XIX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions
First day. 8 form

1. (A.Zaslavsky) Let ABC be an isosceles obtuse-angled triangle, and D be a point on its base AB such that AD equals to the circumradius of triangle BCD. Find the value of $\angle ACD$.

   Answer. 30° or 150°.

   Solution. Let O be the circumcenter of triangle BCD, M be the midpoint of CD, and H be the projection of D to AC. Then $\angle DOM = \angle DOC/2 = \angle DBC = \angle DAC$ and $DO = DA$ (fig. 8.1). Therefore the triangles DAH and DOM are congruent, i.e. $DH = DM = DC/2$ and $\angle DCH = 30^\circ$. Hence the angle ACD equals 30°, if H lies on segment AC, otherwise it equals 150°.

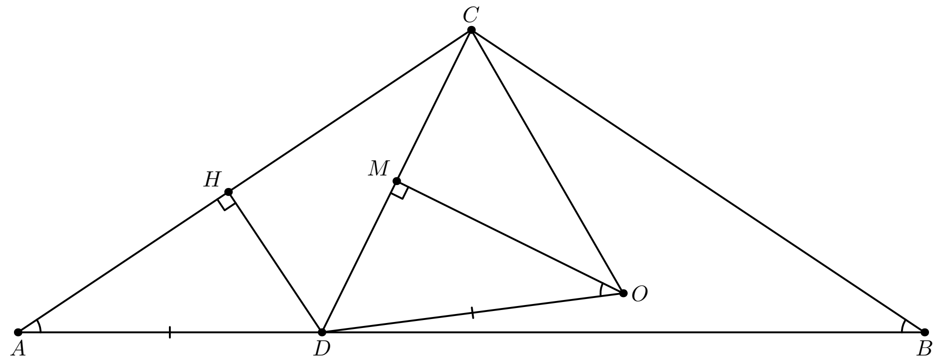

Fig. 8.1

   Remark. The problem may also be solved by the sines law.

2. (A.Teryoshin) The bisectors of angles A, B, and C of triangle ABC meet for the second time its circumcircle at points A_1, B_1, C_1 respectively. Let A_2, B_2, C_2 be the midpoints of segments AA_1, BB_1, CC_1 respectively. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar.

   Solution. Since A_1, B_1, C_1 are the midpoints of arcs BC, CA, AB respectively, the sum of arcs B_1C and A_1C_1 equals a cemicircle, thus CC_1 and A_1B_1 are perpendicular. Hence A_1A, B_1B, C_1C are the altitudes of triangle $A_1B_1C_1$, and their common point I is its orthocenter. The points A_2, B_2, C_2 are the projections of the circumcenter O of ABC to AA_1, BB_1, CC_1, therefore they lie on the circle with diameter OI. Then $\angle B_2A_2C_2 = \angle B_2IC_2 = \angle B_1A_1C_1$

(fig. 8.2). Similarly we obtain that $\angle A_1B_1C_1 = \angle A_2B_2C_2$ and thus the triangles are similar.

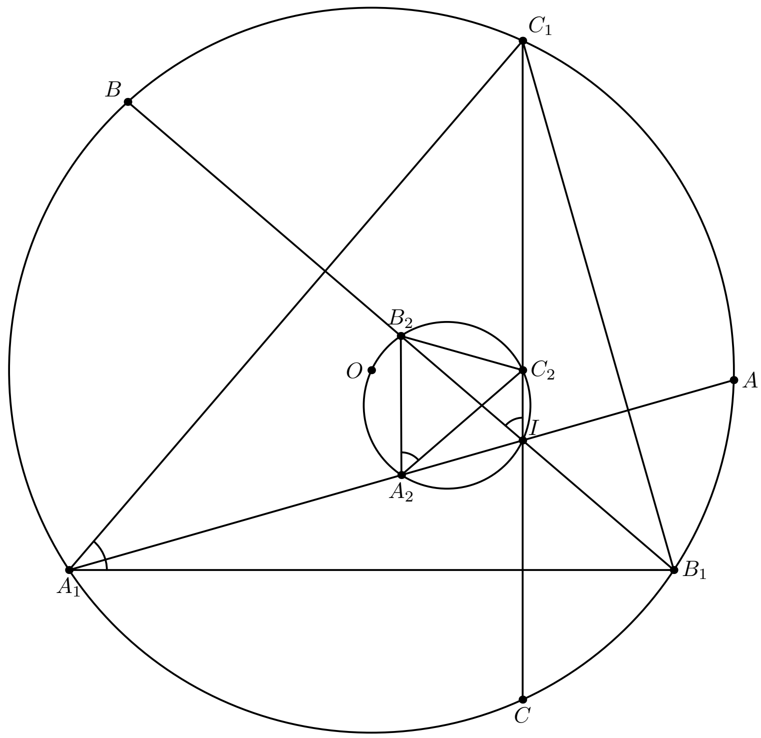

Fig. 8.2

Remark. The required assertion is a partial case of the following fact. If H is the orthocenter of triangle ABC, P is an arbitrary point of the plane, and A', B', C' are the projections of P to AH, BH, CH respectively, then the triangles ABC and $A'B'C'$ are similar.

3. (A.Terteryan) The altitudes of a parallelogram are greater than 1. Does this yield that the unit square may be covered by this parallelogram?

Answer. No.

Solution. Firstly consider how the square may be covered by a strip with width h. Since the projections of diagonals to the perpendicular to the strip can not be greater than h, the angles between the diagonals and this perpendicular have to be sufficiently large. Choose h such that the critical value of this angle equals $44°$, then the angle between the boundary of the strip and some sideline of the square is less than $1°$.

Take now a rhombus with altitude h. If the unit square is covered by this rhombus, then the angles between each sideline of the rhombus and some sideline of the square is less than $1°$. If this sideline is the same for both sideline of the rhombus, then the acute angle of the rhombus is less than 2°. In the opposite case the acute angle is greater than 88°. Therefore the unit square can not be covered by a rhombus with altitude h and acute angle 45°.

Remark. From the solution we see that the square covered by a rhombus with angle 45° may be covered by any parallelogram having greater or equal altitudes. Thus the minimal value of altitude warranting that the unit square can be covered is $\sqrt{2}\sin 67,5^\circ$. The square may be covered by a rhombus with such altitude by two ways (fig. 8.3).

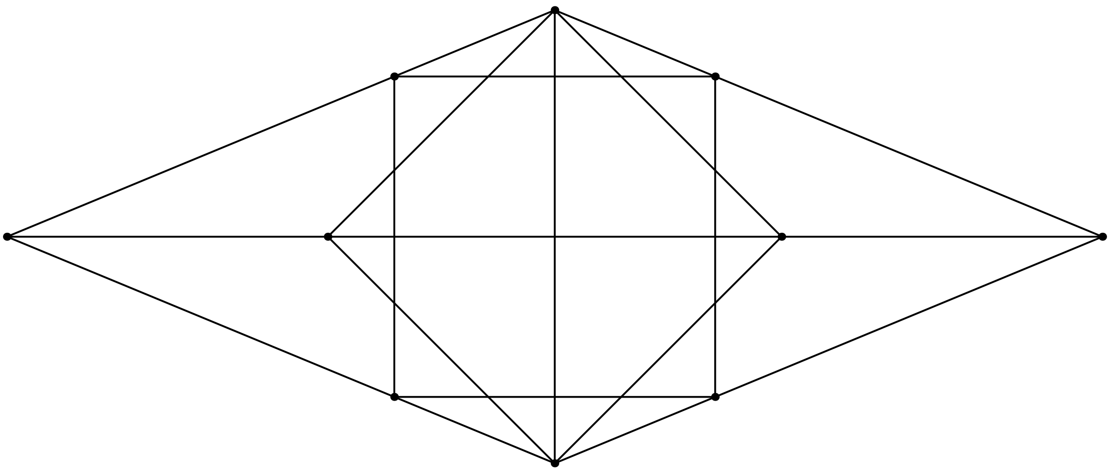

Fig. 8.3

4. (A.Zaslavsky) Let ABC be an acute-angled triangle, O be its circumcenter, BM be a median, and BH be an altitude. Circles AOB and BHC meet for the second time at point E, and circles AHB and BOC meet at point F. Prove that $ME = MF$.

