XVIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (N.Moskvitin, 8) Let O and H be the circumcenter and the orthocenter respectively of triangle ABC. It is known that BH is the bisector of angle ABO. The line passing through O and parallel to AB meets AC at K. Prove that $AH = AK$.

   Solution. Let D be the reflection of H about AC. Since D lies on the circumcircle of ABC, we have $\angle ODB = \angle OBD = \angle HBA$. Thus $OD \parallel AB$, i.e. K lies on OD and $\angle HKA = \angle OKC = \angle BAC$ (fig. 1). On the other hand, $\angle CBO = \angle HBA = 90° - \angle A$, hence $\angle ABC = 3(90° - \angle BAC)$, $\angle ACB = 2\angle BAC - 90°$, and $\angle HAK = 180° - 2\angle BAC$. Therefore $\angle AHK = \angle BAC = \angle AKH$ и $AK = AH$.

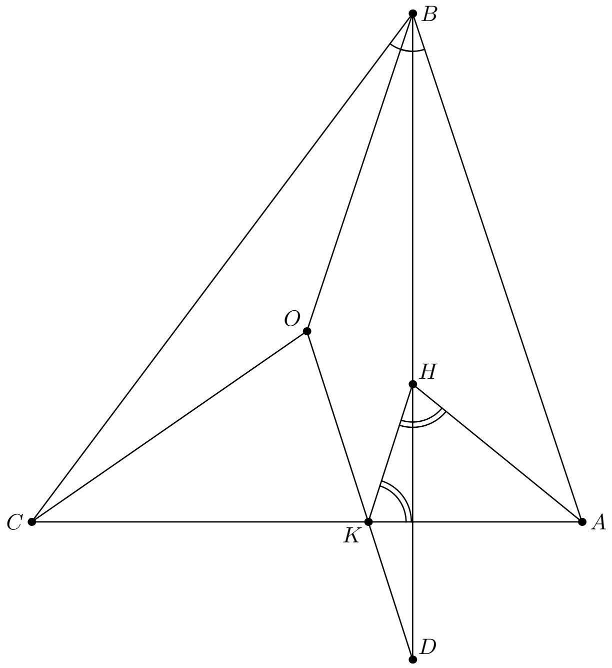

Fig. 1.

2. (A.Salimova, 8) Let $ABCD$ be a curcumscribed quadrilateral with incenter I, and let O_1, O_2 be the circumcenters of triangles AID and CID. Prove that the circumcenter of triangle O_1IO_2 lies on the bisector of angle ABC.

   Solution. Note that O_1O_2 is the perpendicular bisector to DI, $\angle IO_1O_2 = \angle IAD$, $\angle IO_2O_1 = \angle ICD$. Hence we obtain for the center O of circle IO_1O_2: $\angle OIO_1 = 90° - \angle ICD$. On the otyer hand, $\angle O_1IA = 90° - \angle IDA$ и $\angle BIO_1 = 180° - \angle IAB - \angle IBA + 90° - \angle IDA = 90° + \angle ICD$. Therefore B, I, O lie on the bisector of angle B (fig. 2).

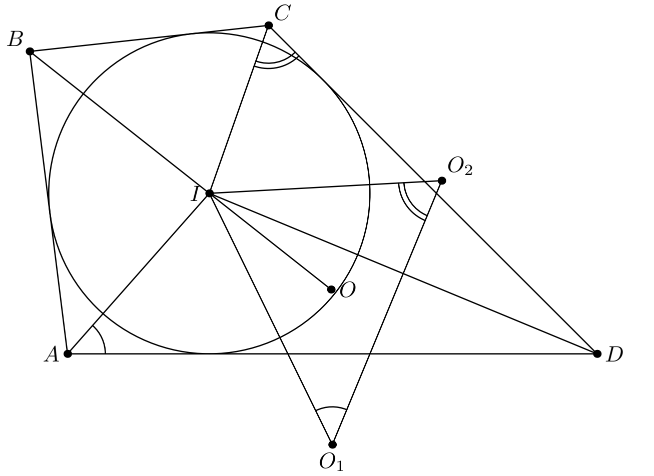

Fig. 2.

3. (N.Moskvitin, 8) Let CD be an altitude of right-angled triangle ABC with $\angle C = 90°$. Regular triangles AED and CFD are such that E lies on the same side from AB as C, and F lies on the same side from CD as B. The line EF meets AC at L. Prove that $FL = CL + LD$.

Solution. By the assumption we obtain that $FD = CD$, $DE = AD$, and $\angle FDE = \angle CDA$. Thus triangles FDE and CDA are congruent and $\angle FLC = 60°$. Take on the ray LC segment $LK = LF$. Since LFK is a regular triangle, we obtain that $FK = FL$ and $\angle KFL = \angle CFD = 60°$. Therefore $\angle KFC = \angle DFL$, i.e. triangles KFC and LFD are congruent (fig. 3). Hence $KC = LD$ which is equivalent to the required assertion.

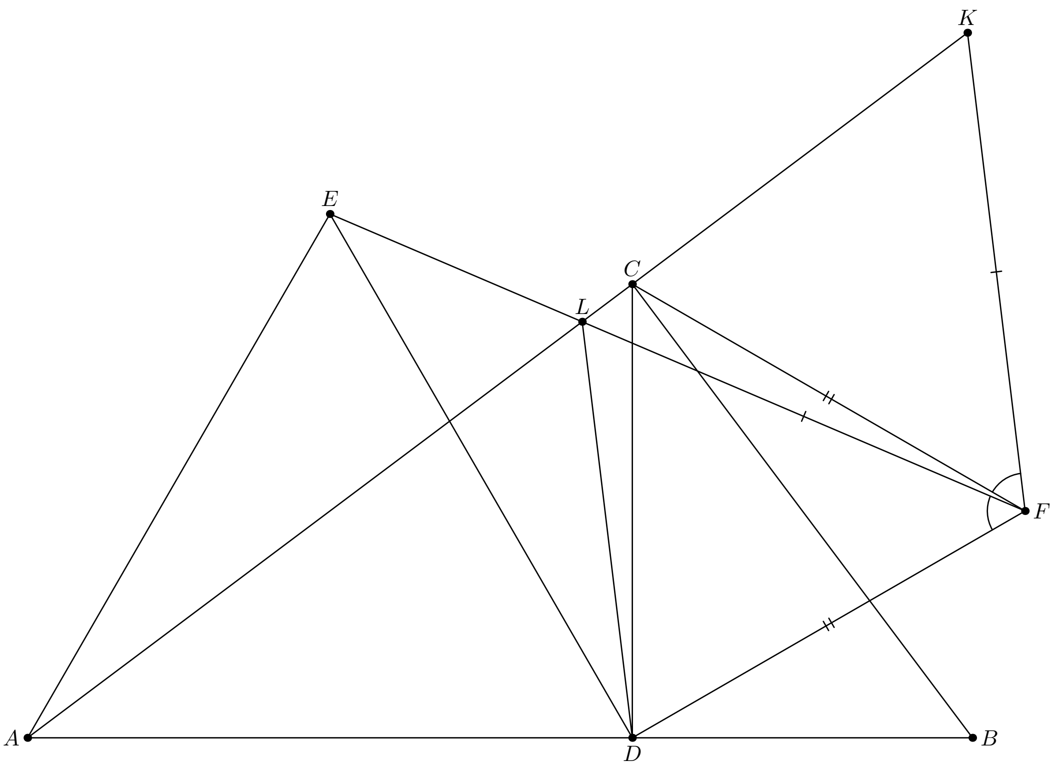

Fig. 3.

