XIX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (A.Mardanov) (8) Let L be the midpoint of the minor arc AC of the circumcircle of an acute-angled triangle ABC. A point P is the projection of B to the tangent at L to the circumcircle. Prove that P, L, and the midpoints of sides AB, BC are concyclic.

   Solution. Let M, N, and K be the midpoints of AB, BC, and AC; let H be the foot of the altitude from B; then H is also the common point of BP and AC. It is clear that $MN \parallel AC \parallel PL$, thus $MPLN$ is a trapezoid. It is known that $MHKN$ is an isosceles trapezoid, hence $\angle MHP = \angle NKL$, $MH = KN$. Also $PH = KL$. Therefore the triangles MHP and NKL are congruent, i.e the trapezoid $MPLN$ is isosceles (fig.1), thus P, L, and the midpoints of AB and BC are concyclic.

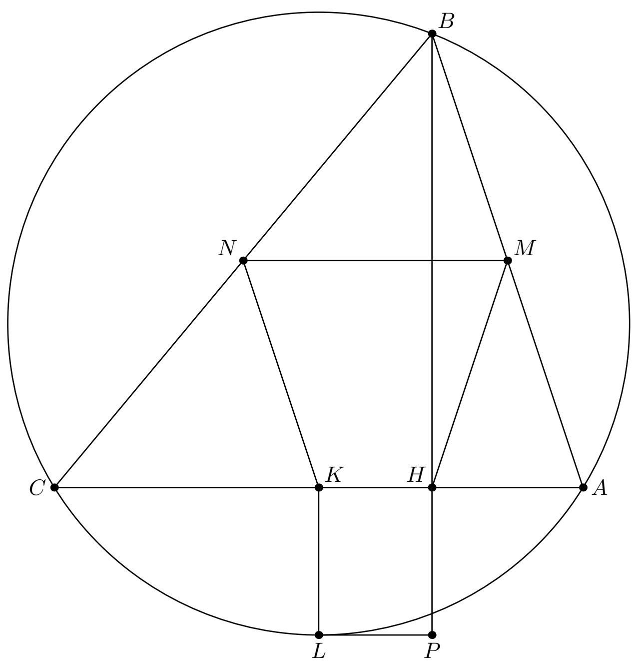

Fig. 1.

2. (N.Moskvitin) (8) The diagonals of a rectangle $ABCD$ meet at point E. A circle centered at E lies inside the rectangle. Let CF, DG, AH be the tangents to this circle from C, D, A; let CF meet DG at point I, EI meet AD at point J, and AH meet CF at point L. Prove that LJ is perpendicular to AD.

   Solution. Since the lines DG and AH are symmetric with respect to the perpendicular bisector to AD, and AH and CF are symmetric with respect to the perpendicular bisector to AC, we obtain that $\angle CID = 2\angle EAD = \angle CED$, i.e. C, D, I, E are concyclic. Hence $\angle AEI = \angle CDI$, and since $\angle AEL = \angle ADC = 90°$, we obtain that $\angle JEL = \angle IDA = \angle JAL$. Therefore A, J, E, L are concyclic and $\angle AJL = 90°$ (fig.2).

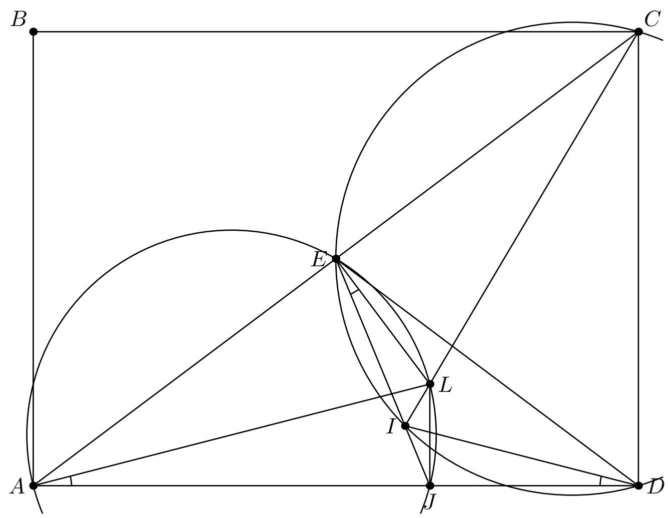

Fig. 2.

3. (D.Mukhin) (8) A circle touches the lateral sides of a trapezoid $ABCD$ at points B and C, and its center lies on AD. Prove that the diameter of the circle is less than the medial line of the trapezoid.

   Solution. Let O be the center of the circle; EF be its diameter lying on AD; G, H be the projections of B and C to AD. Since the arcs BE and CF are equal, we obtain that $\angle ABG = \angle DCH$, i.e. $AG = DH$ and the trapezoid is isosceles. Therefore its medial line equals to $AH = EH + AE = EH + OA - OB$. But from the similarity of triangles OAB and OBG we have $OA - OB > OB - OG = GE = HF$, thus $AH > EH + HF = EF$ (fig.3).

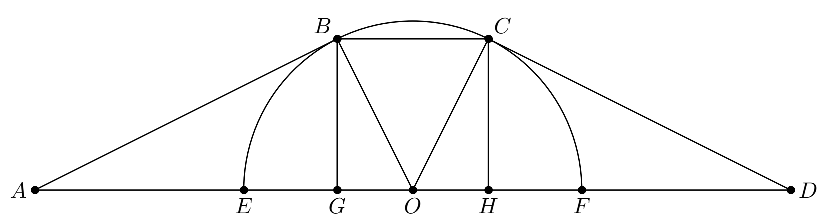

Fig. 3.

4. (F.Ivlev, A.Mardanov) (8) Points D and E lie on the lateral sides AB and BC respectively of an isosceles triangle ABC in such a way that $\angle BED = 3\angle BDE$. Let D' be the reflection of D about AC. Prove that the line $D'E$ passes through the incenter of ABC.