   Solution. Let the extension of BH meet the circumcircle at point D. Prove that E lies also on circles DCO and ADH. In fact let E' be the second common point of circles ABO and DCO. Then $\angle BE'C = 2\pi - \angle BE'O - \angle CE'O = \ BAO + \angle CDO = \pi - (\angle AOB + \angle COD)/2 = \pi/2$, i.e. E' coincide with E. Similarly F lies on circles CHD and AOD.

   Note now that $\angle OEH = 2\pi - \angle OEB - \angle BEH = \angle OAB + \angle BCH = \angle CBH + \angle BCH = \pi/2$, therefore E lies on the circle with diameter OH. Similarly F lies on this circle. It is clear that M also lies on this circle.

   Prove that the reflections of lines AF, BF, CF, DF about the bisectors of angles A, B, C, D respectively meet at E. Let P, Q, R, S be the projections of F to AB, BC, CD, DA respectively. Then $\angle PSR + \angle PQR = \angle FSP + \angle FSR + \angle FQP + \angle FQR = \angle FAB + \angle FRC + \angle FBA + \angle FDC = \pi$, because $\angle AFB = \angle CFD = \pi/2$. Thus P, Q, R, S are concyclic. Then the reflections P', Q', R', S', of F about AB, BC, CD, DA are also concyclic. Since, for example, $AP' = AF = AS'$, the perpendicular bisector to $P'S'$ coincide with the bisector of angle $P'AS'$, which is symmetric to AF about the bisector of angle A. Hence the reflections of AF, BF, CF, DF about the corresponding bisectors meet at the circumcenter E'' of $P'Q'R'S'$. It is easy to see that the angles $BE''C$ and $AE''D$ are right, i.e. E'' coincide with E.

Finally we obtain that $\angle EHM = \angle EBC = \angle FCA = \angle FHM$ and since E, F, H, M are concyclic, $EM = FM$ (fig. 8.4).

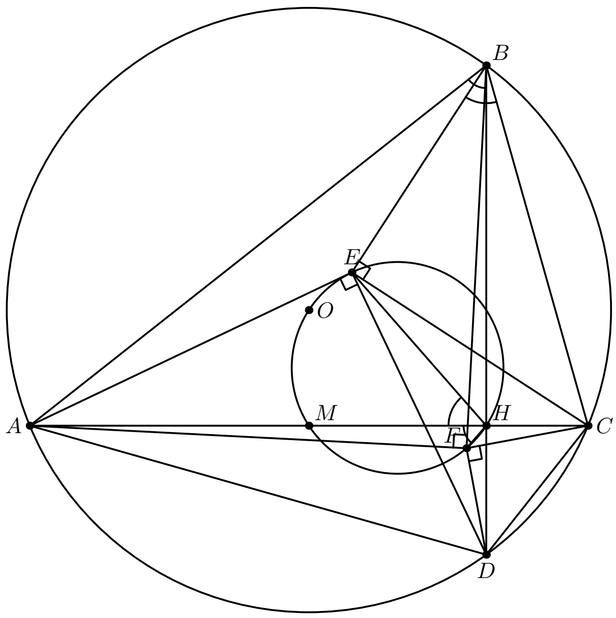

Fig. 8.4

Remark. Since the lines joining E and F with the vertices of $ABCD$ are symmetric about the bisectors of its angles, these points are the foci of an inellipse. Also they are symmetric about the line joining the midpoints of AC and BD.

XIX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions

Second day. 8 form

5. (L.Popov) The median CM and the altitude AH of an acute-angled triangle ABC meet at point O. A point D lies outside the triangle in such a way that $AOCD$ is a parallelogram. Find the length of BD, if $MO = a$, $OC = b$.

   Answer. $2a + b$.

   Solution. Let K be a point on the ray CM such that $CM = MK$. Then $CAKB$ is a parallelogram, i.e. $AK = BC$ and $AK \parallel BC$. Also $AO = CD$ and $\angle BCD = \angle OAK = 90^\circ$ because AH is the altitude (fig. 8.5). Therefore the triangles BCD and KAO are congruent, i.e. $BD = OK = 2CM - CO = 2a + b$.

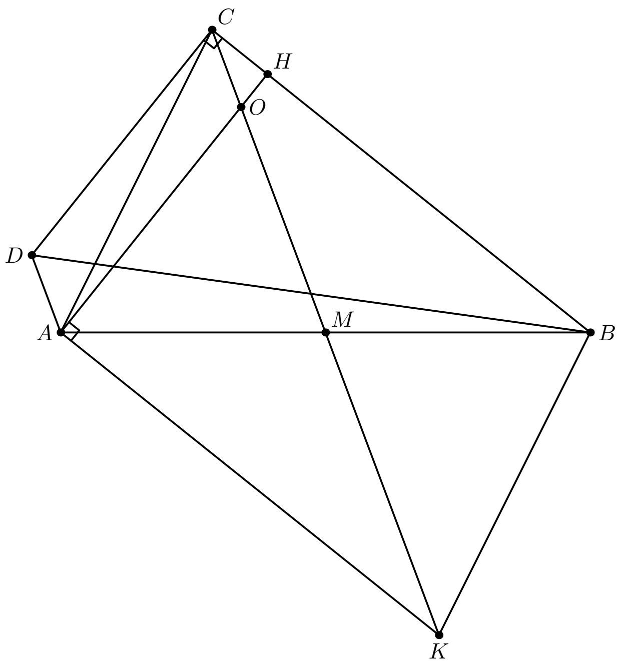

Fig. 8.5

6. (F.Nilov) For which n the plane may be paved by congruent figures bounded by n arcs of circles?

   Answer. $n > 2$.

   Solution. Take a square $ABCD$ and replace the sides AB, AD by equal arcs directed outside it, and replace the sides BC, CD by the same arcs directed inside the square. It is clear that the plane may be paved by the obtained figures bounded by four arcs. Also the plane may be paved by the strip composed from k such figures, and this strip is bounded by $2k+2$ arcs. Finally we can choose the radius of arcs such that the arcs AB and AD form a cemicircle (fig. 8.6). The obtained figure is bounded by three arcs. Composing a strip from k figures we obtain a figure bounded by $2k+1$ arcs.

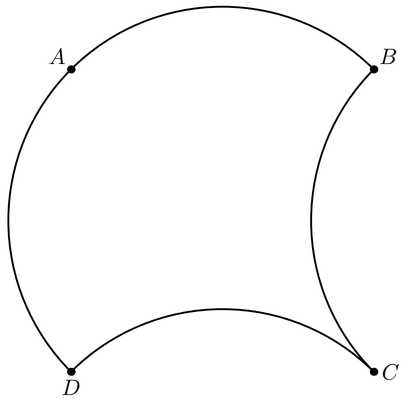

Fig. 8.6

If $n = 2$, then the figure is a crescent and its external arc is longer than the internal one. Hence there exists a point on the external arc belonging to two other crescents. It is clear that the angle formed by their external arcs can not be paved.

7. (G.Filippovsky) The bisector of angle A of triangle ABC meet its circumcircle ω at point W. The circle s with diameter AH (H is the orthocenter of ABC) meets ω for the second time at point P. Restore the triangle ABC if the points A, P, W are given.

   Solution. By the points A, P, W restore ω, its center O and the point A' opposite to A. Since $\angle APA' = \angle APH = 90^\circ$, H lies on PA'. Since $\angle ABA' = \angle ACA' = 90^\circ$, the quadrilateral $HBA'C$ is a parallelogram, i.e. H and A' are symmetric with respect to the midpoint M of BC. Hence we can restore M, as the common point of PA' and OW, finally draw a perpendicular through M to OW and find its common points B, C with ω (fig. 8.7).