Remark. The assertion is correct for an arbitrary triangle ABC and an arbitrary point D on side AB.

4. (D.Shvetsov, 8) Let AA_1, BB_1, CC_1 be the altitudes of acute angled triangle ABC; A_2 be the touching point of the incircle of triangle AB_1C_1 with B_1C_1; points B_2, C_2 be defined similarly. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

   Solution. Since triangles AB_1C_1 and ABC are similar we have $B_1A_2 : A_2C_1 = (p-b) : (p-c)$, where a, b, c, p are the sidelengths and the semiperimeter of ABC. Similarly we obtain that $C_1B_2 : B_2A_1 = (p-c) : (p-a)$ and $A_1C_2 : C_2B_1 = (p-a) : (p-b)$. By the Ceva theorem we obtain the required assertion.

5. (D.Shvetsov, 8) Let the diagonals of cyclic quadrilateral $ABCD$ meet at point P. The line passing through P and perpendicular to PD meets AD at point D_1; a point A_1 is defined similarly. Prove that the tangent at P to the circumcircle of triangle D_1PA_1 is parallel to BC.

   Solution. Let MN be the tangent. Then $\angle NPD = 90^\circ - \angle MPD_1 = 90^\circ - \angle PA_1A = \angle PAD = \angle PBC$ (fig.5), which yields the required assertion.

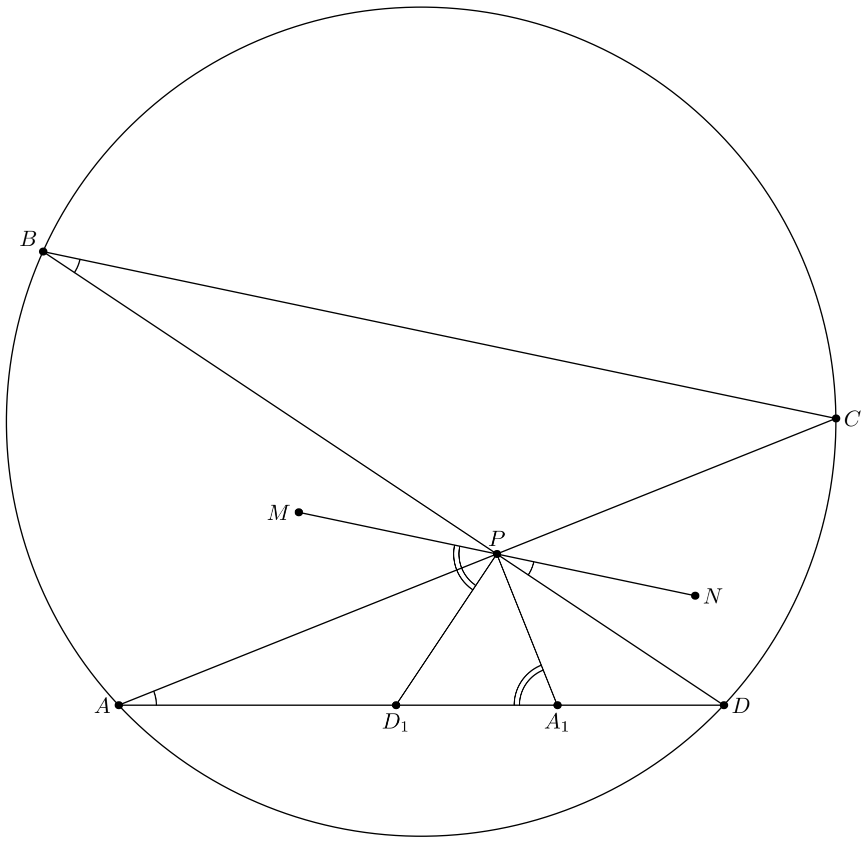

Fig. 5.

6. (F.Ivlev, 8–9) The incircle and the excircle of triangle ABC touch the side AC at points P and Q respectively. The lines BP and BQ meet the circumcircle of triangle ABC for the second time at points P' and Q' respectively. Prove that $PP' > QQ'$.

Solution. Since $CP = AQ$ we have $BP \cdot PP' = AP \cdot PC = BQ \cdot QQ'$. But since $|AP - CP| = |AB - CB| < |(AB^2 - CB^2)/AC|$ point P lies between the midpoint of AC and the foot of the corresponding altitude (fig. 6). Therefore $BP < BQ$ and $PP' > QQ'$.

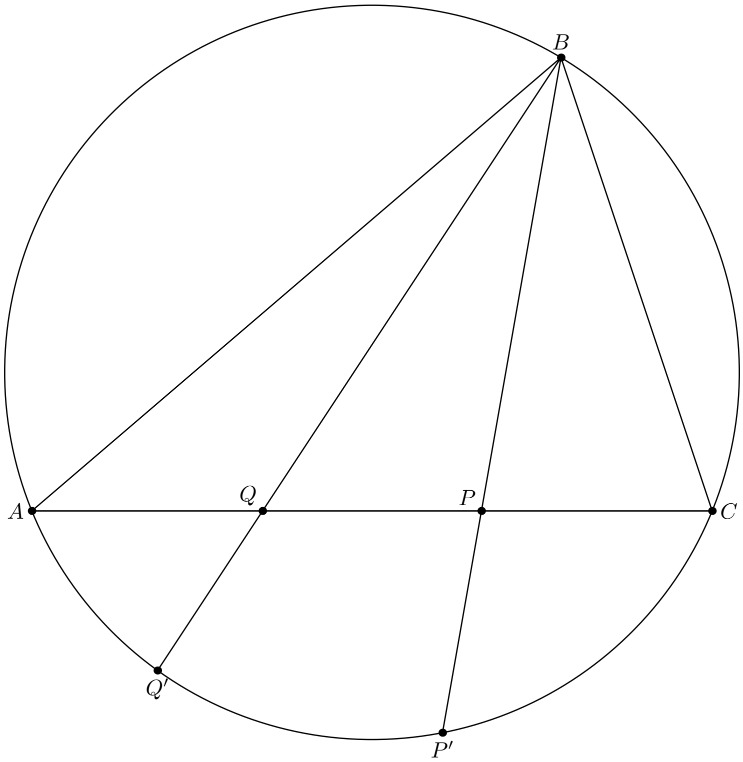

Fig. 6.