   Solution. Let the bisectors AL and CH meet at point I. Since $\angle BLA = \angle BAL + \angle ACL = 3\angle BAL$, we obtain that $DE \parallel AL$. Also $AD' \parallel BL$. Since $LE : AD' = LE : AD = BL : BA = IL : IA$, we obtain that ED' passes through I (fig.4).

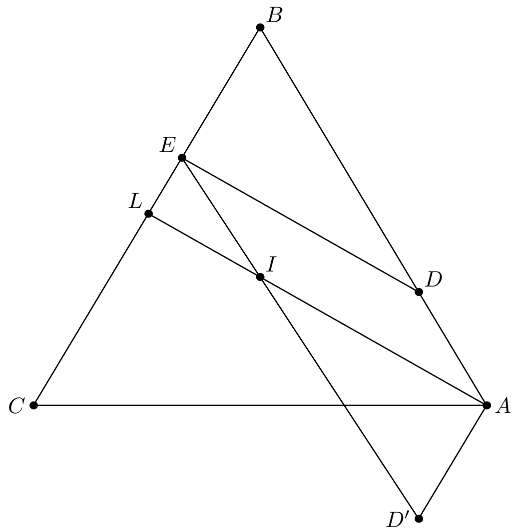

Fig. 4.

5. (I.Kukharchuk) (8) Let $ABCD$ be a cyclic quadrilateral. Points E and F lie on the sides AD and CD in such a way that $AE = BC$ and $AB = CF$. Let M be the midpoint of EF. Prove that $\angle AMC = 90°$.

First solution. In the pentagon $ABCFE$ we have $\angle A + \angle C = 180°$, thus $\angle B + \angle E + \angle F = 360°$. Draw from any point U segments $UX = AB = CF$, $UY = BC = AE$, $UZ = ME = MF$ in such a way that $\angle XUY = \angle B$, $\angle YUZ = \angle E$, $\angle ZUX = \angle F$. Then the triangles UXY, UYZ, UZX, and XYZ are congruent to the triangles BAC, EMA, FMC, and ACM respectively, therefore $\angle AMC = \angle AME + \angle CMF = 90°$ (fig.5).

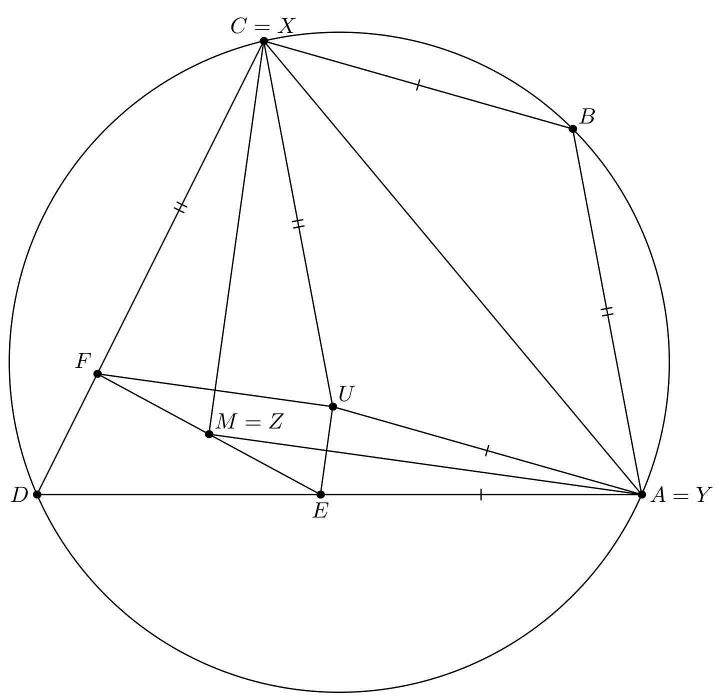

Fig. 5.

Second solution. Construct the parallelogram $ABCU$. The point E, F are the reflections of U about the bisectors of angles BAD, BCD respectively. Since $\angle AUC+\angle ADC = 180°$, these bisectors are perpendicular. Therefore the triangle UEF is right-angled, and the bisectors meet at its orthocenter M.

6. (D.Shvetsov) (8–9) Let A_1, B_1, C_1 be the feet of altitudes of an acute-angled triangle ABC. The incicrle of triangle $A_1B_1C_1$ touches A_1B_1, A_1C_1, B_1C_1 at points C_2, B_2, A_2 respectively. Prove that the lines AA_2, BB_2, CC_2 concur at a point lying on the Euler line of triangle ABC.

   Solution. Since the altitudes of triangle ABC coincide with the bisectors of triangle $A_1B_1C_1$ which are perpendicular to the sidelines of triangle $A_2B_2C_2$, we obtain that the triangles ABC and $A_2B_2C_2$ are homothetic (fig.6). Its homothety center lies on the line passing through the circumcenters of these triangles, i.e. the Euler line of ABC.

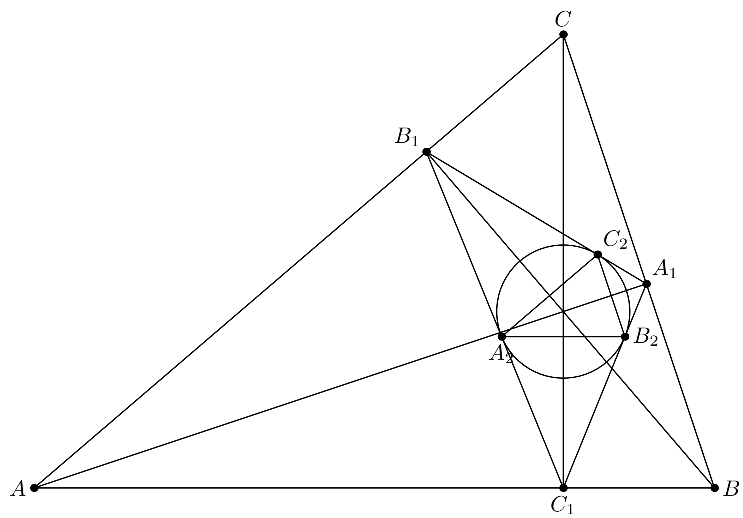

Fig. 6.