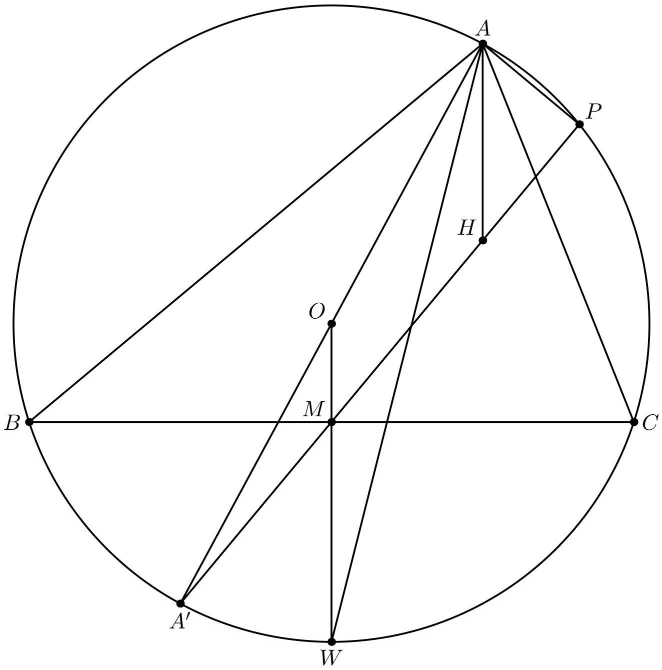

Fig. 8.7

8. (D.Dyomin, I.Kykharchyk) Two circles ω_1 and ω_2 meeting at point A and a line a are given. Let BC be an arbitrary chord of ω_2 parallel to a, and E, F be the second common points of AB and AC respectively with ω_1. Find the locus of common points of lines BC and EF.

Answer. Let X_1X_2 be the diameter of ω_2 perpendicular to a, and Y_1, Y_2 be the second common point of AX_1, AX_2 respectively with ω_1. Then the required locus is the interval bounded by the common points of the tangents to ω_2 at X_1, X_2 and the tangents to ω_1 at Y_1, Y_2 respectively.

Solution. Let B_1C_1, B_2C_2 be two dispositions of BC, and E_1F_1, E_2F_2 be two corresponding dispositions of EF. Since the arcs B_1B_2 and C_1C_2 are equal, the arcs E_1F_1 and E_2F_2 are also equal, i.e. $E_1F_1 \parallel E_2F_2$. Also the chords B_iC_i and E_iF_i bound the arcs with the same angular measure (fig. 8.8). Therefore when the line BC moves uniformly, EF also moves uniformly, and their common point moves on the line. Clearly the boundary dispositions of this points are the common points of the tangents.

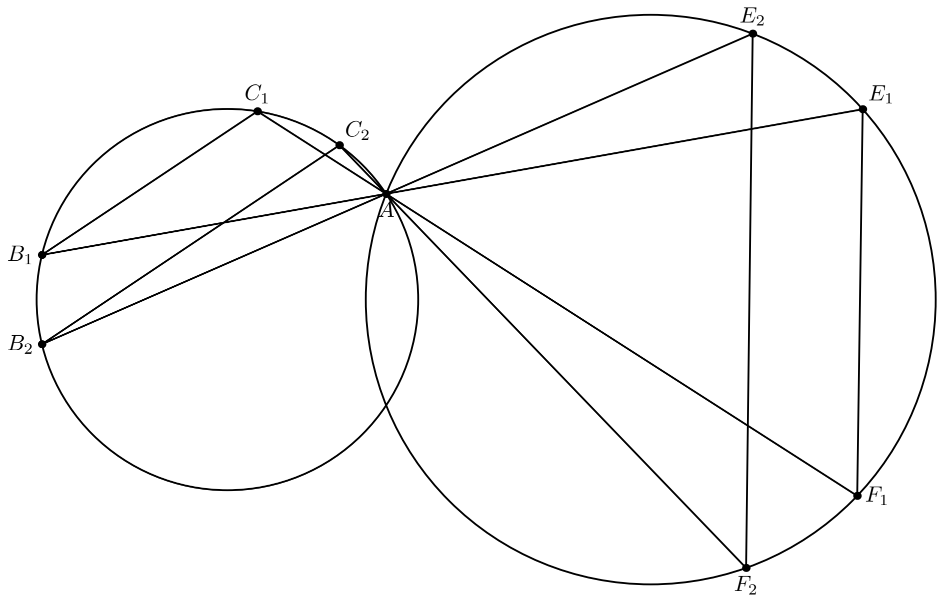

Fig. 8.8

XIX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions
First day. 9 form

1. (E.Bakaev) The ratio of the medianAM of a triangle ABC to the side BC equals $\sqrt{3} : 2$. The points on the sides of ABC dividing these side into 3 equal parts are marked. Prove that some 4 of these 6 points are concyclic.

   First solution. using the median formula we obtain $AM^2 = (2b^2 + 2c^2 - a^2)/4 = 3a^2/4$, i.e. $b^2 + c^2 = 2a^2$. Then the square of the median from B equals $(2a^2 + 2c^2 - b^2)/4 = 3c^2/4$, similarly the square of the median from C equals $3b^2/4$. Therefore the triangle formed by the medians is similar to ABC.

   Now let A_1, A_2 lie on BC, B_1, B_2lie on CA, and C_1, C_2 lie on AB in such a way that $BA_1 = A_1A_2 = A_2C$, $CB_1 = B_1B_2 = B_2A$, $AC_1 = C_1C_2 = C_2B$. Then, for example, the median of triangle BC_1A_2 $C_1A_1 = 2AM/3$, i.e. the triangle $A_1B_1C_1$ is similar to the triangle formed by the medians and the triangle ABC. Hence $\angle A_1B_1C_1 = \angle A = \angle A_1C_2B$ and the circle $A_1B_1C_1$ passes through C_2 (fig. 9.1).

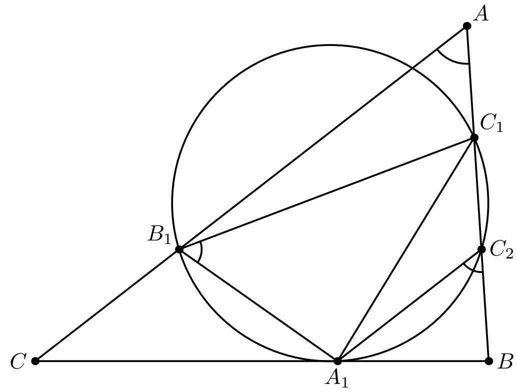

Fig. 9.1

   Second solution. Let segments B_1C_2 and A_1C_1 meet at point O. Prove that these segments are diagonals of a cyclic quadrilateral. Find in which ratios they divide one other: $C_1O = A_1O = x$, $B_1O = 3y$, $C_2O = y$. Also $A_1C_1 = \frac{2}{3}AM$, $B_1C_2 = \frac{2}{3}BC$, thus $A_1C_1 : B_1C_2 = AM : BC = \frac{\sqrt{3}}{2}$. Therefore $\frac{2x}{4y} = \frac{\sqrt{3}}{2}$, i.e. $x^2 = 3y^2$, hence $B_1O \cdot C_2O = C_1O \cdot A_1O$. Thus the quadrilateral $B_1C_1C_2A_1$ is cyclic.

2. (A.Yuran) Can a regular triangle be placed inside a regular hexagon in such a way that all vertices of the triangle were seen from each vertex of the hexagon? (*Point A is seen from B, if the segment AB dots not contain internal points of the triangle.*)

   Answer. No.

   Solution. All points such that the vertices of triangle XYZ are seen from them lie inside three angles vertical to the angles of the triangle. If each of these angles contains exactly two vertices of the hexagon, then its main diagonals can not be concurrent. In the other case two non-adjacent vertices of the hexagon lie on the same angle, for example non-adjacent vertices A and B lie on the angle vertical to the angle X. Then $\angle AXB \leq 60^\circ$, and the arc AXB lies outside the hexagon.

3. (P.Bibikov) Points A_1, A_2, B_1, B_2 lie on the circumcircle of a triangle ABC in such a way that $A_1B_1 \parallel AB$, $A_1A_2 \parallel BC$, $B_1B_2 \parallel AC$. The line AA_2 and CA_1 meet at point A', and the lines BB_2 and CB_1 meet at point B'. Prove that all lines $A'B'$ concur.