7. (G.Filipovsky, 8–9) A square with center F was constructed on the side AC of triangle ABC outside it. After this, everything was erased except F and the midpoints N, K of sides BC, AB. Restore the triangle.

   Solution. Let M be the midpoint of AC. The $FM = AC/2 = KN$ and $FM \perp KN$. Thus we can construct point M and restore the triangle ABC by the midpoints of its sides.

8. (I.Frolov, 8–9) Points P, Q, R lie on the sides AB, BC, CA of triangle ABC in such a way that $AP = PR$, $CQ = QR$. Let H be the orthocenter of triangle PQR, and O be the circumcenter of triangle ABC. Prove that $OH \parallel AC$.

   Solution. Since $\angle ARP = \angle BAC$, $\angle CRQ = \angle BCA$, we obtain that $\angle PRQ = \angle ABC$ and $\angle PHQ = 180^\circ - \angle ABC$, i.e. points B, P, Q, H are concyclic. Also the projections of P, Q to AC bisect segments AR, CR respectively, hence the distance between them equals to a half of AC. Let M, N be the midpoints of AB, BC respectively. Since the projections of PM and QN to AC are equal we have $PM : QN = \cos\angle ACB : \cos\angle BAC = OM : ON$. Therefore the trianglesOPM and OQN are similar, i.e. $\angle POQ = \angle MON = \angle PHQ$. Thus O also lie on the circle $BPHQ$ (fig. 8) and $\angle BOH = \angle BQH = 90^\circ + \angle BCA - \angle BAC$ which is equivalent to the required assertion.

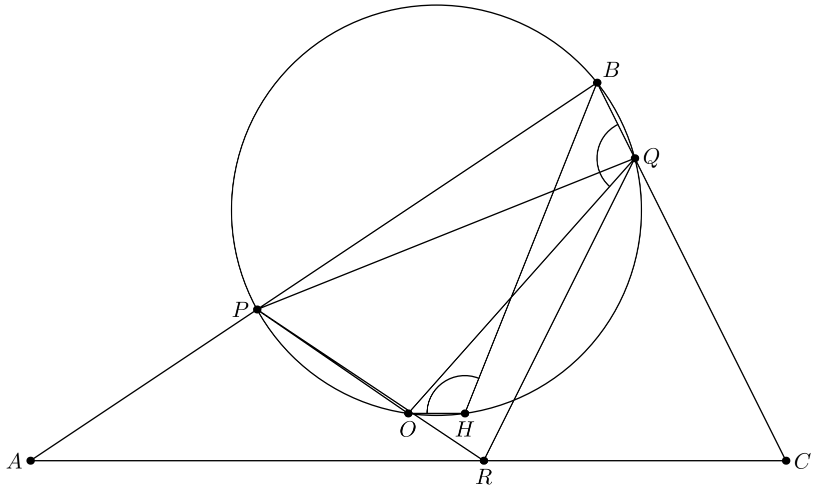

Fig. 8.

9. (F.Ivlev, 8–9) The sides AB, BC, CD and DA of quadrilateral $ABCD$ touch a circle with center I at points K, L, M and N respectively. Let P be an arbitrary point of line AI. Let PK meet BI at point Q, QL meet CI at point R, and RM meet DI at point S. Prove that P, N and S are collinear.

   Solution. By the Menelaos theorem $(BQ : QI)(IP : PA)(AK : KB) = 1$. Similarly $(CR : RI)(IQ : QB)(BL : LC) = 1$ and $(DS : SI)(IR : RC)(CM : MD) = 1$. Multiplying these equalities we obtain from $AK = AN$, $BK = BL$, $CL = CM$, $DM = DN$ that $(IS : SD)(DN : NA)(AP : PI) = 1$ which is equivalent to the required assertion.

   Remark. The assertion is correct if we replace I by an arbitrary point of the plane.

10. (M.Fadin, 8–9) Let ω_1 be the circumcircle of triangle ABC and O be its circumcenter. A circle ω_2 touches the sides AB, AC, and touches the arc $\widehat{BC}$ of ω_1 at point K. Let I be the incenter of ABC. Prove that the line OI contains the symmedian of triangle AIK.

    Solution. Let M, N be the midpoints of arcs BC and ABC of the circumcircle of ABC. It is known that the touching point K of the circumcircle and the cemiincircle lies on NI. Also I lies on AM (fig. 10). Thus triangles IMN and IKA are similar, and the median OI of triangle IMN coincide with the symmedian of triangle IKA.

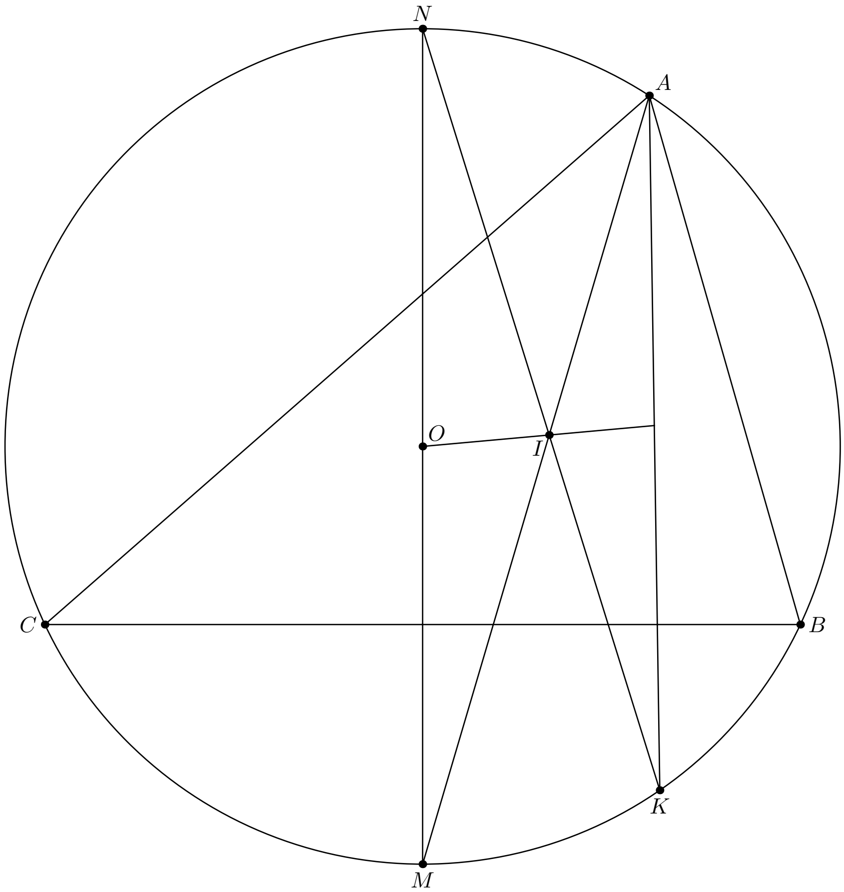

Fig. 10.