7. (D.Demin, I.Kukharchuk) (8–9) Let A be a fixed point of a circle ω. Let BC be an arbitrary chord of ω passing through a fixed point P. Prove that the nine-points circles of triangles ABC touch some fixed circle not depending on BC.

   Solution. The locus of the midpoints of chords BC is the circle with diameter OP, where O is the center of ω. The homothety with center A and coefficient $2/3$ maps this circle to the locus of the centroids of triangles ABC. Applying the homothety with center O and coefficient $3/2$ to this circle, we obtain that the locus of the centers of the nine-points circles is also a circle. Since the radii of all nine-point circles are equal, we obtain that all these circles touch two fixed circles.

8. (G.Filipovsky) (8–9) A triangle ABC ($a > b > c$) is given. Its incenter I and the touching points K, N of the incircle with BC and AC respectively are marked. Construct a segment with length $a - c$ using only a ruler and drawing at most three lines.

   Solution. It is known that the common point T of the line KN and the bisector BI coincide with the projection of A to BI. Thus if AT meets BC at point P, then the altitude BT of triangle BPT coincides with the bisector. Therefore $BP = AB$ and CP is the required segment (fig.8).

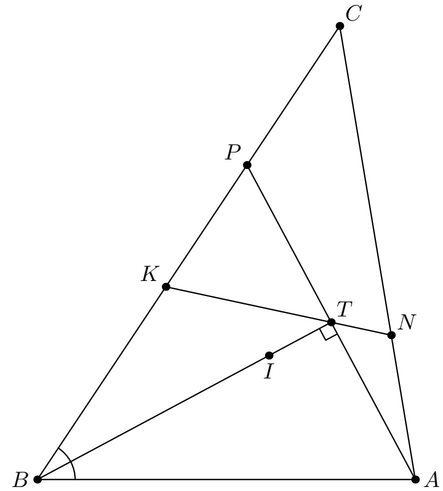

Fig. 8.

9. (S.Gubanov) (8–9) It is known that the reflection of the orthocenter of a triangle ABC about its circumcenter lies on BC. Let A_1 be the foot of the altitude from A. Prove that A_1 lies on the circle passing through the midpoints of the altitudes of ABC.

   Solution. The distance from the circumcenter to the line BC equals to a half of AH, where H is the orthocenter. On the other hand the assumption yields that this distance equals to a half of HA_1. Therefore H is the midpoint of AA_1. Let A_0 be the midpoint of BC. Since the midpoints X, Y of altitudes BB_1, CC_1 lie on the medial lines of the triangle, we obtain that the angles A_0XH and A_0YH are right, i.e. the points X and Y lie on the circle with diameter A_0H. It is clear that A_1 also lies on this circle (fig.9).

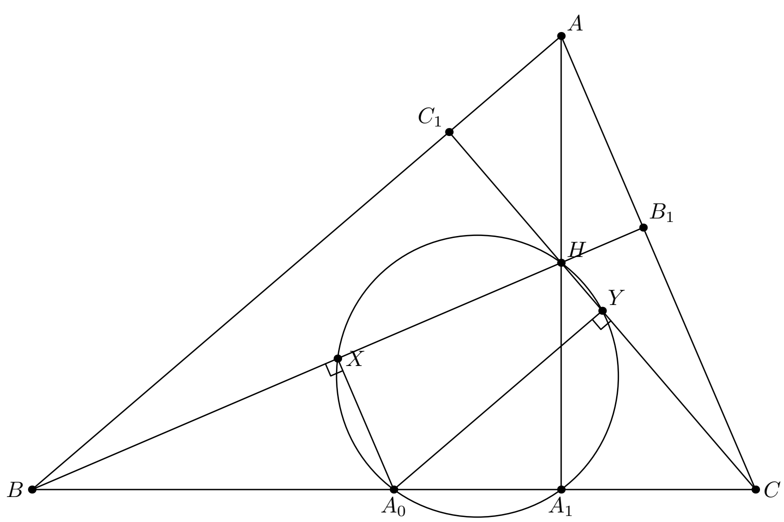

Fig. 9.

10. (G.Zabaznov) (8–9) Altitudes BE and CF of an acute-angled triangle ABC meet at point H. The perpendicular from H to EF meets the line ℓ passing through A and parallel to BC at point P. The bisectors of two angles between ℓ and HP meet BC at points S and T. Prove that the circumcircles of triangles ABC and PST are tangent.

Solution. Let PH meet BC at point M. Since $\angle MPT = \angle APT = \angle MTP$, we have $MT = MP$. Similarly $MS = MP$, i.e. M is the circumcenter of triangle PST. Also since $AO \perp EF$, we obtain that $AO \parallel MP$, where O is the circumcenter of triangle ABC. Since the reflection H' of H about BC lies on the circumcircle of ABC, we obtain that isosceles triangles HMH' and AOH' are similar, thus M lies on the segment OH', and the lines OM and MP form equal angles with BC. Then OM and ℓ meet at the common point of circles ABC and PST, which is the tangency point of these circles (fig.10).

