XX Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2024, July 31

Problems

First day. 8 grade

8.1. A circle ω centered at O and a point P inside it are given. Let X be an arbitrary point of ω, the line XP and the circle XOP meet ω for a second time at points X_1, X_2 respectively. Prove that all lines X_1X_2 are parallel.

8.2. Let CM be the median of an acute-angled triangle ABC, and P be the projection of the orthocenter H to the bisector of angle C. Prove that MP bisects the segment CH.

8.3. Let AD be the altitude of an acute-angled triangle ABC, and A' be the point of its circumcircle opposite to A. A point P lies on the segment AD, and points X, Y lie on the segments AB, AC respectively in such a way that $\angle CBP = \angle ADY$, $\angle BCP = \angle ADX$. Let PA' meet BC at point T. Prove that D, X, Y, T are concyclic.

8.4. A square with sidelength 1 is cut from the paper. Construct a segment with length $1/2024$ using at most 20 folds. No instruments are available, it is allowed only to fold the paper and to mark the common points of folding lines.

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Problems

First day. 9 grade

9.1. Let H be the orthocenter of an acute-angled triangle ABC; A_1, B_1, C_1 be the touching points of the incircle with BC, CA, AB respectively; E_A, E_B, E_C be the midpoints of AH, BH, CH respectively. The circle centered at E_A and passing through A meets for the second time the bisector of angle A at A_2; points B_2, C_2 are defined similarly. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar.

9.2. Points A, B, C, D on the plane do not form a rectangle. Let the sidelengths of triangle T equal $AB+CD$, $AC+BD$, $AD+BC$. Prove that the triangle T is acute-angled.

9.3. Let (P, P') and (Q, Q') be two pairs of points isogonally conjugated with respect to a triangle ABC, and R be the common point of lines PQ and $P'Q'$. Prove that the pedal circles of points P, Q, and R are coaxial.

9.4. For which $n > 0$ it is possible to mark several different points and several different circles on the plane in such a way that:
— exactly n marked circles pass through each marked point;
— exactly n marked points lie on each marked circle;
— the center of each marked circle is marked?

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Final round. Ratmino, 2024, July 31

Problems

First day. 10 grade

10.1. The diagonals of a cyclic quadrilateral $ABCD$ meet at point P. The bisector of angle ABD meets AC at point E, and the bisector of angle ACD meets BD at point F. Prove that the lines AF and DE meet on the median of triangle APD.

10.2. For which greatest n there exists a convex polyhedron with n faces having the following property: for each face there exists a point outside the polyhedron such that the remaining $n-1$ faces are seen from this point?

10.3. Let BE and CF be the bisectors of a triangle ABC. Prove that $2EF \leq BF + CE$.

10.4. Let I be the incenter of a triangle ABC. The lines passing through A and parallel to BI, CI meet the perpendicular bisector to AI at points S, T respectively. Let Y be the common point of BT and CS, and A^* be a point such that $BICA^*$ is a parallelogram. Prove that the midpoint of segment YA^* lies on the excircle of the triangle touching the side BC.

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Final round. Ratmino, 2024, August 1

Problems

Second day. 8 grade

8.5. The vertices M, N, K of rectangle $KLMN$ lie on the sides AB, BC, CA respectively of a regular triangle ABC in such a way that $AM = 2$, $KC = 1$, the vertex L lies outside the triangle. Find the value of angle KMN.

8.6. A circle ω touches lines a and b at points A and B respectively. An arbitrary tangent to the circle meets a and b at X and Y respectively. Points X' and Y' are the reflections of X and Y about A and B respectively. Find the locus of projections of the center of the circle to the lines $X'Y'$.

8.7. A convex quadrilateral $ABCD$ is given. A line $l \parallel AC$ meets the lines AD, BC, AB, CD at points X, Y, Z, T respectively. The circumcircles of triangles XYB and ZTB meet for the second time at point R. Prove that R lies on BD.

8.8. Two polygons are cut from the cartboard. Is it possible that for any disposition of these polygons on the plane they have common inner point or have only finite number of common points?

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Problems

Second day. 9 grade

9.5. Let ABC be an isosceles triangle ($AC = BC$), O be its circumcenter, H be the orthocenter, and P be a point inside the triangle such that $\angle APH = \angle BPO = \pi/2$. Prove that $\angle PAC = \angle PBA = \angle PCB$.

9.6. The incircle of a triangle ABC centered at I touches the sides BC, CA, and AB at points A_1, B_1, and C_1 respectively. The excircle centered at J touches the side AC at point B_2 and touches the extensions of AB, BC at points C_2, A_2 respectively. Let the lines IB_2 and JB_1 meet at point X, the lines IC_2 and JC_1 meet at point Y, the lines IA_2 and JA_1 meet at point Z. Prove that if one of points X, Y, Z lies on the incircle then two remaining points also lie on it.

9.7. Let P and Q be arbitrary points on the side BC of triangle ABC such that $BP = CQ$. The common points of segments AP and AQ with the incircle form a quadrilateral $XYZT$. Find the locus of common points of diagonals of such quadrilaterals.

9.8. Let points P and Q be isogonally conjugated with respect to a triangle ABC. The line PQ meets the circumcircle of ABC at point X. The reflection of BC about PQ meets AX at point E. Prove that A, P, Q, E are concyclic.

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10.5. The incircle of a right-angled triangle ABC touches the hypothenuse AB at point T. The squares $ATMP$ and $BTNQ$ lie outside the triangle. Prove that the areas of triangles ABC and TPQ are equal.

10.6. A point P lies on one of medians of triangle ABC in such a way that $\angle PAB = \angle PBC = \angle PCA$. Prove that there exists a point Q on another median such that $\angle QBA = \angle QCB = \angle QAC$.

10.7. Let ABC be a triangle with $\angle A = 60^\circ$; AD, BE, and CF be its bisectors; P, Q be the projections of A to EF and BC respectively; and R be the second common point of the circle DEF with AD. Prove that P, Q, R are collinear.

10.8. The common tangents to the circumcircle and an excircle of triangle ABC meet BC, CA, AB at points A_1, B_1, C_1 and A_2, B_2, C_2 respectively. The triangle Δ_1 is formed by the lines AA_1, BB_1, and CC_1, the triangle Δ_2 is formed by the lines AA_2, BB_2, and CC_2. Prove that the circumradii of these triangles are equal.

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