11. (D.Prokopenko, 8–10) Let ABC be a triangle with $\angle A = 60^\circ$, and T be a point such that $\angle ATB = \angle BTC = \angle ATC$. A circle passing through B, C and T meets AB and AC for the second time at points K and L. Prove that the distances from K and L to AT are equal.

    Solution. Note that AT passes through the vertex of a regular triangle with base BC. Since $\angle A = 60^\circ$ this vertex is a common point of tangents to the circumcircle of ABC at B and C, i.e. AT is a symmedian of the triangle. On the other hand it is clear that triangles ABL and ACK are regular, i.e. the triangles ABC and ALK are symmetric with respect to the bisector of angle A and AT is a median of triangle AKL (fig. 11).

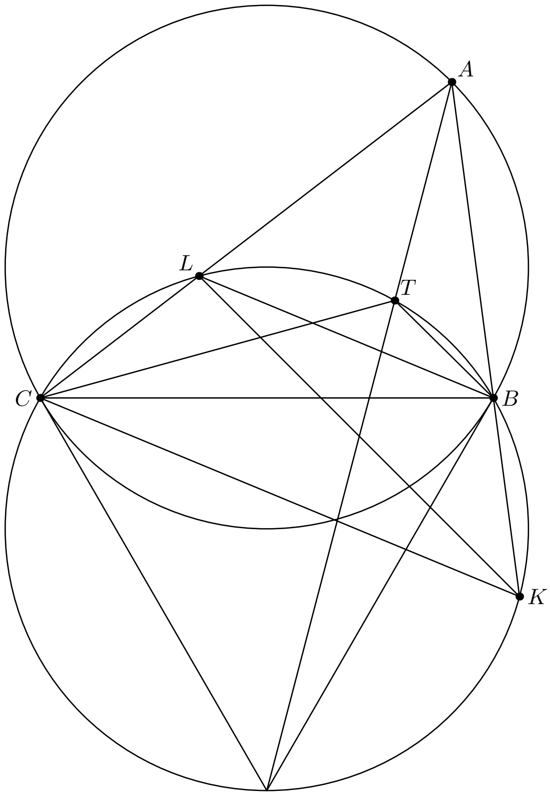

Fig. 11.

12. (L.Emelyanov 8–11) Let K, L, M, N be the midpoints of sides BC, CD, DA, AB respectively of a convex quadrilateral $ABCD$. The common points of segments AK, BL, CM, DN divide each of them into three parts. It is known that the ratio of the length of the medial part to the length of the whole segment is the same for all segments. Does this yield that $ABCD$ is a parallelogram?

Answer. No.

Solution. Let $PQRS$ be a trapezoid with the ratio of bases PS and QR less than 2. Take on the extensions of PQ beyond P and Q such points A and K respectively that $PA = PQ = 2QK$. Take on the extensions of RS beyond R and S such points M and C respectively that $CR = RS = 2SM$. Let CK and QR meet at point B, AM and PS meet at point D. Then it easy to see that $AM = MD$, $BK = KC$, the midpoints N, L of AB, CD lie on PS, QR respectively, and $QR : BL = PS : DN = PQ : AK = RS : CM = 2/5$ (fig. 12).

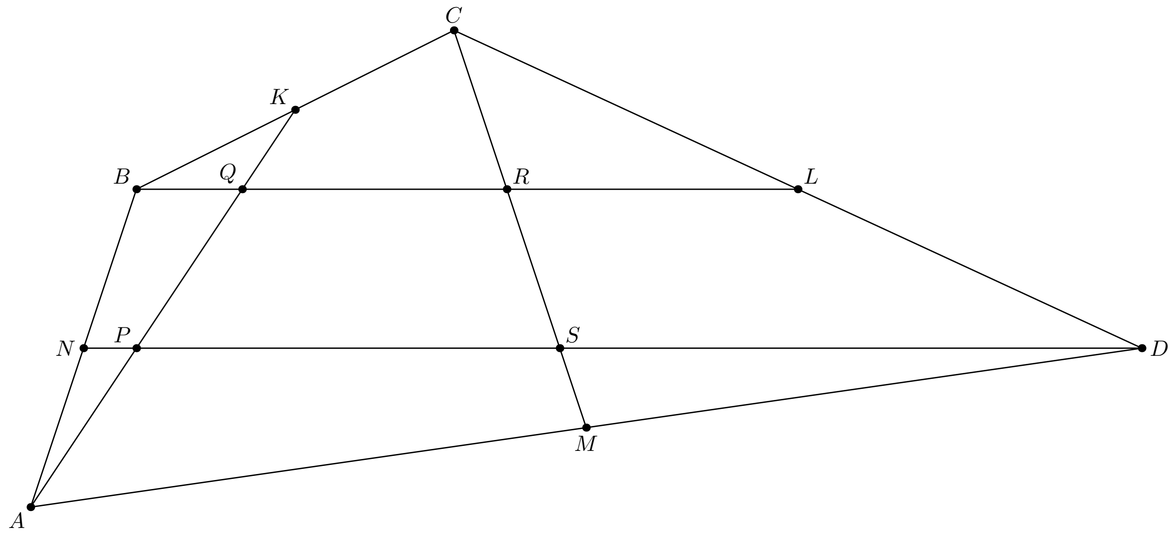

Fig. 12.

Remark. It may be proved that all quadrilaterals satisfying to the conditions can be constructed by the indicated way and thus the length of the medial part is $2/5$ of the length of whole segment.

13. (M.Saghafian, 8–11) Eight points in a general position are given in the plane. The areas of all 56 triangles with vertices at these points are written in a row. Prove that it is possible to insert the symbols "+" and "−" between them in such a way that the obtained sum is equal to zero.

    Solution. Note that the required assertion is correct for any four points. In fact, if these points are vertices of a convex quadrilateral $ABCD$ then $S_{ABC}+S_{ACD}-S_{BCD}-S_{ABD}=0$. And if point D lies inside triangle ABC then $S_{ABC}-S_{ABD}-S_{ACD}-S_{BCD}=0$.

    Now take a correspondence between the given points and vertices of cube $ABCDA'B'C'D'$ and consider the following 14 quadruples of the vertices: six faces of the cube, six sections passing through the pairs of opposite edges, and two tetrahedrons $AB'CD'$, $A'BC'D$. Any two of these quadruples have at most two common points, thus each of 56 triangles is included exactly in one quadruple. Hence choosing the corresponding symbols for each quadruple we obtain the required result.

14. (L.Emelyanov) A triangle ABC is given. Let C' and C'_a be the touching points of sideline AB with the incircle and with the excircle touching the side BC. Points C'_b, C'_c, A', A'_a, A'_b, A'_c, B', B'_a, B'_b, B'_c are defined similarly. Consider the lengths of 12 altitudes of triangles $A'B'C'$, $A'_aB'_aC'_a$, $A'_bB'_bC'_b$, $A'_cB'_cC'_c$.