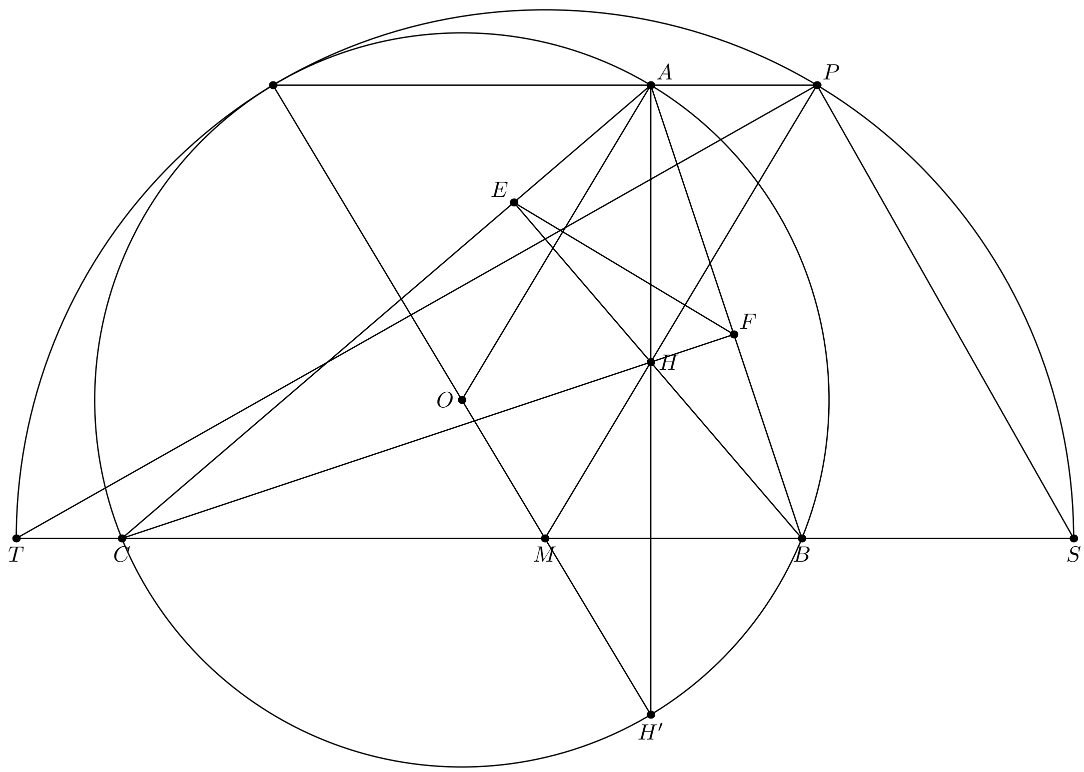

Fig. 10.

11. (M.Kursky) (8–10) Let H be the orthocenter of an acute-angled triangle ABC; E, F be points on AB, AC respectively, such that $AEHF$ is a parallelogram; X, Y be the common points of the line EF and the circumcircle ω of triangle ABC; Z be the point of ω opposite to A. Prove that H is the orthocenter of triangle XYZ.

    Solution. The assumption yields that $\angle BHE = \angle CHF = \pi/2$, therefore the triangles BHE and CHF are similar and $AF : EB = EH : EB = HF : FC = AE : EC$. Hence $AE \cdot EB = AF \cdot FC$, i.e. the powers of E and F with respect to the circumcircle are equal and the midpoint D of AH is also the midpoint of XY. Thus the medial line OD of triangle AHZ is perpendicular to XY. Therefore ZH is the altitude of triangle XYZ, and since the reflection A of H with respect to the midpoint of XY lies on the circumcircle, we obtain that H is the orthocenter (fig.11).

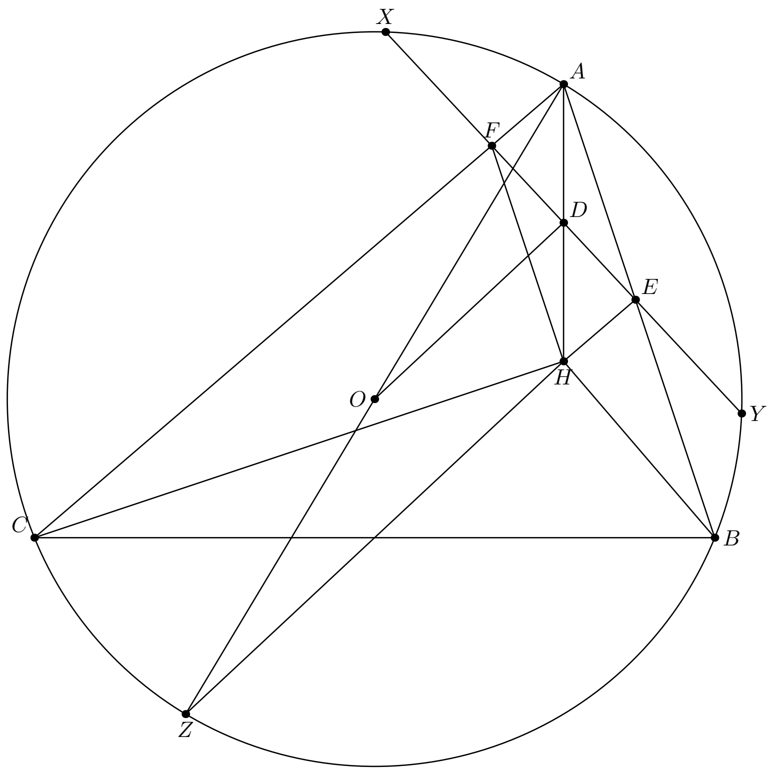

Fig. 11.

12. Let ABC be a triangle with obtuse angle B, and P, Q lie on AC in such a way that $AP = PB$, $BQ = QC$. The circle BPQ meets the sides AB and BC at points N and M respectively.

    (a) (P.Ryabov, 8–9) Prove that the distances from the common point R of PM and NQ to A and C are equal.

    (b) (A.Zaslavsky, 10–11) Let BR meet AC at point S. Prove that $MN \perp OS$, where O is the circumcenter of ABC.

    Solution. (a) Let O be the circumcenter of triangle ABC. Then $OP \perp AB$ and $OQ \perp BC$. Also $\angle NQA = \angle NBP = \angle A$ and similarly $\angle MPC = \angle C$. Hence $\angle PRQ + \angle POQ = \pi$ and $OPRQ$ is a cyclic quadrilateral. Therefore $\angle PRO = \angle PQO = \pi/2 - \angle C$, i.e. the diagonals of this quadrilateral are perpendicular and $AR = AC$ (fig.12).

