XVIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XVIII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade (at the start of the correspondent round) have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. For complete solution of both parts of Problem 14, a participant from a younger grade (in Russian system, 9 at most) obtains 12 points. (A participant from an elder grade obtains points only for part 14b.)

A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. **If a problem has an explicit answer, this answer must be presented distinctly**. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The **solutions** for the problems (in Russian or in English) must be **delivered not before December 1, 2021 and not later than on March 1, 2022**. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, press «Registration» in the left upper part, and follow the instructions.

Attention:

1. The solution of each problem (and of each part of it if any) must be contained in a **separate** pdf, doc, docx or jpg file. If the solution is contained in several files then pack them to an **archive** (zip or rar) and load it.

2. We recommend to prepare the paper using computer or to scan it rather than to photograph it. **In all cases, please check readability of the file before uploading**.

3. If you upload the solution of some problem more than once then only the last version is retained in the checking system. **Thus if you need to change something in your solution then you have to upload the whole solution again.**

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round will be held in July–August 2022 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before the round. The cutoff is determined after the examination of the papers of the correspondence round according to the number of participants with any given score. The graduates who are winners of the correspondence round will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** up to June 1, 2022. If you want to know your detailed results, please apply to **geomshar@yandex.ru** after publication of the list.

1. (8) Let O and H be the circumcenter and the orthocenter respectively of triangle ABC. It is known that BH is the bisector of angle ABO. The line passing through O and parallel to AB meets AC at K. Prove that $AH = AK$.

2. (8) Let $ABCD$ be a curcumscribed quadrilateral with incenter I, and let O_1, O_2 be the circumcenters of triangles AID and CID. Prove that the circumcenter of triangle O_1IO_2 lies on the bisector of angle ABC.

3. (8) Let CD be an altitude of right-angled triangle ABC with $\angle C = 90°$. Regular triangles AED and CFD are such that E lies on the same side from AB as C, and F lies on the same side from CD as B. The line EF meets AC at L. Prove that $FL = CL + LD$.

4. (8) Let AA_1, BB_1, CC_1 be the altitudes of acute angled triangle ABC; A_2 be the touching point of the incircle of triangle AB_1C_1 with B_1C_1; points B_2, C_2 be defined similarly. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

5. (8) Let the diagonals of cyclic quadrilateral $ABCD$ meet at point P. The line passing through P and perpendicular to PD meets AD at point D_1; a point A_1 is defined similarly. Prove that the tangent at P to the circumcircle of triangle D_1PA_1 is parallel to BC.

6. (8–9) The incircle and the excircle of triangle ABC touch the side AC at points P and Q respectively. The lines BP and BQ meet the circumcircle of triangle ABC for the second time at points P' and Q' respectively. Prove that $PP' > QQ'$.

7. (8–9) A square with center F was constructed on the side AC of triangle ABC outside it. After this, everything was erased except F and the midpoints N, K of sides BC, AB. Restore the triangle.

8. (8–9) Points P, Q, R lie on the sides AB, BC, CA of triangle ABC in such a way that $AP = PR$, $CQ = QR$. Let H be the orthocenter of triangle PQR, and O be the circumcenter of triangle ABC. Prove that $OH \parallel AC$.

9. (8–9) The sides AB, BC, CD and DA of quadrilateral $ABCD$ touch a circle with center I at points K, L, M and N respectively. Let P be an arbitrary point of line AI. Let PK meet BI at point Q, QL meet CI at point R, and RM meet DI at point S. Prove that P, N and S are collinear.

10. (8–9) Let ω_1 be the circumcircle of triangle ABC and O be its circumcenter. A circle ω_2 touches the sides AB, AC, and touches the arc $\widehat{BC}$ of ω_1 at point K. Let I be the incenter of ABC. Prove that the line OI contains the symmedian of triangle AIK.

11. (8–10) Let ABC be a triangle with $\angle A = 60°$, and T be a point such that $\angle ATB = \angle BTC = \angle ATC$. A circle passing through B, C and T meets AB and AC for the second time at points K and L. Prove that the distances from K and L to AT are equal.