    (a) (8–9) Find the maximal number of different lengths.

    (b) (10–11) Find all possible numbers of different lengths.

    Answer. (a) 6. (b) From 2 to 6.

    Solution. (a) Note that the lines $A'B'$ and $A'_cB'_c$ are perpendicular to the bisector of angle C. Also $BA'=AB'_c=p-b$, and $AB'=BA'_c=p-a$, where a, b, c, p are the sidelengths and the semiperimeter of ABC. Hence the distances from A to $A'B'$ and from B to $A'_cB'_c$ are equal. Similarly the distances from B to $A'B'$ and from A to $A'_cB'_c$ are equal. Since $AC' = BC'_c$, we obtain that the altitudes of triangles $A'B'C'$, $A'_cB'_cC'_c$ from C', C'_c respectively are also equal (fig. 14). Similarly we have the equalities of five other pairs of altitudes, i.e. between 12 altitudes there are six pairs of equal segments.

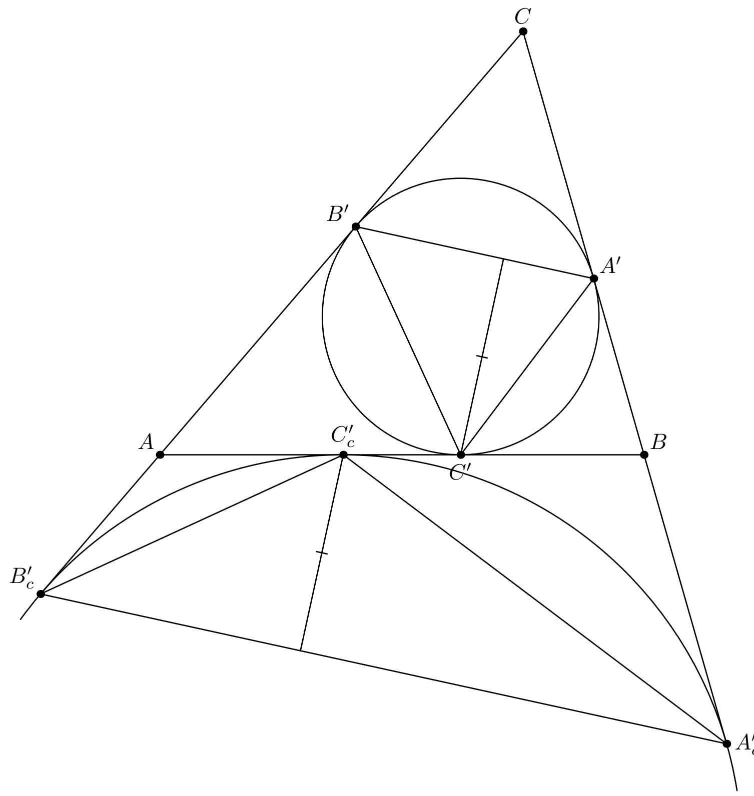

Fig. 14.

(b) The angles of triangle $A'B'C'$ are equal $(\pi - A)/2$, $(\pi - B)/2$, $(\pi - C)/2$, and the angles of triangle $A'_cB'_cC'_c$ are equal $A/2$, $B/2$, $(\pi+C)/2$. By the sinus law the altitudes of these triangle are equal $2r\cos(A/2)\cos(B/2)$, $2r\cos(A/2)\cos(c/2)$, $2r\cos(B/2)\cos(C/2)$, and $2r_c\sin(A/2)\sin(B/2)$, $2r_c\sin(A/2)\cos(C/2)$, $2r_c\sin(B/2)\cos(C/2)$, where r, r_c are the inradius and the exradius. Similarly we obtain the formulas for the altitudes of two remaining triangles. From (a) we have $r : r_c = \mathrm{tg}(A/2)\,\mathrm{tg}(B/2)$. Hence if $B = \pi/2$ we obtain that $2r\cos(A/2) = 2r_c\sin(A/2)$, i.e. we have four equal altitudes. So we have five different lengths for a nonisosceles right angled triangle, four lengths for an isosceles not right angled triangle, three lengths for an isosceles right angled triangle, and two lengths for a regular triangle.

15. (I.Mikhaylov, 9–11) A line ℓ parallel to the side BC of triangle ABC touches its incircle and meets its circumcircle at points D and E. Let I be the incenter of ABC. Prove that $AI^2 = AD \cdot AE$.

    Solution. Since $DE \parallel BC$ we have $\smile BD = \smile CE$, i.e. $\angle BAD = \angle CAE$ and $\angle DAI = \angle EAI$. Now note that the tangents from D and E to the incircle meet at a point F lying on the circumcircle hence DI and EI bisect angles FDE and FED respectively.

    Let A', D', E' be the second common points of AI, DI, EI respectively with the circumcircle. Since these points are the midpoints of arcs DFE, EF, DE respectively we obtain that $\smile D'A' = \smile DE'$ (fig.15). Therefore $\angle AID = \angle AEI$, i.e. the triangles AID and AEI are similar and $AI : AD = AE : AI$.

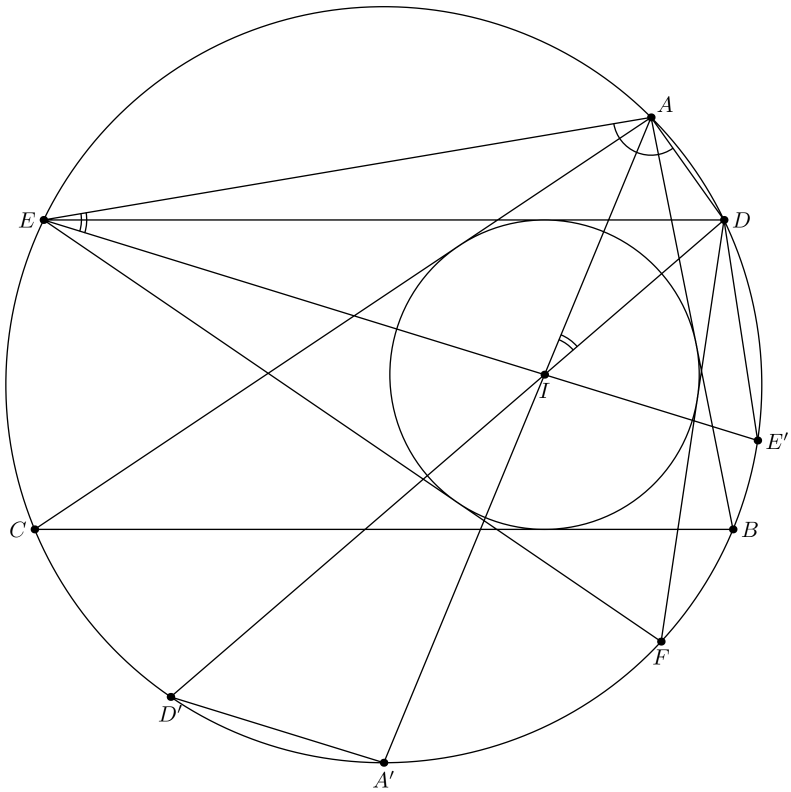

Fig. 15.

