triangle ABC touches the hypothenuse AB at point T. The squares $ATMP$ and $BTNQ$ lie outside the triangle. Prove that the areas of triangles ABC and TPQ are equal.

   Solution. Let $BC = a$, $AC = b$, $AB = c$, $AT = p - a$, $BT = p - b$. Then the inradius of ABC equals $p - c$, and its area

   $$S = p(p - c) = \frac{S^2}{(p - a)(p - b)} = (p - a)(p - b).$$

   Since PTQ is a right-angled triangle with cathetus $PT = \sqrt{2}(p - a)$, $TQ = \sqrt{2}(p - b)$, its area also equals $(p - a)(p - b)$.

6. (A.Zaslavsky) A point P lies on one of medians of triangle ABC in such a way that $\angle PAB = \angle PBC = \angle PCA$. Prove that there exists a point Q on another median such that $\angle QBA = \angle QCB = \angle QAC$.

   Solution. Let P lie on the median from B. Then AC touches the circle PB and the circle APB, because the median is the radical axis of these circles. On the other hand the circle APB touches BC (fig.10.6), therefore $AC = BC$ and the reflection of P about the symmetry axis of the triangle is the required point Q.

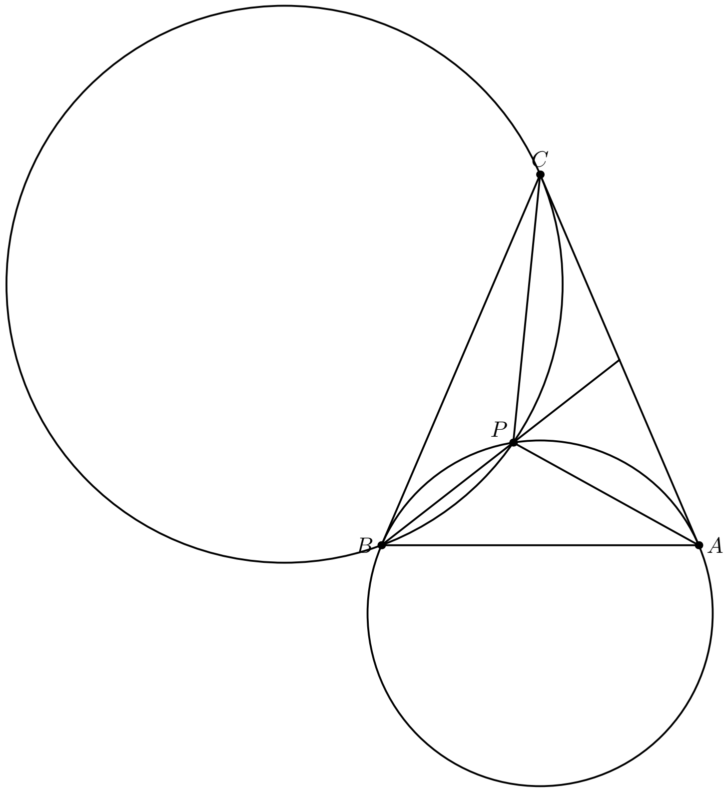

Fig. 10.6.

Remark. The assertion of the problem may be reformulated: if a Brocard point lies on a median (a symedian), then the triangle is isosceles and this point coincides with a Humpty and a Dumpty points.

7. (K.Belsky) Let ABC be a triangle with $\angle A = 60^\circ$; AD, BE, and CF be its bisectors; P, Q be the projections of A to EF and BC respectively; and R be the second common point of the circle DEF with AD. Prove that P, Q, R are collinear.

   Solution. It is known that the circle passing through the feet of bisectors passes also through the Feuerbach point. Also, since $\angle A = 60^\circ$, the orthocenter and the circumcenter of the triangle are symmetric with respect to the bisector of angle A. Hence the center of the nine-points circle lies on AD, i.e. the Feuerbach point coincides with R. Also, if I, r are the incenter and the inradius then $AI = 2r = 2IR$. Thus we have to prove that PQ bisects AI. Let us prove this for an arbitrary triangle.

   Let EF meet AD and BC at points S, T respectively. Since the quadruple A, I, S, D is harmonic, the inversion about the circle with diameter AI swaps points S and D. On the other hand, T is the foot of the external bisector of angle A, therefore AQ and AP are the altitudes of right-angled triangles

DAT and SAT. Hence $TS \cdot TP = TD \cdot TQ = TA^2$ and the inversion with center T and radius TA swaps S and P, T and Q. Since this circle and the circle with diameter AI are perpendicular, the inversion about the last circle swaps P and Q, i.e. PQ passes through its center R (fig. 10.7).

