XIX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XIX Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade (at the start of the correspondence round) have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. **If a problem has an explicit answer, this answer must be presented distinctly**. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The **solutions** for the problems (in Russian or in English) must be **delivered not before December 1, 2022 and not later than on March 1, 2023**. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, press «Registration» in the left upper part, and follow the instructions.

Attention:

1. The solution of each problem (and of each part of it if any) must be contained in a **separate** pdf, doc, docx or jpg file. If the solution is contained in several files then pack them to an **archive** (zip or rar) and load it.
2. We recommend to prepare the paper using computer or to scan it rather than to photograph it. **In all cases, please check readability of the file before uploading**.
3. If you upload the solution of some problem more than once then only the last version is retained in the checking system. **Thus if you need to change something in your solution then you have to upload the whole solution again.**

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round will be held in July–August 2023 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before the round. The cutoff is determined after the examination of the papers of the correspondence round according to the number of participants with any given score. The graduates who are winners of the correspondence round will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** up to June 1, 2023. If you want to know your detailed results, please apply to **geomshar@yandex.ru** after publication of the list.

1. (8) Let L be the midpoint of the minor arc AC of the circumcircle of an acute-angled triangle ABC. A point P is the projection of B to the tangent at L to the circumcircle. Prove that P, L, and the midpoints of sides AB, BC are concyclic.

2. (8) The diagonals of a rectangle $ABCD$ meet at point E. A circle centered at E lies inside the rectangle. Let CF, DG, AH be the tangents to this circle from C, D, A; let CF meet DG at point I, EI meet AD at point J, and AH meet CF at point L. Prove that LJ is perpendicular to AD.

3. (8) A circle touches the lateral sides of a trapezoid $ABCD$ at points B and C, and its center lies on AD. Prove that the diameter of the circle is less than the medial line of the trapezoid.

4. (8) Points D and E lie on the lateral sides AB and BC respectively of an isosceles triangle ABC in such a way that $\angle BED = 3\angle BDE$. Let D' be the reflection of D about AC. Prove that the line $D'E$ passes through the incenter of ABC.

5. (8) Let $ABCD$ be a cyclic quadrilateral. Points E and F lie on the sides AD and CD in such a way that $AE = BC$ and $AB = CF$. Let M be the midpoint of EF. Prove that $\angle AMC = 90°$.

6. (8–9) Let A_1, B_1, C_1 be the feet of altitudes of an acute-angled triangle ABC. The incicrle of triangle $A_1B_1C_1$ touches A_1B_1, A_1C_1, B_1C_1 at points C_2, B_2, A_2 respectively. Prove that the lines AA_2, BB_2, CC_2 concur at a point lying on the Euler line of triangle ABC.

7. (8–9) Let A be a fixed point of a circle ω. Let BC be an arbitrary chord of ω passing through a fixed point P. Prove that the nine-points circles of triangles ABC touch some fixed circle not depending on BC.

8. (8–9) A triangle ABC ($a > b > c$) is given. Its incenter I and the touching points K, N of the incircle with BC and AC respectively are marked. Construct a segment with length $a - c$ using only a ruler and drawing at most three lines.

9. (8–9) It is known that the reflection of the orthocenter of a triangle ABC about its circumcenter lies on BC. Let A_1 be the foot of the altitude from A. Prove that A_1 lies on the circle passing through the midpoints of the altitudes of ABC.

10. (8–9) Altitudes BE and CF of an acute-angled triangle ABC meet at point H. The perpendicular from H to EF meets the line ℓ passing through A and parallel to BC at point P. The bisectors of two angles between ℓ and HP meet BC at points S and T. Prove that the circumcircles of triangles ABC and PST are tangent.

11. (8–10) Let H be the orthocenter of an acute-angled triangle ABC; E, F be points on AB, AC respectively, such that $AEHF$ is a parallelogram; X, Y be the common points of the line EF and the circumcircle ω of triangle ABC; Z be the point of ω opposite to A. Prove that H is the orthocenter of triangle XYZ.

12. Let ABC be a triangle with obtuse angle B, and P, Q lie on AC in such a way that $AP = PB$, $BQ = QC$. The circle BPQ meets the sides AB and BC at points N and M respectively.