16. (M.Plotnikov, D.Khilko, P.Kozhevnikov, 9–11) Let $ABCD$ be a cyclic quadrilateral, $E = AC \cap BD$, $F = AD \cap BC$. The bisectors of angles AFB and AEB meet CD at points X, Y. Prove that A, B, X, Y are concyclic.

    Solution. Let U be the common point of AB and CD (fig. 16). Then $DY : YC = DE : EC = AD : BC = UD : UB$ (the second equality follows from the similarity of triangles EAD and EBC, the third one follows from the similarity of triangles UDA and UBC). Therefore $UY = UD + UD \cdot CD/(UD + UB) = UD(UC + UB)/(UD + UB)$. Similarly we obtain that $UX = UC(UD + UB)/(UC + UB)$. Thus $UX \cdot UY = UC \cdot UD = UA \cdot UB$. The reasoning is similar for another configurations.

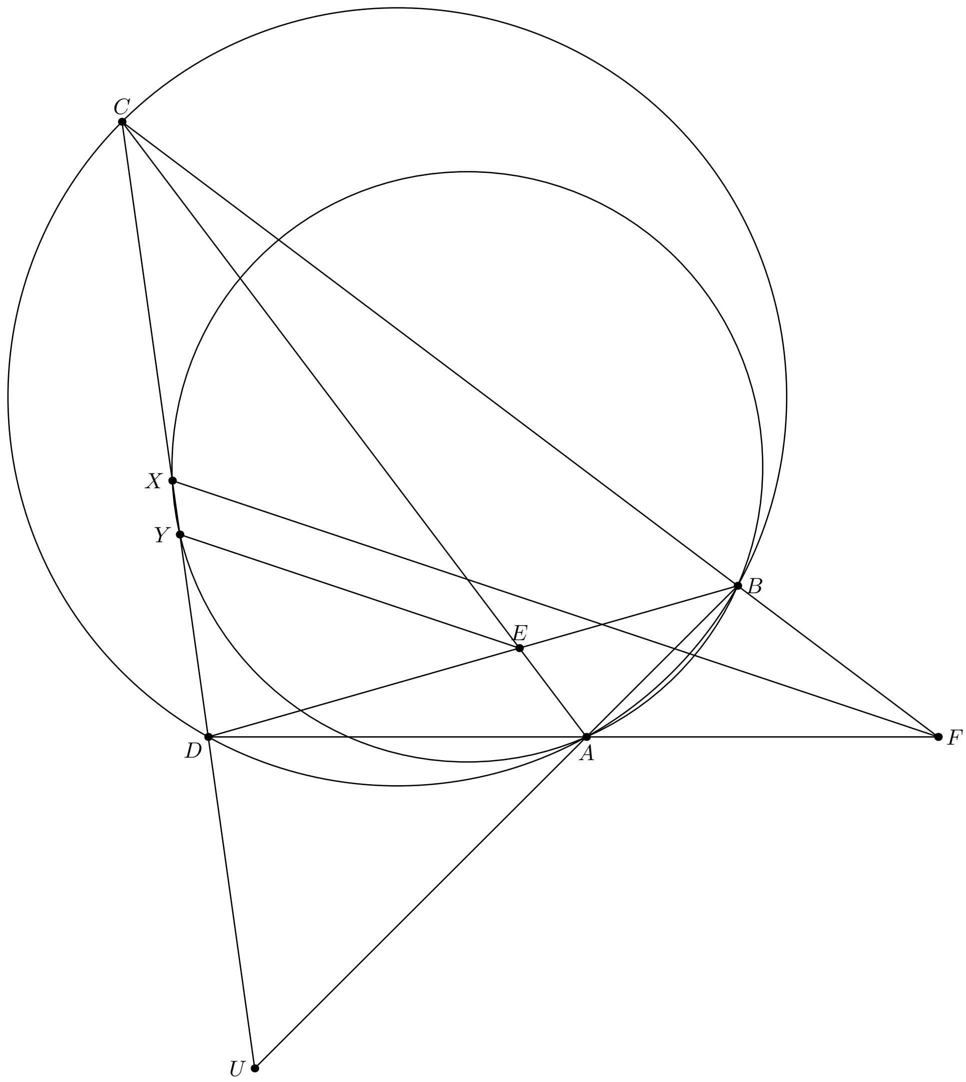

Fig. 16.

17. (I.Kukharchuk, 9–11) Let a point P lie inside a triangle ABC. The rays starting at P and crossing the sides BC, AC, AB under the right angle meet the circumcircle of ABC at A_1, B_1, C_1 respectively. It is known that lines AA_1, BB_1, and CC_1 concur at point Q. Prove that all such lines PQ concur.

    Solution. Consider a point R isogonally conjugated to Q. Let AR, BR, CR meet the circumcircle for the second time at points A_2, B_2, C_2 respectively. Let also A_1P, B_1P, C_1P meet the circumcircle at points A_3, B_3, C_3. Finally let O be the circumcenter of ABC. The line QR passes through the common point X of B_1A_2 and A_1B_2 (Pascal theorem for $BB_2A_1AA_2B_1$). Similarly it passes through the common point Y of B_1C_2 and C_1B_2. Therefore the lines XY and QR coincide.

    The line PO passes through X (Pascal theorem for $B_3B_1A_2A_3A_1B_2$ and A_3A_2 is a diameter because A_1 and A_2 are symmetric with respect to the perpendicular bisector to BC), similarly passes through Y. Hence PO coincide with XY and with QR. Therefore all lines PQ pass through O.

18. (A.Zaslavsky, 10–11) The products of the opposite sidelengths of a cyclic quadrilateral

$ABCD$ are equal. Let B' be the reflection of B about AC. Prove that the circle passing through A, B', D touches AC.

Solution. Construct two circles passing through B' and touching AC at points A and C respectively. Let D' be the second common point of these circles. Since $\angle CAD' + \angle ACD' = \angle AB'D' + \angle CB'D' = \angle B$, we obtain that D' lies on the circumcircle of the quadrilateral. Also the midpoint of AC lies on $B'D'$ because the tangents from it to the circles are equal (fig. 18). Hence $AD'/CD' = \sin\angle ACD' / \sin\angle CAD' = \sin\angle CB'D' \sin\angle AB'D' = AB'/CB' = AC/BC$, i.e. D' coincides with c D.