   First solution. Let CA_1, CB_1 meet AB at points X, Y respectively. Since the arcs CA_2, BA_1, AB_1, and CB_2 are equal, we obtain that $AA' \parallel B'Y$, $BB' \parallel A'X$, and the triangles $AA'X$ and $YB'B$ are homothetic. Their homothety center Z lies on the line AB and satisfies to $ZX \cdot ZY = ZA \cdot ZB$. Since $\angle ACX = \angle BCY$, the circles ABC and CXY are tangent. Therefore Z lies on their common tangent and do not depend on A', B' (fig.9.3).

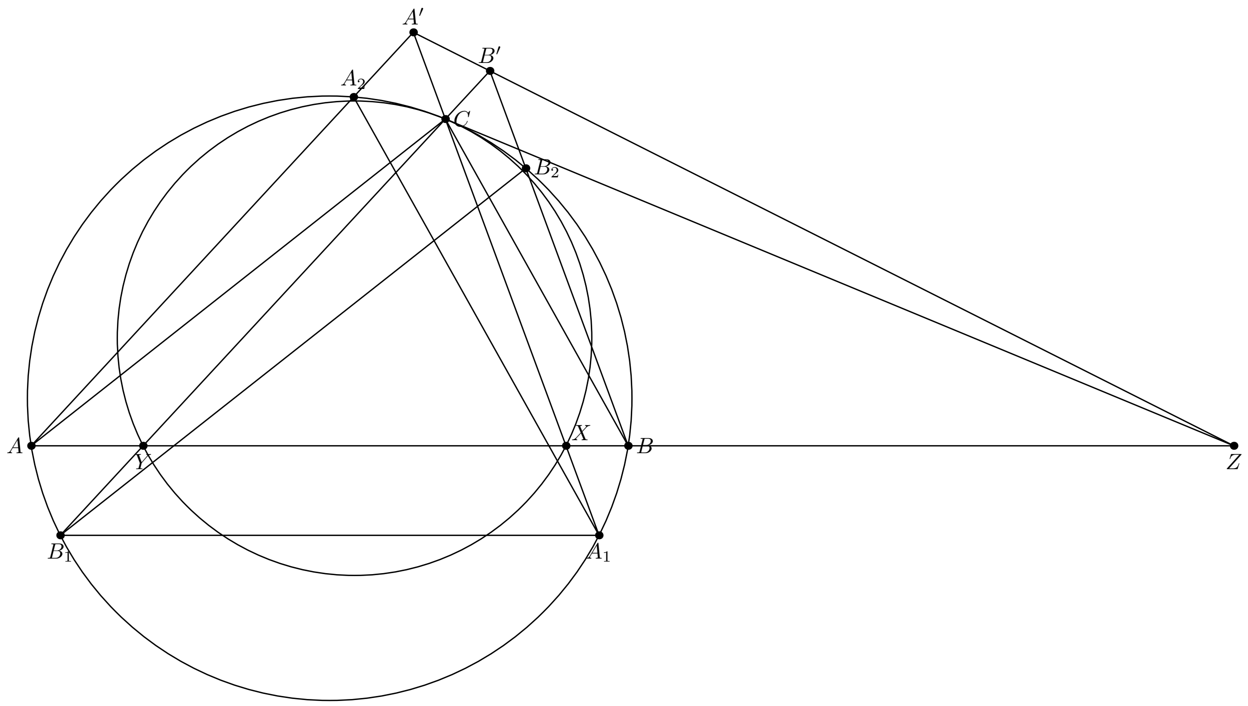

Fig. 9.3

Second solution (sketch). Since the correspondence between A_1 and A_2 is projective, the locus of A' is some conic passing through A and C. When A_1 lies on the internal or the external bisector of angle C, the lines AA_2 and CA_1 are parallel, therefore this conic is an equilateral hyperbola with asymptotes parallel to these bisectors. Similarly the locus of B' is an equilateral hyperbola passing through B and C with asymptotes parallel to the bisectors. The correspondence between A' and B' is also projective, and both points coincide with C and infinite points at the same time. Therefore all lines $A'B'$ concur at the fourth common point of two hyperbolas.

4. (G.Galyapin) The incircle ω of a triangle ABC centered at I touches BC at point D. Let P be the projection of the orthocenter of ABC to the median from A. Prove that the circle AIP and ω cut off equal chords on AD.

   Solution. Let M be the midpoint of BC, N be the midpoint of AD, E be the second common point of AD and ω, and F b the common point of MI with the circle DIE. It is known that the radii of circles BCP and ABC are equal, hence $MP{\cdot}MA = MB^2$. Also M, I, N are collinear (on the Gauss line of the degenerated quadrilateral $ABDC$). Finally the quadrilateral $DB'EC'$ is harmonic, where B' and C' are the touching points of ω with AC and AB, hence the tangent to ω at E passes through the point $Z = B'C' \cap BC$, forming a harmonic quadruple with B, C, D. Then each circle passing through D and

Z is orthogonal to the circle with diameter BC, partially we have this for the circle DIE (with diameter IZ). Therefore $MI \cdot MF = MB^2 = MP \cdot MA$, i.e. $AFIP$ is a cyclic quadrilateral (fig. 9.4). Then the degrees of N with respect to ω and (AIP) are equal which yields the required equality.

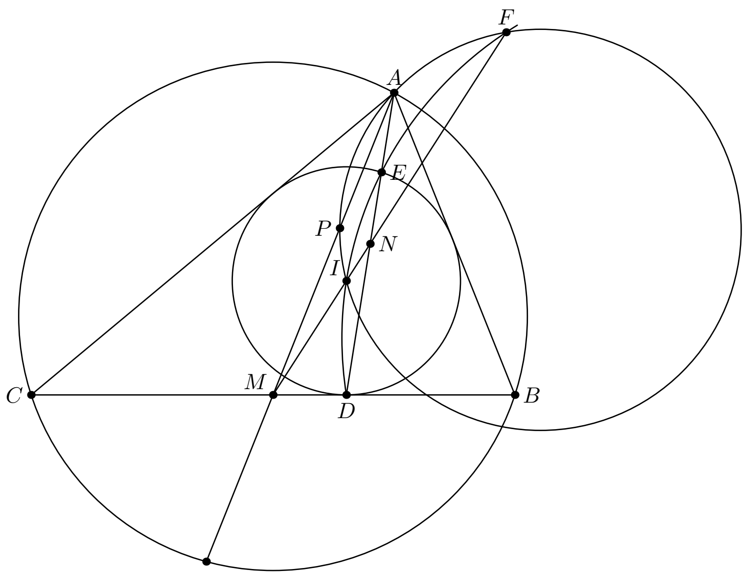

Fig. 9.4

Remark. Point F is the inversion image of N and I with respect to the circle with diameter BC and ω respectively.

XIX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions

Second day. 9 form

5. (A.Mardanov) A point D lie on the lateral side BC of an isosceles triangle ABC. The ray AD meets the line passing through B and parallel to the base AC at point E. Prove that the tangent to the circumcicle of triangle ABD at B bisects EC.

   Solution. Let M be the common point of the tangent with CE. Then $\angle CBM = \angle DAB$, thus $\angle MBE = \angle CAD$. On the other hand $BC : BE = (BC : AC)(AC : BE) = (AB : AC)(CD : BD) = \sin\angle DAC : \sin\angle DAB$. Therefore BM is a median of triangle BCE (fig.9.5).