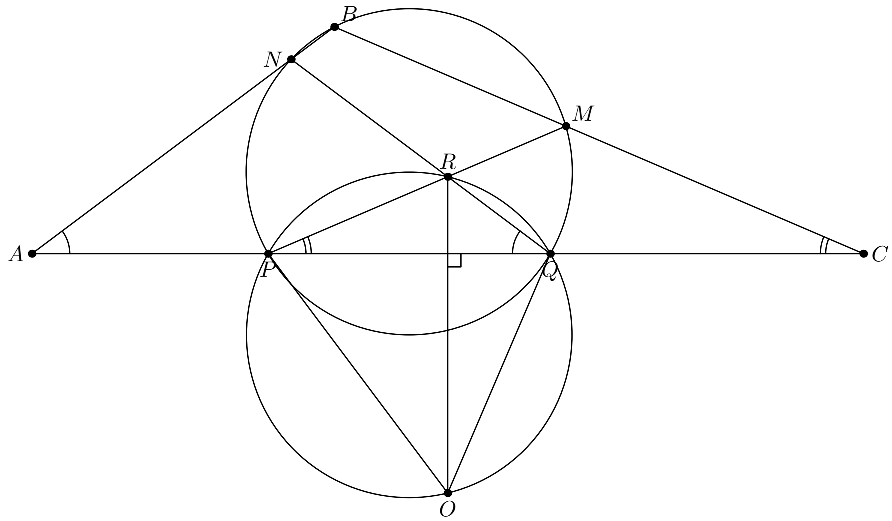

Fig. 12.

(b) By (a) the triangles QRP and ABC are orthologic with center O. Also they are perspective with center S. By the Sondat theorem OS is perpendicular to the perspective axis MN.

13. (A.Mardanov) (8–11) The base AD of a trapezoid $ABCD$ is twice greater than the base BC, and the angle C equals one and a half of the angle A. The diagonal AC divides angle C into two angles. Which of them is greater?

Solution. Let the lateral sidelines meet at point P and the perpendicular bisector to PD meet AP at point Q. Then $\angle AQD = 2\angle QPD = \angle QAD$ and $AD = QD > CD$ (a hypothenuse is greater than a cathetus). Thus $\angle ACD > \angle CAD = \angle BCA$ (fig.13).

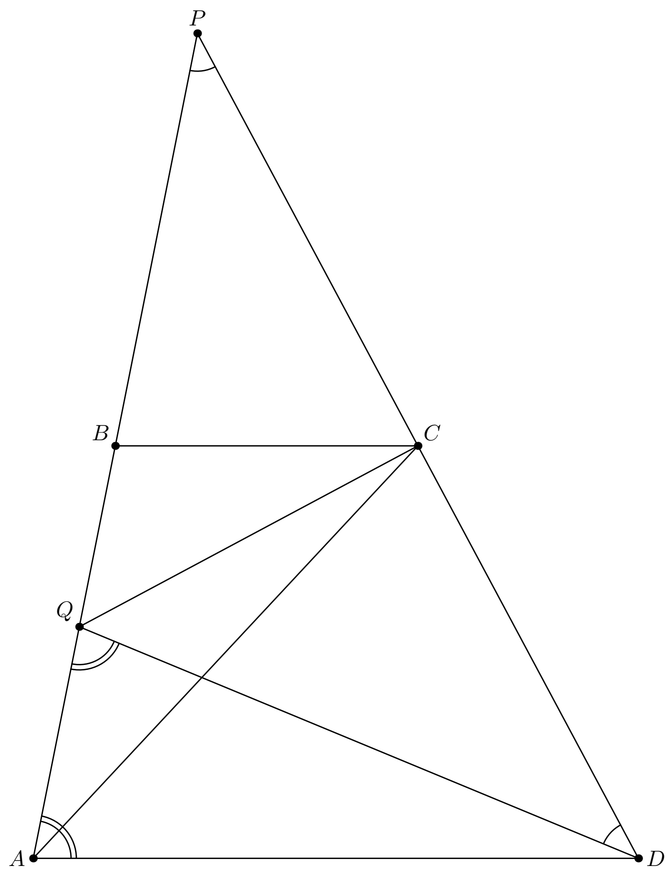

Fig. 13.

14. (A.Skopenkov) (8–11) Suppose that a closed oriented polygonal line l in the plane does not pass through a point O, and is symmetric with respect to O. Prove that the winding number of l around O is odd.

    The winding number of l around O is defined to be the following sum of the oriented angles divided by 2π:

$$\deg_O l := \frac{\angle A_1OA_2 + \angle A_2OA_3 + \ldots + \angle A_{n-1}OA_n + \angle A_nOA_1}{2\pi}.$$

    Solution. It is clear that the number of links is even: $n = 2k$. Since A_1 and A_{k+1} are symmetric with respect to O, we obtain that between A_1 and A_{k+1} the vector OA_i rotates by angle $m\pi$, where m is odd. But it rotates by the same angle between A_{k+1} and A_1. Therefore the winding number equals $2\pi m$.

15. (A.Matveev) (9–10) Let $ABCD$ be a convex quadrilateral. Points X and Y lie on the extensions beyond D of the sides CD and AD respectively in such a way that $DX = AB$ and $DY = BC$. Similarly points Z and T lie on the extensions beyond B of the sides CB and AB respectively in such a way that $BZ = AD$ and $BT = DC$. Let M_1 be the midpoint of XY, and M_2 be the midpoint of ZT. Prove that the lines DM_1, BM_2, and AC concur.

    Solution. Let DM_1 meet AC at point P. Then $\sin\angle ADP : \sin\angle CDP = \sin\angle YDM : \sin\angle XDM = XD : YD = AB : BC$. Therefore $AP : CP = (AB \cdot AD) : (CB \cdot CD)$. We obtain the same ratio for the meeting point of AC with the line BD_2.

16. (P.Kozhevnikov) (9–11) Let AH_A and BH_B be the altitudes of a triangle ABC. The line H_AH_B meets the circumcircle of ABC at points P and Q. Let A' be the reflection of A about BC, and B' be the reflection of B about CA. Prove that A', B', P, Q are concyclic.