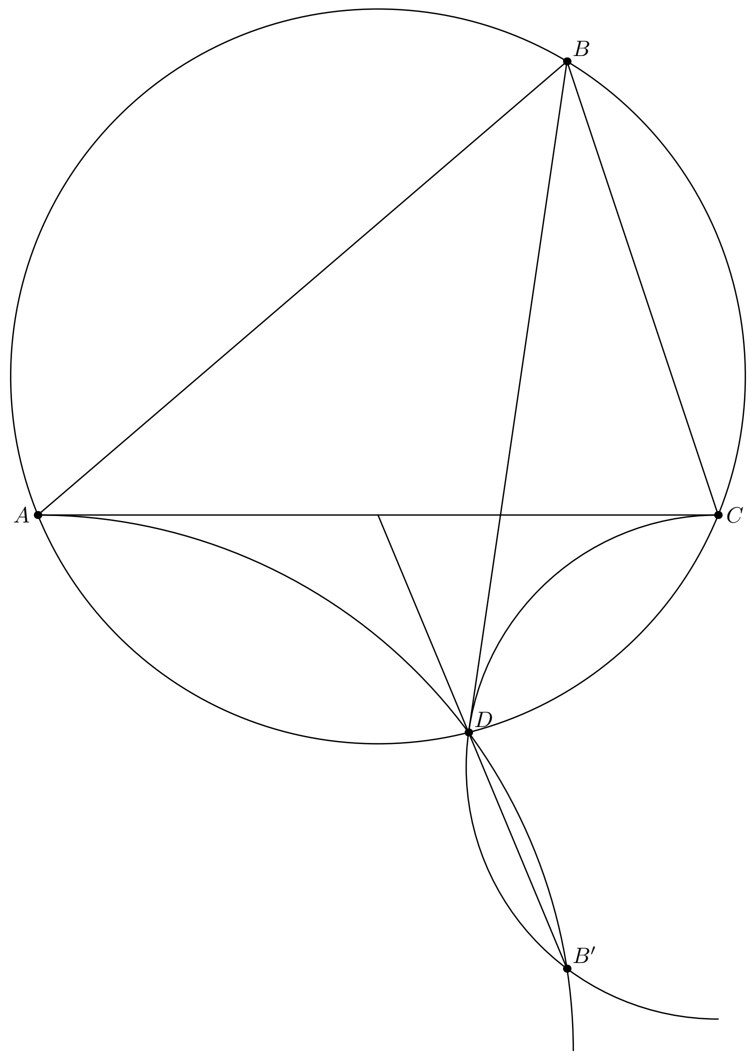

Fig. 18.

19. (M.Didin, 10–11) Let I be the incenter of triangle ABC, and K be the common point of BC with the external bisector of angle A. The line KI meets the external bisectors of angles B and C at points X and Y. Prove that $\angle BAX = \angle CAY$.

Solution. Let I_a, I_b, I_c be the excenters of ABC. The I is the orthocenter of triangle $I_aI_bI_c$ and I_aA is its altitude. Points A, K, I_b, I_c form a harmonic quadruple, thus the quadruple of points I, K, Y, X and the quadruple of lines AI, AK, AY, AX are also harmonic. Since $AI \perp AK$, we obtain that AI is the bisector of angle XAY.

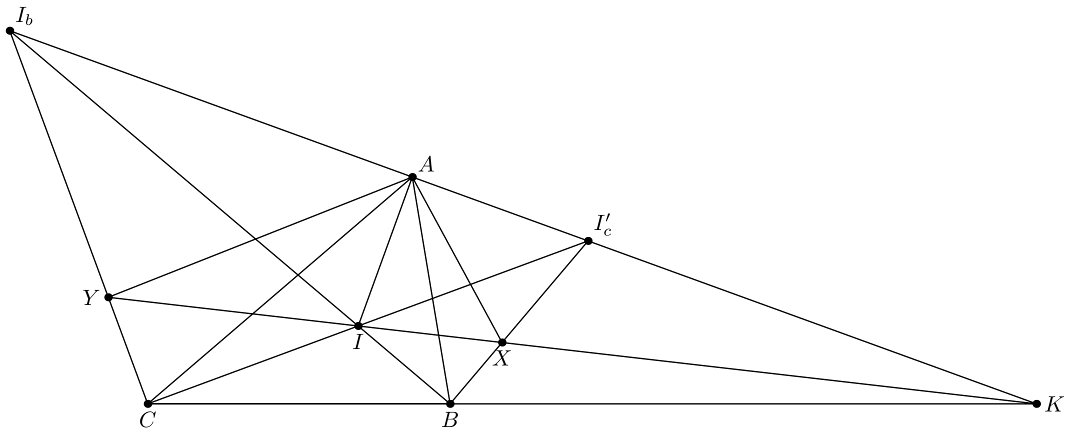

Fig. 19.

20. (I.Kukharchuk, 10–11) Let O, I be the circumcenter and the incenter of triangle ABC; R, r be the circumradius and the inradius; D be the touching point of the incircle with BC; and N be an arbitrary point of segment ID. The perpendicular to ID at N meets the circumcircle of ABC at points X and Y. Let O_1 be the circumcircle of triangle XIY. Find the product $OO_1 \cdot IN$.

Answer. Rr.

First solution. Let K be the midpoint of XY. Then $O_1X^2 - OX^2 = O_1K^2 - OK^2$. On the other hand $O_1X^2 = O_1I^2 = OO_1^2 + OI^2 - 2OO_1 \cdot OI \cos\angle O_1OI$ (fig. 20). Therefore $2Rr = R^2 - OI^2 = 2OO_1(OK - OI\cos\angle O_1OI) = 2OO_1 \cdot IN$. The solution is similar for another configurations.

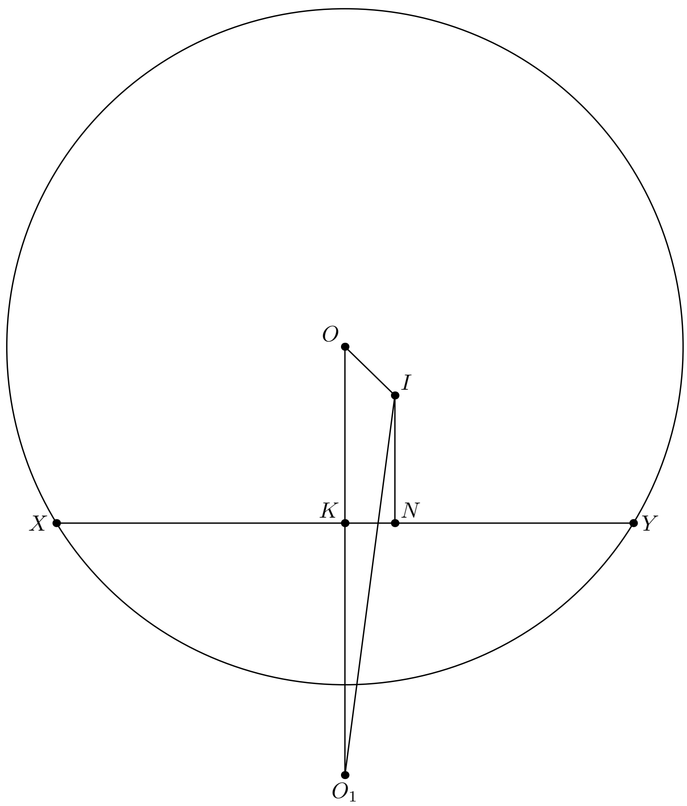

Fig. 20.