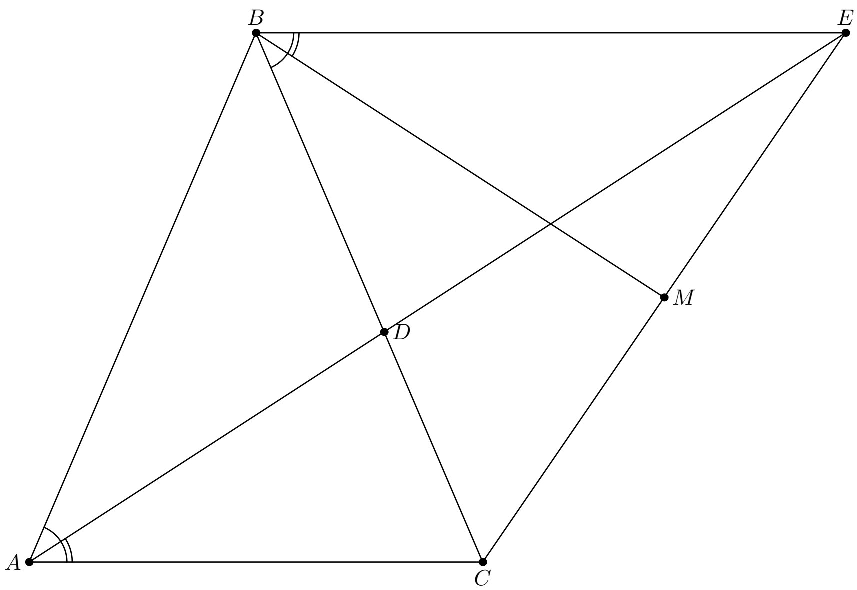

Fig. 9.5

6. (G.Zabaznov) Let ABC be an acute-angled triangle with circumcircle Ω. Points H and M are the orthocenter and the midpoint of BC respectively. The line HM meets the circumcircle ω of triangle BHC at point $N \neq H$. Point P lies on the arc BC of ω not containing H in such a way that $\angle HMP = 90°$. The segment PM meets Ω at point Q. Points B' and C' are the reflections of A about B and C respectively. Prove that the circumcircles of triangles $AB'C'$ and PQN are tangent.

   Solution. Let T be the point of Ω opposite to A (T is the circumcenter of triangle $AB'C'$ and lie on MH), and Q' be the reflection of A about Q.

Since the circle Ω and ω are symmetric with respect to M, $MQ \cdot MP = MH \cdot MN = MT \cdot MN$, i.e. T lies on the circle PQN. Also the triangles MQN and MHP (congruent to MTP) are similar, thus $\angle NHP = \angle NQM$ and the radii of circles PQN and ω are equal. Therefore the circle PQN is the reflection of Ω about QT and touches the circle $AB'C'$ at Q' (fig. 9.6).

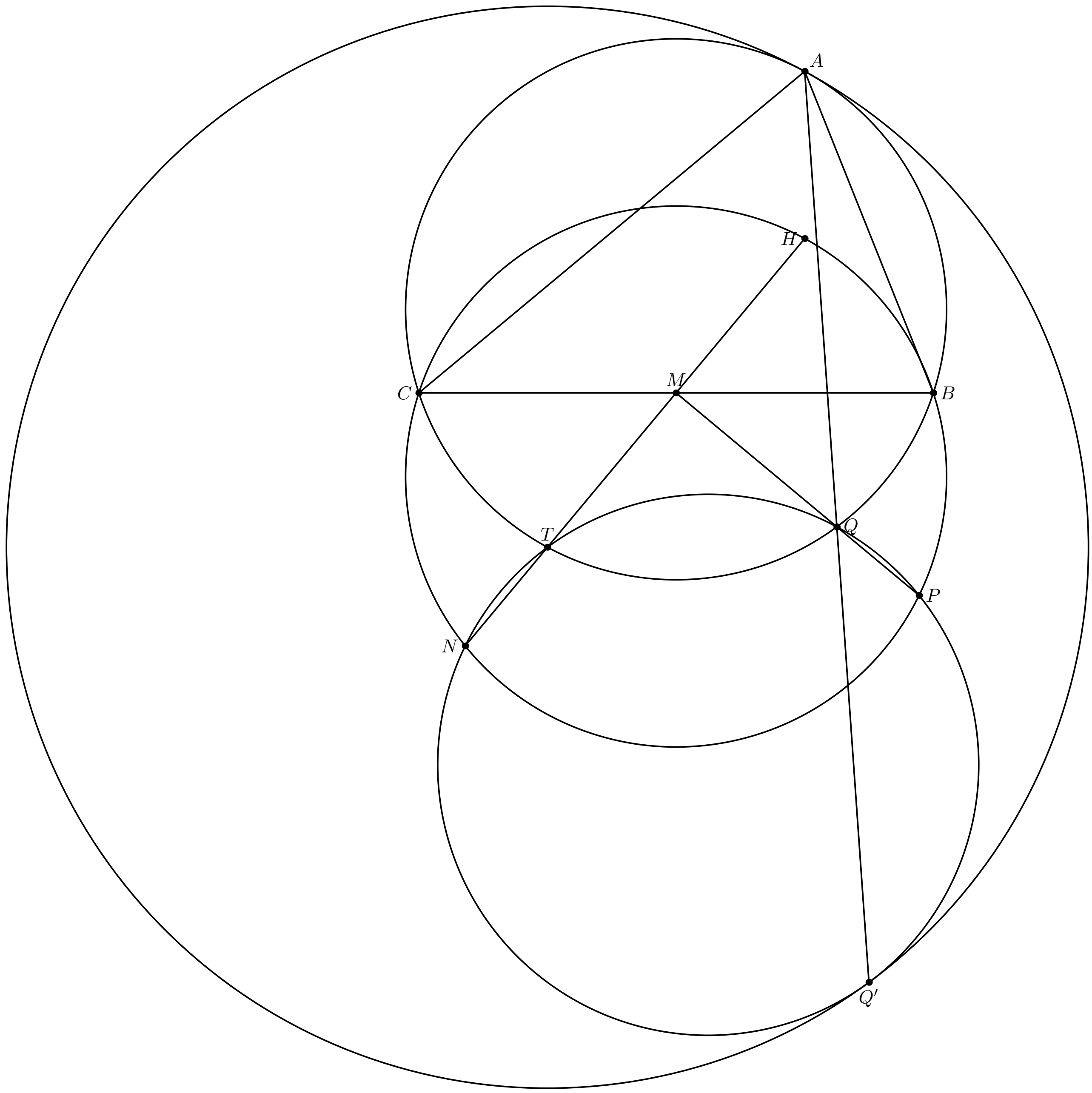

Fig. 9.6

7. (F.Bakharev) Let H be the orthocenter of triangle T. The sidelines of triangle T_1 pass through the midpoints of T and are perpendicular to the corresponding

bisectors of T. The vertices of triangle T_2 bisect the bisectors of T. Prove that the lines joining H with the vertices of T_1 are perpendicular to the sidelines of T_2.

Solution. Prove that the lines joining H with the vertices of T_1 are the radical axes of circles having the bisectors of T as diameters. This yields the required assertion. Since these radical axes are perpendicular to the lines joining the centers, it is sufficient to prove that the vertices of T_1 have the same degrees with respect to the corresponding pairs of circles. We prove that the vertex of T_1 is the radical center of two circles having bisectors as diameters and the incircle of T. Since the sideline of T_1 is perpendicular to the bisector of T (i.e the centers line of he incircle and the circle having this bisector as diameter), it is sufficient to prove that the degrees of the midpoint of the side with respect to these circles are equal.

Denote the vertices of T by A, B, and C, the midpoint of BC by M, the foot of the corresponding bisector by L, the foot of the altitude by D, and the touching point with the incircle by T. We have to prove that $MT^2 = ML \cdot MD$. If the sidelengths are a, b и c, then $MT = |b-c|/2$, $ML = a|b-c|/2(b+c)$, $MD = |b^2-c^2|/2a$, which yields the required equality.

8. (M.Didin, I.Frolov) Let ABC be a triangle with $\angle A = 120^\circ$, I be the incenter, and M be the midpoint of BC. The line passing through M and parallel to AI meets the circle with diameter BC at points E and F (A and E lie on the same semiplane with respect to BC). The line passing through E and perpendicular to FI meets AB and AC at points P and Q respectively. Find the value of $\angle PIQ$.

   Answer. 90°.