Solution. Since the reflections of the orthocenter about the sidelines lie on the circumcircle, we have $H_AH \cdot H_AA' = H_AB \cdot H_AC = H_AP \cdot H_AQ$, therefore P, Q, H, A' are concyclic. Similarly P, Q, H, B' are concyclic (fig.16).

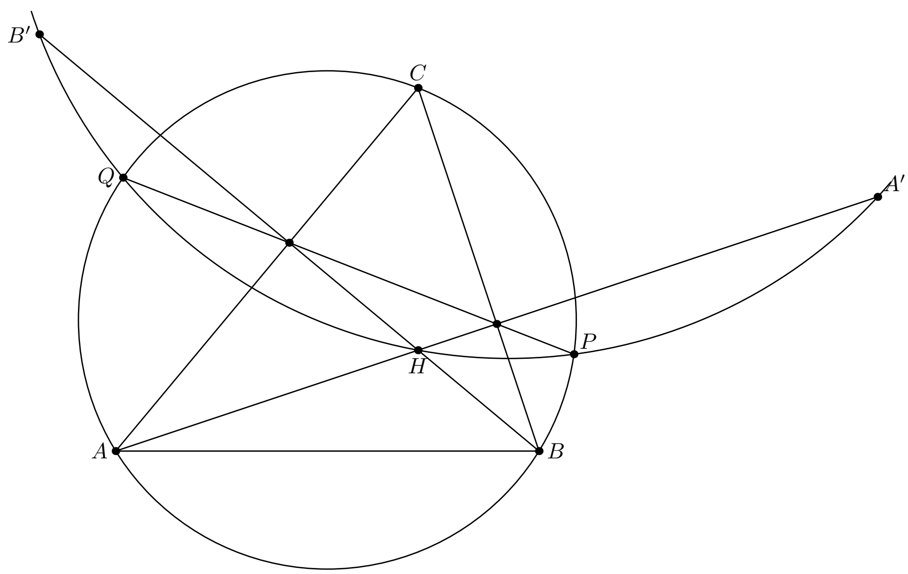

Fig. 16.

17. (L.Shatunov) (9–11) A common external tangent to circles ω_1 and ω_2 touches them at points T_1, T_2 respectively. Let A be an arbitrary point on the extension of T_1T_2 beyond T_1, and B be a point on the extension of T_1T_2 beyond T_2 such that $AT_1 = BT_2$. The tangents from A to ω_1 and from B to ω_2 distinct from T_1T_2 meet at point C. Prove that all nagelians of triangles ABC from C have a common point.

Solution. Let us prove that all nagelians pass through the center of internal homothety of the circles. Reformulate the problem: let a triangle ABC be given; T_1, T_2 be two points on the side AB, symmetric with respect to its midpoint; and two circles inscribed into angles A, B touch AB at T_1, T_2 respectively. Then the internal homothety center of these circles lies on the nagelian CD.

When T_1, T_2 move along AB, the centers of the circles move along the bisectors of angles A and B respectively, and the ratio of their radii is constant and equals $\cot\frac{A}{2} : \cot\frac{B}{2} = AD : BD$. Therefore the homothety center moves along some line passing through D. Also since $AC + AD = BC + BD$, we obtain that the incircle of triangle ACD corresponds to the incircle of triangle BCD (fig.17). It is clear that the homothety center of these circles lies on CD. Thus this is correct for any pair of corresponding circles.

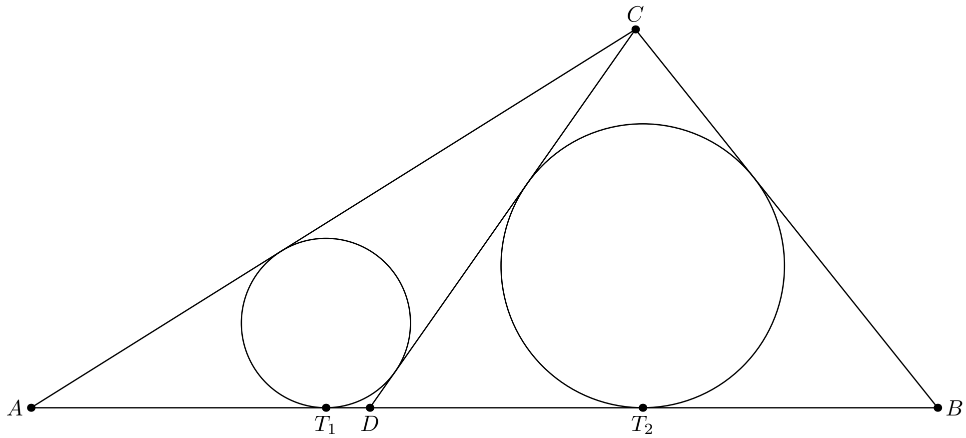

Fig. 17.

18. (A.Zaslavsky) (9–11) Restore a bicentral quadrilateral $ABCD$ if the midpoints of the arcs AB, BC, CD of its circumcircle are given.

    Solution. The circumcircle of the quadrilateral passes through given points, and its chords joining the midpoints of the opposite arcs are perpendicular. Hence we can restore the midpoint of the arc DA. The tangents to the circumcircle at the midpoints of arcs are parallel to the sidelines of the required quadrilateral, and two quadrilaterals are homothetic because they are circumscribed. Therefore we can draw the circumcircle of the quadrilateral formed by the tangents, and applying the homothety mapping this circle to the circumcircle we restore the quadrilateral.

19. (A.Zaslavsky) (10–11) A cyclic quadrilateral $ABCD$ is given. An arbitrary circle passing through C and D meets AC, BC at points X, Y respectively. Find the locus of common points of circles CAY and CBX.

    Answer. The line CE, where $AEBD$ is a harmonic quadrilateral.

    Solution. Consider a composition of an inversion centered at C and the reflection about the bisector of angle BCA, swapping A and B. It swaps also the line AB and the circumcircle of $ABCD$, i.e. maps D to some point D' on AB; the image of any circle passing through C and D is a line passing through D' and meeting AC, BC at points Y', X' respectively. The circles CAY and CBX are transformed to the lines AX', BY', and the common point of these lines lies on the line passing through C and meeting AB at point E' such that A, B, D', E' is a harmonic quadruple. Repeated applying of the inversion and the reflection maps this line to CE (fig.19).