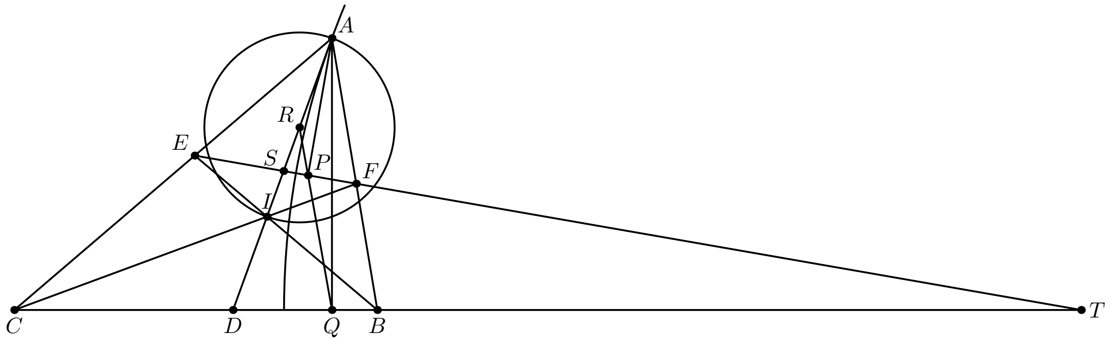

Fig. 10.7.

8. (G.Galyapin) The common tangents to the circumcircle and an excircle of triangle ABC meet BC, CA, AB at points A_1, B_1, C_1 and A_2, B_2, C_2 respectively. The triangle Δ_1 is formed by the lines AA_1, BB_1, and CC_1, the triangle Δ_2 is formed by the lines AA_2, BB_2, and CC_2. Prove that the circumradii of these triangles are equal.

   Solution.

   Lemma. Let J be the center of the excircle, and P be the touching point of the line $A_1B_1C_1$ with the circumcircle. Then C, J, C_1, P are concyclic.

   Proof. Let W be the second common point of CJ with the circumcircle. Then $UW \parallel C_1J$, and $\angle(PC, CJ) = \angle(PC_1, C_1J)$ (fig. 10.8).

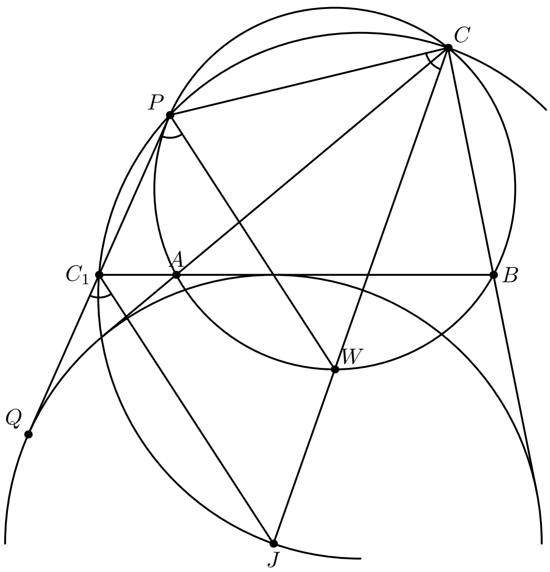

Fig. 10.8.

The assertion of the lemma may be reformulated: AA_1, AA_2 are isogonal in angle BAC, and $2\varphi = 2\angle(A_2A, AA_1) = \angle(JP, PQ)$, where P,Q are the touching points of the circumcircle and the excircle with their common tangent. Similar equalities are correct for B and C.

Since $\angle(A_2A, AA_1) = \angle(B_2B, BB_1) = \angle(C_2C, CC_1)$, we have $\angle(B_2B, C_2C) = \angle(BB_1, CC_1)$. Therefore B, C, J, $A_3 = BB_1 \cap CC_1$, and $A_4 = BB_2 \cap CC_2$ are concyclic, J is the midpoint of arc A_3A_4, and $\angle(A_4J, JA_3) = 2\varphi$. Similar equalities we obtain for the remaining vertices of Δ_1 and Δ_2, therefore the rotation with center J by angle 2φ maps one of these triangles to the second one, which yields the required assertion.