   Solution. Note that the circumcircle of ABC forms equal angles with circles BCI and BCE. Therefore the inversion about the circumcircle transposes these circles, and the circumcenter O of ABC is their internal homothety center. Since AI passes through the center of circle BIC and is parallel to EF, we obtain that O lies on FI.

   Let P', Q' be the reflections of C and B about BI and CI respectively, E' and M be the midpoints of $P'Q'$ and BC respectively. Then BIQ', CIP' are regular triangles, and the vector $E'M$ equals a semisum of vectors $Q'B$ and $P'C$. Since the angle between these vectors equals $30^\circ = \pi - \angle BIC$, $E'M = BC/2$. Also $E'M$ and the altitude of triangle BIC form equal angles with the bisector of angle BIC, hence $E'M \parallel AI$, and E' coincide with c E.

Now use next

Lemma. Let the sides of triangle XYZ be the bases of isosceles triangles XYZ', YZX', ZXY' lying outside XYZ and such that $\angle X'ZY = \angle Y'ZX = \pi/2 - \angle Z'XY$. Then $ZZ' \perp X'Y'$.

Proof. Let X'', Y'' be the reflections of Z about X, Y, and P be the projection of Z to $X''Y''$. Then $ZPX''Y$, $ZPY''X$ are cyclic quadrilaterals, therefore $\angle YPX'' = \angle XPY'' = \angle X'ZY$ and the bisector ZP of angle XPY meets the perpendicular bisector to XY at point Z'.

□

Applying the lemma to triangles BIQ', CIP', and BOC we obtain that $OI \perp P'Q'$ (fig. 9.8). Hence P', Q' coincide with P, Q, and $\angle PIQ = 90°$.

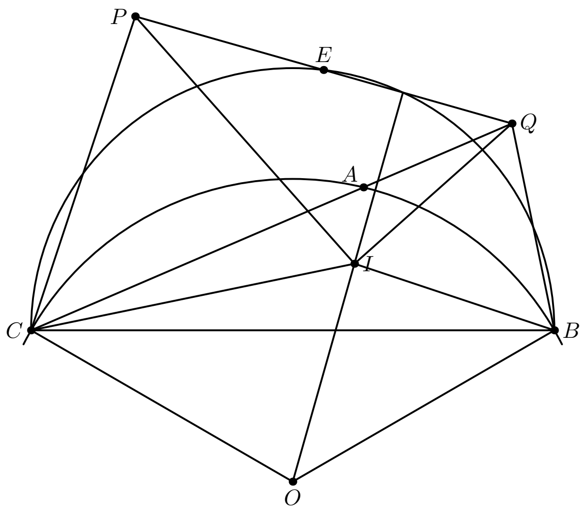

Fig. 9.8

Remark. The lemma was proposed as a problem on XXX Tournament of Towns.

XIX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions
First day. 10 form

1. (A.Mardanov) Let M be the midpoint of cathetus AB of triangle ABC with right angle A. Point D lies on the median AN of triangle AMC in such a way that the angles ACD and BCM are equal. Prove that the angle DBC is also equal to these angles.

   First solution. Since CM is a median, $AC : BC = \sin\angle MCB : \sin\angle MCA = \sin\angle ACD : \sin\angle DAC = AD : CD$, i.e. $AC : AD = BC : CD$. Also $\angle CAD = \angle ACM = \angle BCD$. Therefore the triangles ACD and BDC are similar, and $\angle DBC = \angle ACD$ (fig. 10.1).

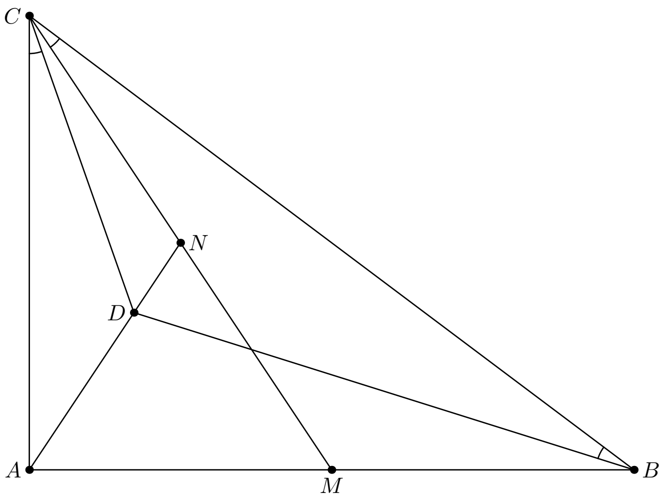

Fig. 10.1

   Second solution. A point D' isogonally conjugated to D with respect to ABC is the projection of A to CM. Hence $MB^2 = MA^2 = MD' \cdot MC$, the triangles BMC and $D'MB$ are similar, and $\angle DBC = \angle D'BM = \angle BCM$.

   Remark. the point D' is the projection of the orthocenter of ABC to the median, i.e. a Humpty point. Thus D is a Dumpty point.

2. (M.Plotnikov, B.Frenkin) The Euler line of a scalene triangle touches its incircle. Prove that this triangle is obtuse-angled.

   First solution. Let H be the orthocenter of triangle ABC, O be its circumcenter, I be the incenter, and A', B', C' be the touching points of the incircle with BC, CA, AB respectively. Suppose that ABC is not an obtuse-angled triangle, and the touching point of OH with the incircle lies on the arc $B'C'$. The O, H lie inside or on the boundary of triangle ABC, thus they lie inside the quadrilateral $IB'AC'$. therefore the projections of O and H to AB lie on the segment AC'. But the touching point of the incircle with any side lies between the midpoint of this side and the foot of the corresponding altitude — a contradiction.

Second solution (sketch). Use the following fact: The Euler line of an acute-angled scalene triangle intersects the longest and the shortest sides, and the Euler line of an obtuse-angled triangle intersects two major sides. Let the Euler line of a triangle divides it into a triangle and a quadrilateral. Prove that I lies inside the quadrilateral if and only if the original triangle is obtuse-angled.

Fix the circumcircle and the incircle of a triangle and "rotate" the triangle between them. The form of a part containing I may change when I lies on the Euler line, or when this line passes through one of vertices of the triangle. But in the first case the triangle is isosceles, i.e. one vertex lie on the line OI, and when the triangle passes through this position the configuration changes to the symmetric one. Therefore the form of the part does not change. In the second case the Euler line is a median of a right-angled triangle, and I lies in the part containing its minor cathetus. For an acute-angled triangle this part is also a triangle, and for an obtuse-angled triangle it is a quadrilateral.

3. (M.Didin, I.Frolov) Let ω be the circumcircle of triangle ABC O be its center, A' be the point of ω opposite to A, and D be a point on a minor arc BC of ω. A point D' is the reflection of D about BC. The line $A'D'$ meets ω for the second time at point E. The perpendicular bisector to $D'E$ meets AB and AC at points F and G respectively. Prove that $\angle FOG = 180° - 2\angle BAC$.

   First solution. Let the line passing through D' and perpendicular to $A'D'$ meet AB and AC at points F' and G' respectively. Since $\angle AEA' = 90°$, we have $AF = FF'$, $AG = GG'$, and $\angle FOG = \angle F'A'G'$. Since $\angle ABA' = \angle ACA' = 90°$, $A'BF'D'$ and $A'G'CD'$ are cyclic quadrilaterals, therefore $\angle F'A'G' = \angle F'A'D' + \angle D'A'G' = \angle ABD' + \angle D'CA = \angle CD'B - \angle CAB = 180° - 2\angle CAB$. (fig. 10.3).