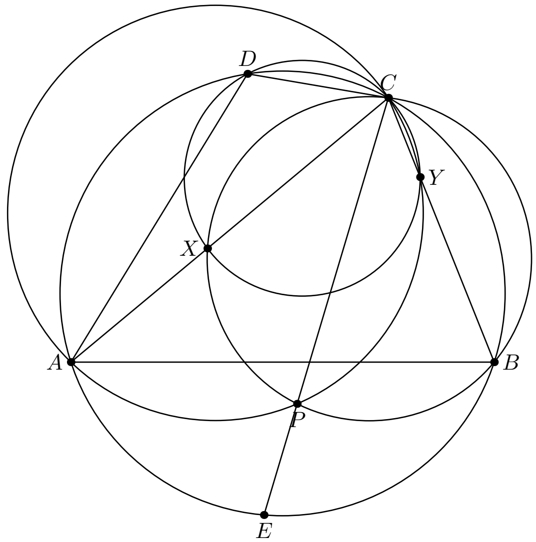

Fig. 19.

20. (A.Shevtsov) (10–11) Let a point D lie on the median AM of a triangle ABC. The tangents to the circumcircle of triangle BDC at points B and C meet at point K. Prove that DD' is parallel to AK, where D' is isogonally conjugated to D with respect to ABC.

First solution. If D moves along AM, then D' and K move along a symedian and the perpendicular bisector to BC. The correspondence between D' and D is projective, and K depend on D quadratically (it is the pole of fixed line BC with respect to the circle BDC, and the coefficients of equation of this circle are quadratic functions of D). Hence we have to prove the assertion for five positions of D. If D coincides with A, then D' is the foot of the symedian, and K is the common point of tangents to the circumcircle of ABC, i.e., all three points are collinear. If D is the second common point of AM with the circumircle, then K also lies on the symedian, and D' is its infinite point. If D coincides with M, then K also coincides with M, and D' coincides with A. If $ABDC$ is a parallelogram, then D' is the common point of the tangents to the circumcircle at B and C, and K is the reflection of D' about M. Finally if D is infinite, then K coincides with M. The required assertion is correct for all cases.

Second solution

(N. Beluhov, https://artofproblemsolving.com/community/c6h3025566p27515578)

Denote the circle BDC by Γ and let AB and AC meet Γ again at B' and C'. Also let M' be the midpoint of $B'C'$ and let the tangents to Γ at B' and C' meet at K'.

Notice that A, K, and K' are collinear. (All three lie on the polar of $BC \cap B'C'$ with respect to Γ.) So what we want becomes $DD' \parallel KK'$. We will show that both of DD' and KK' are antiparallel to MM' within $\angle BAC$.

We begin with DD'. Let AD meet Γ again at E. Then $ABD' \sim AEC$, and so $AD' \cdot AE = AB \cdot AC$. Hence, $AD : AD' = (AD \cdot AE) : (AD' \cdot AE) = (AB \cdot AB') : (AB \cdot AC) = AB' : AC$.

On the other hand, $ABCM \sim AC'B'M'$, and so $AM : AM' = AC : AB' = AD' : AD$. Furthermore, A, D, and M are collinear, and the same similarity yields also $\angle BAM = \angle C'AM'$, so that A, D', and M' are collinear as well. Thus D, D', M, and M' are concyclic. From here, a straightforward angle chase shows that DD' and MM' are indeed antiparallel within $\angle BAC$, as claimed.

We continue with KK'. Let O and R be the center and radius of Γ. Then O, M, and K are collinear, O, M', and K' are collinear as well, and $OM \cdot OK = R^2 = OM' \cdot OK'$. Consequently, K, K', M, and M' are concyclic. From here, a straightforward angle chase shows that KK' and MM' are indeed antiparallel within $\angle BAC$, as claimed. The solution is complete.

21. (I.Mikhaylov) (10–11) Let $ABCD$ be a cyclic quadrilateral; M_{ac} be the midpoint of AC; H_d, H_b be the orthocenters of $\triangle ABC, \triangle ADC$ respectively; P_d, P_b be the projections of H_d and H_b to BM_{ac} and DM_{ac} respectively. Define similarly P_a, P_c for the diagonal BD. Prove that P_a, P_b, P_c, P_d are concyclic.

Solution. It is known that A, C, P_d, H_d lie on the circle symmetric to the circumcircle of ABC with respect to M_{ac}. Hence $MA \cdot MC = MP_d \cdot MB' = MP_d \cdot MB$, where B' is the vertex of a parallelogram $ABCB'$. Similarly $MA \cdot MC = MD \cdot MP_b$, thus B, D, P_b, P_d are concyclic. Also since A, C, P_b, P_d are concyclic, we obtain that BD, AC, and P_bP_d concur at the radical center L (fig.21). Similarly P_aP_c passes through L, and $LP_a \cdot LP_c = LA \cdot LC = LB \cdot LD = LP_b \cdot LP_d$ which yields the required assertion.