12. (8–11) Let K, L, M, N be the midpoints of sides BC, CD, DA, AB respectively of a convex quadrilateral $ABCD$. The common points of segments AK, BL, CM, DN divide each of them into three parts. It is known that the ratio of the length of the medial part to

the length of the whole segment is the same for all segments. Does this yield that $ABCD$ is a parallelogram?

13. (8–11) Eight points in a general position are given in the plane. The areas of all 56 triangles with vertices at these points are written in a row. Prove that it is possible to insert the symbols "+" and "−" between them in such a way that the obtained sum is equal to zero.

14. A triangle ABC is given. Let C' and C'_a be the touching points of sideline AB with the incircle and with the excircle touching the side BC. Points C'_b, C'_c, A', A'_a, A'_b, A'_c, B', B'_a, B'_b, B'_c are defined similarly. Consider the lengths of 12 altitudes of triangles $A'B'C'$, $A'_aB'_aC'_a$, $A'_bB'_bC'_b$, $A'_cB'_cC'_c$.

    (a) (8–9) Find the maximal number of different lengths.

    (b) (10–11) Find all possible numbers of different lengths.

15. (9–11) A line ℓ parallel to the side BC of triangle ABC touches its incircle and meets its circumcircle at points D and E. Let I be the incenter of ABC. Prove that $AI^2 = AD{\cdot}AE$.

16. (9–11) Let $ABCD$ be a cyclic quadrilateral, $E = AC \cap BD$, $F = AD \cap BC$. The bisectors of angles AFB and AEB meet CD at points X, Y. Prove that A, B, X, Y are concyclic.

17. (9–11) Let a point P lie inside a triangle ABC. The rays starting at P and crossing the sides BC, AC, AB under the right angle meet the circumcircle of ABC at A_1, B_1, C_1 respectively. It is known that lines AA_1, BB_1, and CC_1 concur at point Q. Prove that all such lines PQ concur.

18. (10–11) The products of the opposite sidelengths of a cyclic quadrilateral $ABCD$ are equal. Let B' be the reflection of B about AC. Prove that the circle passing through A, B', D touches AC.

19. (10–11) Let I be the incenter of triangle ABC, and K be the common point of BC with the external bisector of angle A. The line KI meets the external bisectors of angles B and C at points X and Y. Prove that $\angle BAX = \angle CAY$.

20. (10–11) Let O, I be the circumcenter and the incenter of triangle ABC; R, r be the circumradius and the inradius; D be the touching point of the incircle with BC; and N be an arbitrary point of segment ID. The perpendicular to ID at N meets the circumcircle of ABC at points X and Y. Let O_1 be the circumcircle of triangle XIY. Find the product $OO_1 \cdot IN$.

21. (10–11) The circumcenter O, the incenter I, and the midpoint M of a diagonal of a bicentral quadrilateral were marked. After this the quadrilateral was erased. Restore it.

22. (10–11) Chords A_1A_2, A_3A_4, A_5A_6 of a circle Ω concur at point O. Let B_i be the second common point of Ω and the circle with diameter OA_i. Prove that chords B_1B_2, B_3B_4, B_5B_6 concur.

23. (10–11) An ellipse with focus F is given. Two perpendicular lines passing through F meet the ellipse at four points. The tangents to ellipse at these points form a quadrilateral circumscribed around the ellipse. Prove that this quadrilateral is inscribed into a conic with focus F.

24. (11) Let $OABCDEF$ be a hexagonal pyramid with base $ABCDEF$ circumscribed around a sphere ω. The plane passing through the touching points of ω with faces OFA, OAB and $ABCDEF$ meets OA at point A_1; points B_1, C_1, D_1, E_1 and F_1 are defined similarly. Let ℓ, m и n be the lines A_1D_1, B_1E_1 and C_1F_1 respectively. It is known that ℓ and m are coplanar, also m and n are coplanar. Prove that ℓ and n are coplanar.