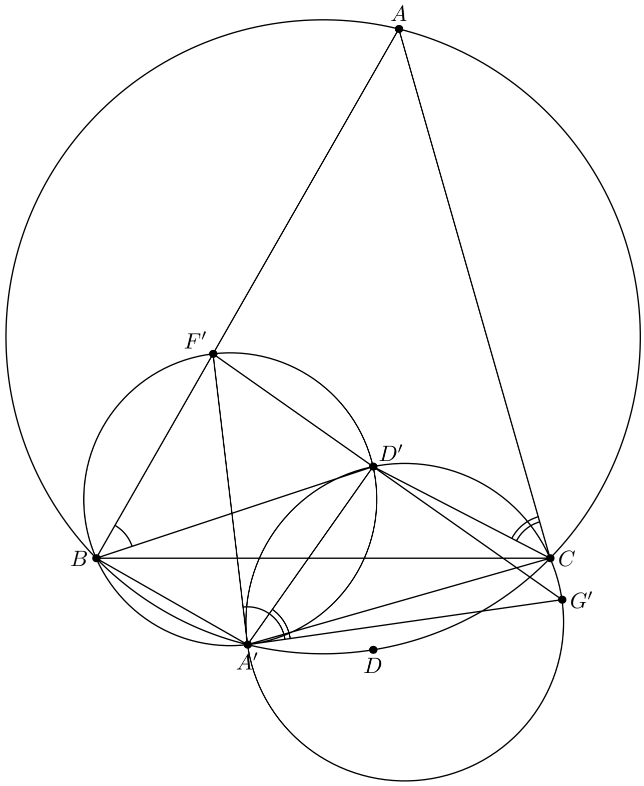

Fig. 10.3

Second solution. The equality $\angle FOG + \angle BOC = \pi$ is equivalent to the existence of point isogonally conjugated to O with respect to the quadrilateral $BFGC$, which means that the projections of O to the sidelines are concyclic. Since $FG \parallel AE$, the projection of O to FG lies on the perpendicular bisector to AE, i.e. coincide with the midpoint of AD'. But D' lies on the circle BCH (H is the orthocenter of ABC), and the homothety centered at A with coefficient $1/2$ maps his circle to the nine-points-circle of ABC.

Remark. Since O and H are isogonally conjugated with respect to ABC, they are the foci of an inellipse. The line FG also touches this ellipse, hence the projection of H to FG lies on the nine-points-circle, and $\angle FHG = \angle BAC$.

4. (D.Reznik, A.Zaslavsky, D.Brodsky) Let ABC be a Poncelet triangle, A_1 is the reflection of A about the incenter I, A_2 is isogonally conjugated to A_1 with respect to ABC. Find the locus of points A_2.

   Answer. The radical axis of I and the circumcircle of ABC.

   First solution. Let P be the common point of BA_2 and the circumcircle of ABC, N be the midpoint of the minor arc AC, N_1 be the reflection of N about I, and S be the midpoint of the minor arc BC. Then the quadrilateral

N_1BA_1S is cyclic. Let R be the common point of N_1S and (ABC). It is easy to see that the arcs RP and AN are equal.

Denote as Q the common point of the circle (SBA_1) and BC. Then we have $\angle QSR = \angle NBC = \smile NC/2 = \smile RP/2 = \angle RSP$, therefore P, Q, and S are collinear. Thus $\angle IN_1S = \angle BQP = (\smile CS + \smile BP)/2 = \smile SP/2 = \angle N_1RP$. Also we have $N_1I = NI = NA = PR$, hence N_1PIR is an isosceles trapezoid, and $PI \parallel N_1S$. Finally $\angle PIA = \angle N_1SA = \angle IBA_1 = \angle IBP$, i.e the circle IBP touches IA_2, and A_2 lies on the radical axis of I and (ABC) (Fig/ 10.4).

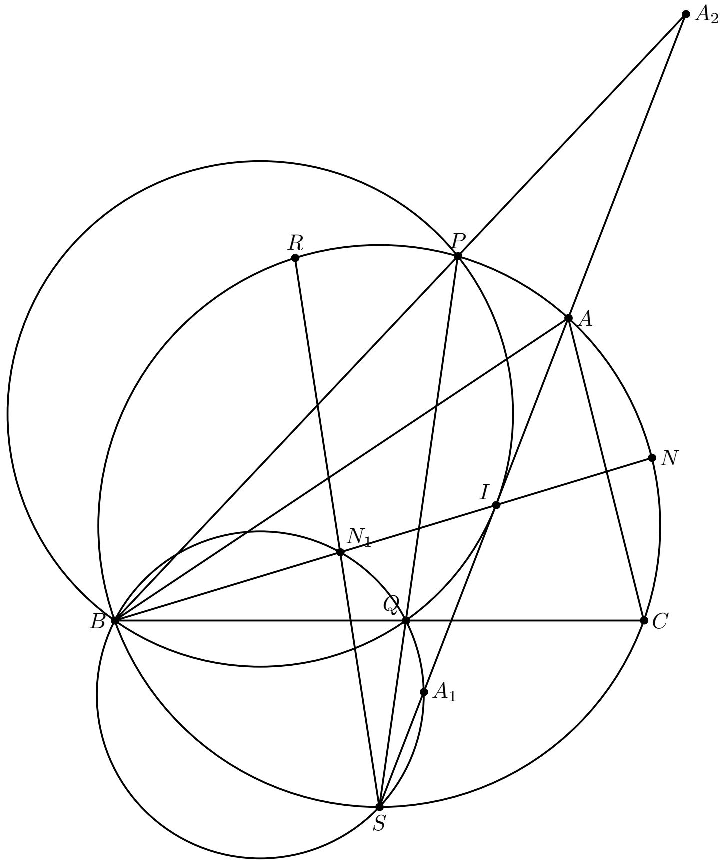

Fig. 10.4

Second solution. Note that A_1 is the inverse map of A_2 with respect to circle BIC. In fact the isogonal conjugation and the inversion give projective maps on the line AI, these maps conserve the incenter and the excenter of triangle ABC, and transform the midpoint S of arc BC to the infinite point.

Let $SI = 1$, $SA_1 = x$. Then $SA = 2 - x$, $SA_2 = 1/x$, and the degree of A_2 with respect to the circumcircle of ABC equals $A_2A{\cdot}A_2S = (1/x{-}2{+}x)/x = (1/x - 1)^2 = A_2I^2$.

Third solution. (S.Shestakov) Let B', C' be the reflections of A_1 about BI, CI respectively. Then $IA = IB_1 = IC_1$. Also, by the trident theorem $SI = SB = SC$, thus $\angle BIB' = \angle BIS = \angle IBS$ and $\angle AIB' = \angle ASB$. Therefore $IB' \parallel SB$. Similarly $IC' \parallel SC$, i.e. quadrilaterals $SBIC$ and $IB'AC'$ are homothetic. Their homothety center coincide with A_2, because the lines BB' and BA_1 are symmetric with respect to AI. Hence $A_2A : A_2I = A_2I : A_2S$ and $A_2I^2 = A_2A \cdot A_2S$.

Remark. The assertion of the problem is a partial case of next fact: if ℓ is a fixed line then the isogonal maps of ℓ with respect to triangles ABC are the conics twice touching two fixed circles. In the considered case these two circles are concentric at I.

XIX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions

Second day. 10 form

5. (A.Teryoshin) The incircle of a triangle ABC touches BC at point D. Let M be the midpoint of arc BAC of the circumcircle, and P, Q be the projections of M to the external bisectors of angles B and C respectively. Prove that the line PQ bisects AD.

   First solution. Let I_c, I_b be the centers of the excircles touching the sides AB, AC respectively. Since M bisects I_aI_b and $MP \parallel BI_b$, we obtain that P bisects BI_c. Hence if P', P'' are the projections of P to BC and AB respectively, we have $P''B = P'B = (p-a)/2$. Since $AB - BD = p - a$, $P'D = P'B + BD = AB - P''B = P''A$. Therefore $PD = PA$. Similarly $QD = QA$, i.e. PQ is the perpendicular bisector to AD (fig. 10.5).