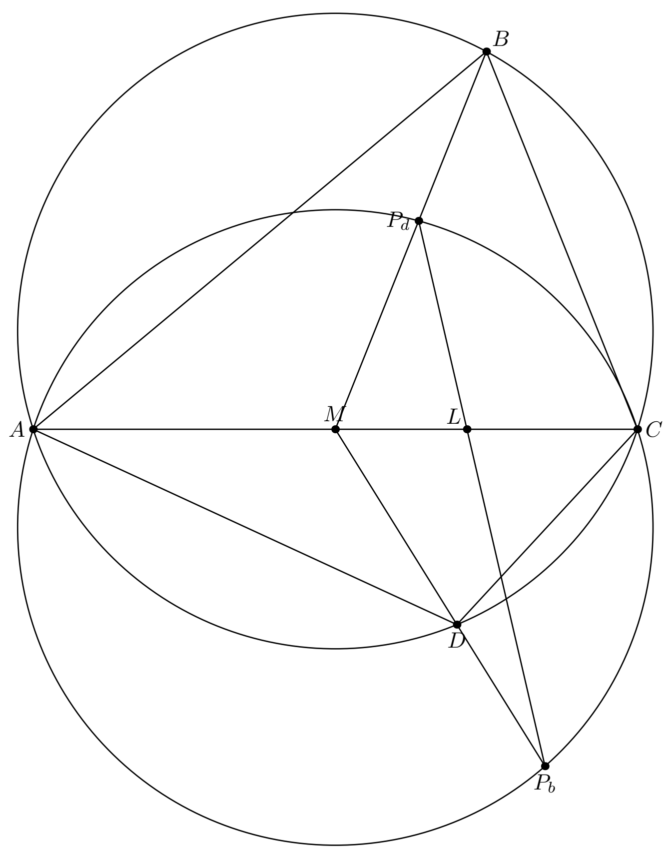

Fig. 21.

22. (A.Mudgal, P.Srivastava) (10–11) Let ABC be a scalene triangle, M be the midpoint of BC, P be the common point of AM and the incircle of ABC closest to A, and Q be the common point of the ray AM and the excircle farthest from A. The tangent to the

incircle at P meets BC at point X, and the tangent to the excircle at Q meets BC at Y. Prove that $MX = MY$.

Solution. Consider the homothety centered at A and mapping the excircle to the incircle. Let B', C', M', Q', Y' be the images of B, C, M, Q, Y respectively. Then $BB'C'C$ is a trapezoid circumscribed around the incircle of ABC, denote the touching points of BC, $B'C'$, BB', CC' as K, L, U, V respectively. By the Brianchon theorem the common point J of segments KL and UV coincides with the common point of the diagonals of the trapezoid, therefore J lies on AM (fig.22). Also $JK : JL = BC : B'C'$. The lines PX, BC, $Q'Y'$, and $B'C'$ form a quadrilateral circumscribed around the same circle, thus its diagonal XY' also passes through J and $XM : Y'M' = JK : JL$. Using the inverse homothety we obtain the required equality.

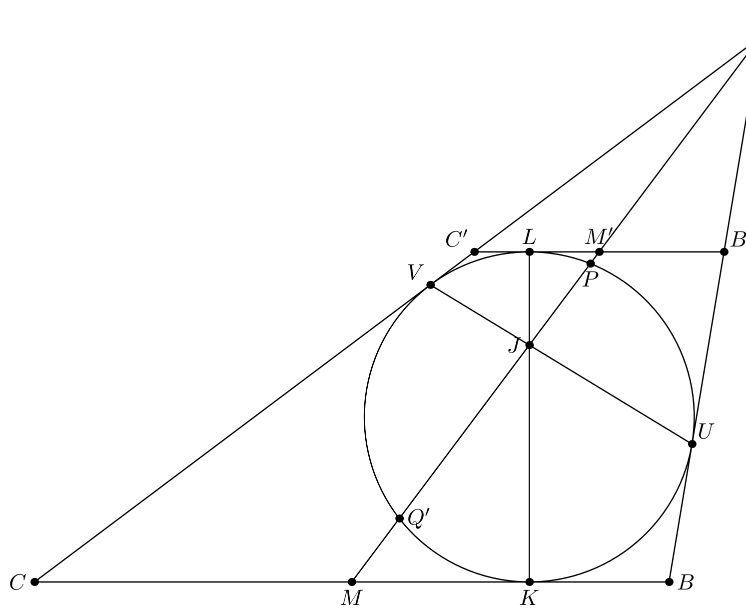

Fig. 22.

23. (A.Mardanov) (10–11) An ellipse Γ_1 with foci at the midpoints of sides AB and AC of a triangle ABC passes through A, and an ellipse Γ_2 with foci at the midpoints of AC and BC passes through C. Prove that the common points of these ellipses and the orthocenter of triangle ABC are collinear.

Solution. Let B_0 be the midpoint of AC. The directrices d_1, d_2 of ellipses Γ_1, Γ_2 corresponding to the focus B_0 are parallel to the altitudes AH, CH, therefore the distances from H to d_1 and d_2 are equal to the distances from A and C respectively to these lines. Since $AB_0 = CB_0$, the ratio of these distances is inverse to the ratio of the excentricities. Since this ratio is the same for the common points of the ellipses, these three points are collinear.

24. (Tran Quang Hung) (11) A tetrahedron $ABCD$ is given. A line ℓ meets the planes ABC, BCD, CDA, DAB at points D_0, A_0, B_0, C_0 respectively. Let P be an arbitrary point not lying on ℓ and the planes of the faces, and A_1, B_1, C_1, D_1 be the second common points of lines PA_0, PB_0, PC_0, PD_0 with the spheres $PBCD$, $PCDA$, $PDAB$, $PABC$ respectively. Prove that P, A_1, B_1, C_1, D_1 lie on a circle.

Solution. Let S be the circumsphere of $ABCD$, and ω be the circumcircle of triangle PA_1B_1. Then BCD is the radical plane of S and the circumsphere of $PBCD$; since A_0 lies on the plane BCD, the powers of A_0 with respect to S and the sphere $PBCD$ are equal, i.e. $A_0P \cdot A_0A_1$. This product equals also to the power of A_0 with respect to the circumcircle ω of triangle PA_1B_1. Thus A_0 lies on the radical axis of S and ω. Similarly we obtain that B_0 lies on the radical axis of S and ω. Therefore this radical axis coincides with ℓ.

Since C_0 lies on the radical plane DAB of S and the circumsphere of $PDAB$, the powers of C_0 with respect to these spheres are equal. Hence PC_0 is the radical axis of ω and the sphere $PDAB$, but PC_0 meets this sphere at C_1, therefore C_1 lies on ω. Similarly D_1 lies on ω. Thus P, A_1, B_1, C_1, D_1 are concyclic.