(a) (8–9) Prove that the distances from the common point R of PM and NQ to A and C are equal.

(b) (10–11) Let BR meet AC at point S. Prove that $MN \perp OS$, where O is the circumcenter of ABC.

13. (8–11) The base AD of a trapezoid $ABCD$ is twice greater than the base BC, and the angle C equals one and a half of the angle A. The diagonal AC divides angle C into two angles. Which of them is greater?

14. (8–11) Suppose that a closed oriented polygonal line l in the plane does not pass through a point O, and is symmetric with respect to O. Prove that the winding number of l around O is odd.

    The winding number of l around O is defined to be the following sum of the oriented angles divided by 2π:

$$\deg_O l := \frac{\angle A_1OA_2 + \angle A_2OA_3 + \ldots + \angle A_{n-1}OA_n + \angle A_nOA_1}{2\pi}.$$

15. (9–10) Let $ABCD$ be a convex quadrilateral. Points X and Y lie on the extensions beyond D of the sides CD and AD respectively in such a way that $DX = AB$ and $DY = BC$. Similarly points Z and T lie on the extensions beyond B of the sides CB and AB respectively in such a way that $BZ = AD$ and $BT = DC$. Let M_1 be the midpoint of XY, and M_2 be the midpoint of ZT. Prove that the lines DM_1, BM_2, and AC concur.

16. (9–11) Let AH_A and BH_B be the altitudes of a triangle ABC. The line H_AH_B meets the circumcircle of ABC at points P and Q. Let A' be the reflection of A about BC, and B' be the reflection of B about CA. Prove that A', B', P, Q are concyclic.

17. (9–11) A common external tangent to circles ω_1 and ω_2 touches them at points T_1, T_2 respectively. Let A be an arbitrary point on the extension of T_1T_2 beyond T_1, and B be a point on the extension of T_1T_2 beyond T_2 such that $AT_1 = BT_2$. The tangents from A to ω_1 and from B to ω_2 distinct from T_1T_2 meet at point C. Prove that all nagelians of triangles ABC from C have a common point.

18. (9–11) Restore a bicentral quadrilateral $ABCD$ if the midpoints of the arcs AB, BC, CD of its circumcircle are given.

19. (10–11) A cyclic quadrilateral $ABCD$ is given. An arbitrary circle passing through C and D meets AC, BC at points X, Y respectively. Find the locus of common points of circles CAY and CBX.

20. (10–11) Let a point D lie on the median AM of a triangle ABC. The tangents to the circumcircle of triangle BDC at points B and C meet at point K. Prove that DD' is parallel to AK, where D' is isogonally conjugated to D with respect to ABC.

21. (10–11) Let $ABCD$ be a cyclic quadrilateral; M_{ac} be the midpoint of AC; H_d, H_b be the orthocenters of $\triangle ABC, \triangle ADC$ respectively; P_d, P_b be the projections of H_d and H_b to BM_{ac} and DM_{ac} respectively. Define similarly P_a, P_c for the diagonal BD. Prove that P_a, P_b, P_c, P_d are concyclic.

22. (10–11) Let ABC be a scalene triangle, M be the midpoint of BC, P be the common point of AM and the incircle of ABC closest to A, and Q be the common point of the ray AM and the excircle farthest from A. The tangent to the incircle at P meets BC at point X, and the tangent to the excircle at Q meets BC at Y. Prove that $MX = MY$.

23. (10–11) An ellipse Γ_1 with foci at the midpoints of sides AB and AC of a triangle ABC passes through A, and an ellipse Γ_2 with foci at the midpoints of AC and BC passes through C. Prove that the common points of these ellipses and the orthocenter of triangle ABC are collinear.

24. (11) A tetrahedron $ABCD$ is given. A line ℓ meets the planes ABC, BCD, CDA, DAB at points D_0, A_0, B_0, C_0 respectively. Let P be an arbitrary point not lying on ℓ and the planes of the faces, and A_1, B_1, C_1, D_1 be the second common points of lines PA_0, PB_0, PC_0, PD_0 with the spheres $PBCD$, $PCDA$, $PDAB$, $PABC$ respectively. Prove that P, A_1, B_1, C_1, D_1 lie on a circle.