Seconf solution. There is a general fact: let P be an arbitrary point inside ABC; R and S be such points on segments BP and CP respectively that $RS \parallel BC$; Q and L be the common points of the circumcircle of ABC with RS; Obe the circumcenter of ABC; O_1 be the circumcenter of BPC; O_2 be the circumcenter of QPL. Then $OO_1 : OO_2 = RS : BC$.

Prove it. Let BP the circles QPL, ABC at points F, G respectively. Project O, O_1, and O_2 to BP and take a homothety with center P and coefficient 2. The image of O_1 is B, the image of O_2 is F, and the image of O is such point X on BG that $XG = BP$. We have to prove that $GB : BX = BR : RP$. This is equivalent (since $BX = GP$) to $GR/RX = BR/RP$, which is correct because the degrees of R with respect to circles ABC and QPL are equal.

21. (A.Zaslavsky, 10–11) The circumcenter O, the incenter I, and the midpoint M of a diagonal of a bicentral quadrilateral were marked. After this the quadrilateral was erased. Restore it.

    Solution. Diagonals of the quadrilateral meet at point L lying on OI. Also $OM \perp ML$ which allows to construct the point L. Now let AB be the diameter of the circumcicle passing through I, and C be such point of the circumcircle that $CL \perp AB$. Then CO, CI, CL are the median the bisector and the altitude of a right angled triangle ABC. Therefore CI is the bisector of angle OCL. This allows to construct C as a common point of the perpendicular from L to OI and the Appolonius circle for points O and L, thus we can construct the circumcircle. Finally note that the midpoint N of the second diagonal is the common point of the line MI and the circle with diameter OL, This allows to construct the vertices of the quadrilateral.

22. (P.Kozhevnikov, 10–11) Chords A_1A_2, A_3A_4, A_5A_6 of a circle Ω concur at point O. Let B_i be the second common point of Ω and the circle with diameter OA_i. Prove that chords B_1B_2, B_3B_4, B_5B_6 concur.

    Solution. Consider a composition of the inversion centered at O with radius $\sqrt{OA_1 \cdot OA_2}$ and the reflection with about O. It transposes A_1, A_2, conserves Ω and maps the circle with diameter OA_1 to a line perpendicular to A_1A_2. Therefore it maps B_1, B_2 to the points C_1, C_2 opposite to A_2, A_1, and so it maps the line B_1B_2 to the circle C_1C_2O. Let this circle meet for the second hand A_2A_1 and A_2C_1 at points X, Y respectively (fig. 22). Then $A_2X = OA_1$, i.e. the degree of A_2 with respect to the circle does not depend on the choice of the chord. Hence the degree of the center O_1 of Ω also does not depend on the chord and all circles meet OO_1 at the same point.

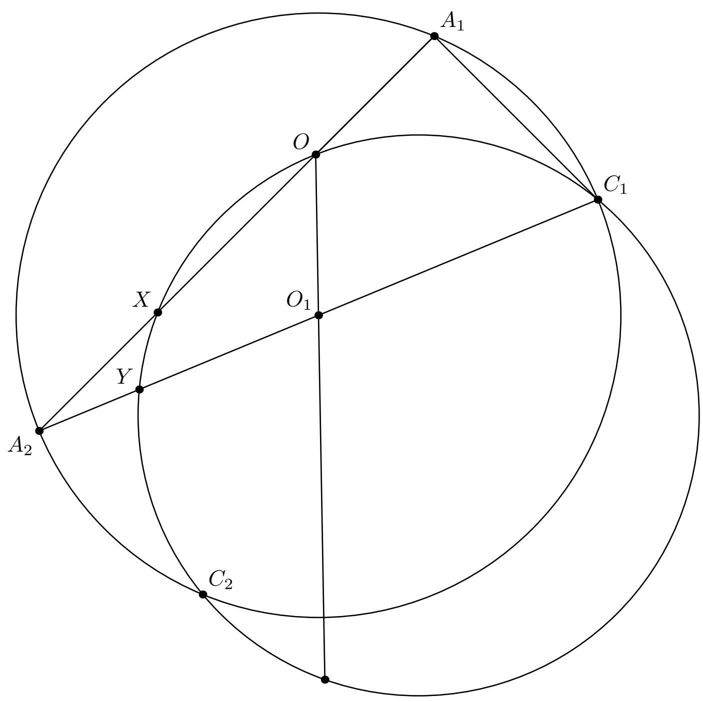

Fig. 22.

23. (A.Sgibnev, 10–11) An ellipse with focus F is given. Two perpendicular lines passing through F meet the ellipse at four points. The tangents to ellipse at these points form a quadrilateral circumscribed around the ellipse. Prove that this quadrilateral is inscribed into a conic with focus F.

    Solution. Prove general assertion: let two perpendicular rays with origin F meet ellipse at points A, B. Then the locus of common points of tangents to the ellipse at A and B is a conic with focus F.

    First way. A polar transformation centered at F maps the ellipse to a circle, also it maps A, B to two perpendicular tangents to this circle. The chords joining the touching points touche several circle hence their images lie on the conic with focus F. Note that this reasoning is also correct for an arbitrary constant angle AFB. The ratio of excentricities of the original ellipse and the obtained conic equals $\cos(\angle AFB/2)$.

    Second way. Use the following fact: let AB be a chord of the ellipse passing through its focus F and C be the common point of tangents at A, B. Then $CF \perp AB$.

    Consider a projective transformation mapping the ellipse to a circle and mapping F to its center. It maps AB and C to a diameter of the circle and the infinite point perpedicular to it, also it maps CF to a perpendicular diameter. The common points of tangents at the endpoints of these diameters lie on a concentric circle. The inverse transformation maps this circle its center, and the infinite line to a conic, its center, and its directrix respectively.

24. (O.Smirnov, 11) Let $OABCDEF$ be a hexagonal pyramid with base $ABCDEF$ circumscribed around a sphere ω. The plane passing through the touching points of ω with faces OFA, OAB and $ABCDEF$ meets OA at point A_1; points B_1, C_1, D_1, E_1 and F_1 are defined similarly. Let ℓ, m и n be the lines A_1D_1, B_1E_1 and C_1F_1 respectively. It is known that ℓ and m are coplanar, also m and n are coplanar. Prove that ℓ and n are coplanar.

Solution. A cone with vertex O circumscribed around the sphere meets the base of the pyramid by an ellipse inscribed into $ABCDEF$. By the Brianchon theorem the lines AD, BE, and CF concur at some point L. Then the common point of A_1D_1 and B_1E_1 lies on OL. Similarly the common point of B_1E_1 and C_1F_1 lies on this line. Therefore ℓ, m, n, and OL concur.