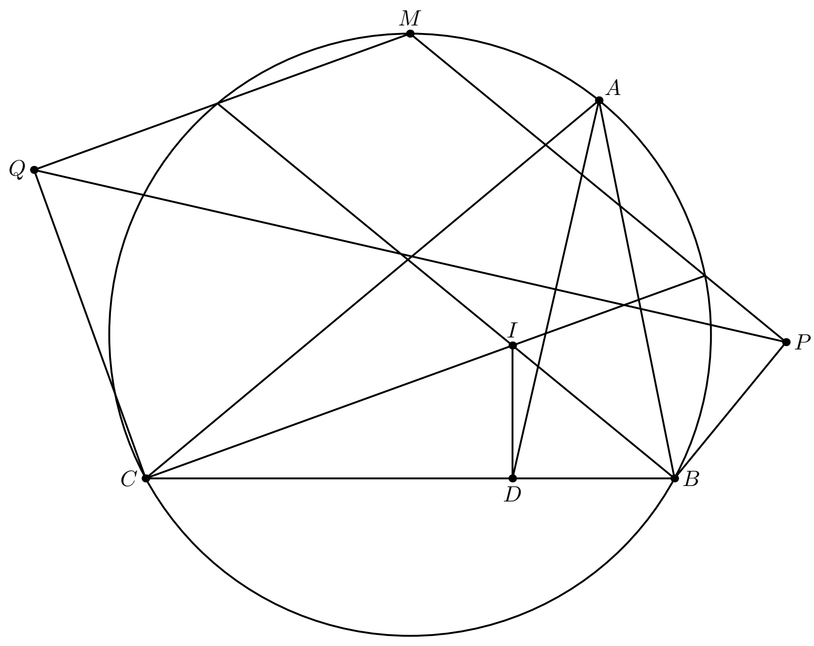

Fig. 10.5

   Second solution. Let a point X move uniformly from B to C, and Y move uniformly from I_c to I_b. Then the midpoint of XY also moves uniformly on PQ. Since IA and ID are the altitudes of similar triangles II_cI_b and IBC, X and Y coincide with D and A respectively at the same time. Therefore the midpoint of AD lies on PQ.

6. (Tran Quang Hung) Let E be the projection of the vertex C of a rectangle $ABCD$ to the diagonal BD. Prove that the common external tangents to the circles AEB and AED meet on the circle AEC.

First solution. Let ω_1 and ω_2 be the circumcircles of triangles AEB and AED, respectively. Let R_1 and R_2 be the circumradii of ω_1 and ω_2, respectively, and X be their external similitude center. Then

$$Power(X, \omega_1) : Power(X, \omega_2) = R_1^2 : R_2^2. \tag{1}$$

Let lines CB and CD meet for the second time ω_1 and ω_2 again at points M and N respectively. It is clear that AN and AM are diameters of ω_1 and ω_2, also the triangle AND is similar to AMB, and the triangle AMN is similar to ABD. From these we have

$$\frac{R_1}{R_2} = \frac{AM}{AN} = \frac{CD}{CB}, \tag{2}$$

On the other hand

$$\frac{CM}{CN} = \frac{\sin \angle ENC}{\sin \angle EMC} = \frac{\sin \angle EAD}{\sin \angle EAB} = \frac{ED}{EB}\frac{AB}{AD} = \frac{CD^3}{CB^3}.$$

Therefore

$$\frac{Power(C, \omega_1)}{Power(C, \omega_2)} = \frac{CB \cdot CM}{CD \cdot CN} = \frac{R_1^2}{R_2^2}. \tag{3}$$

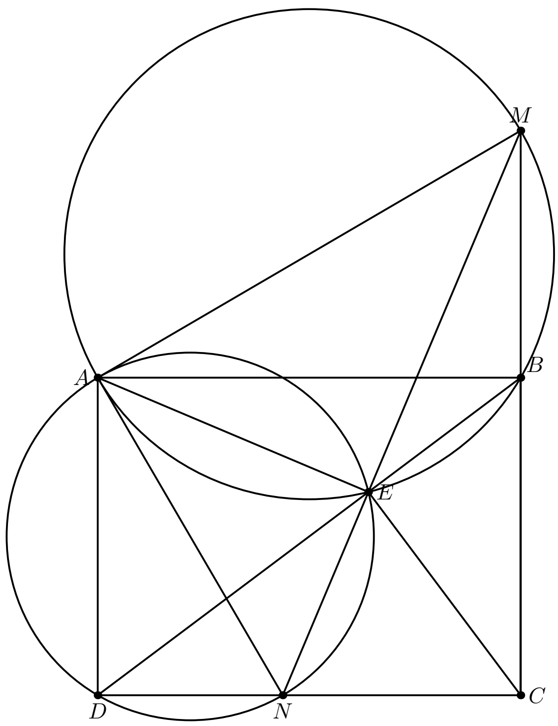

Рис. 10.6

From (1) and (3), we deduce that X and C lie on a circle coaxial with ω_1 and ω_2 or four points X, C, A, and E are concyclic. This completes the proof.

Remark. Similarly, we can prove that internal similitude center of ω_1 and ω_2 also lies on the circumcircle of triangle AEC.

Second solution. Let $AB > BC$ and so $DE > BE$. Let S be the midpoint of arc ACE, and the inversion centered at S with radius SA map D to a point T. Then we have to prove that T lies on the circle AEB. We have $\angle ATE = \angle ATS + \angle STE = \angle SAD + \angle DES = \angle EDA - \angle ESA = \angle BAC - \angle ECA = \angle ABE$, q.e.d.

7. (A.Skopenkov, I.Bogdanov) There are 43 points in the space: 3 yellow and 40 red. Any four of them are not coplanar. May the number of triangles with red vertices hooked with the triangle with yellow vertices be equal to 2023? *Yellow triangle is hooked with the red one if the boundary of the red triangle meet the part of the plane bounded by the yellow triangle at the unique point. The triangles obtained by the transpositions of vertices are identical.*

   Answer. No.

   Solution. Draw all segments joining the pairs of red points lying on the different sides from the yellow plane (passing through three yellow points) and color all segments intersecting the yellow triangle at internal point black, color all remaining segments white. It is clear that the number of red triangles hooked with the yellow one equals to the number of pairs of segments with common vertex colored differently. Call such pair of segments *a jackdaw*. If the number of red points lying on each side from the yellow plane is odd, then the numbers of black and white segments having each red point as a vertex have different parities, therefore the number of jackdaws is even. If the number of red points on the each side of yellow plane is even, consider a graph having red points as vertices and black segments as edges. The number of its vertices with odd degree is even, therefore the common number of jackdaws is also even.

8. (L.Shatunov) A triangle ABC is given. Let ω_1, ω_2, ω_3, ω_4 be circles centered at points X, Y, Z, T respectively such that each of lines BC, CA, AB cuts off on them four equal chords. Prove that the centroid of ABC divides the segment joining X and the radical center of ω_2, ω_3, ω_4 in the ratio $2:1$ from X.

   Solution. Prove that the circumcircle Ω of triangle ABC is the nine-points-circle of triangle YZT. In fact, let M be the midpoint of YZ, M_a, M_b, M_c be the projections of M to BC, CA, AB respectively, Y_a, Z_a be the projections of Y, Z to BC. Then M_a is the midpoint of Y_aZ_a and since BC cuts off equal chords on ω_2, ω_3, we obtain that the degrees of M_a with respect to these circles are equal. Similarly the degrees of M_b, M_c with respect to these circles are equal. Therefore the projections of M to the sideline of ABC are collinear (on the radical axis), i.e. M lies on Ω. Similarly the midpoints of segments YT, ZT, XY, XZ, XT lies on Ω. Thus the quadruple X, Y, Z, T is orthocentric, and Ω is the nine-points-circle of triangles XYZ, YZT, XZT, XYT.

Now let O be the center of Ω, H be the orthocenter of ABC, H' be the center of circle YZT, X' be the radical center of ω_2, ω_3, ω_4, and H_t, R_t be the projections of H and X' respectively to YZ. Then HH_t is parallel to the Simson line $X'R_t$ of M and passes through the orthocenter of ABC. Hence HH_t is the Steiner line of M, and X' is the midpoint of HH'. Also O is the midpoint of XH' (because X is the orthocenter of YZT). Therefore the centroid G of ABC is also the centroid of points H, X, H', i.e G lies on XX' and $GX = 2GX'$.

Remark. We can consider as a partial case of ω_1, ω_2, ω_3, ω_4 the incircle and three excircles of triangle. In this case the assertion of the problem is well known.