XVII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XVII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. **If a problem has an explicit answer, this answer must be presented distinctly**. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The **solutions** for the problems (in Russian or in English) must be **delivered not before December 1, 2020 and not later than on March 1, 2021**. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, press "Registration"in the left upper part, and follow the instructions.
Attention:

1. The solution of each problem (and of each part of it if any) must be contained in a **separate** pdf, doc, docx or jpg file. If the solution is contained in several files then pack them to an archive (zip or rar) and load it.
2. We recommend to prepare the paper using computer or to scan it rather than to photograph it. **In all cases, please check readability of the file before uploading**.
3. If you upload the solution of some problem more than once then only the last version is retained in the checking system. **Thus if you need to change something in your solution then you have to upload the whole solution again.**

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round is to be be held in July–August 2021 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before the round. The cutoff is determined after the examination of the papers of the correspondence round according to the number of participants with any given score. The graduates who are winners of the correspondence round will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2021 at latest. If you want to know your detailed results, please use e-mail **geomshar@yandex.ru**.

1. (8) Let ABC be a triangle with $\angle C = 90^\circ$. A line joining the midpoint of its altitude CH and the vertex A meets CB at point K. Let L be the midpoint of BC, and T be a point of segment AB such that $\angle ATK = \angle LTB$. It is known that $BC = 1$. Find the perimeter of triangle KTL.

2. (8) A perpendicular bisector to the side AC of triangle ABC meets BC, AB at points A_1 and C_1 respectively. Points O, O_1 are the circumcenters of triangles ABC and A_1BC_1 respectively. Prove that $C_1O_1 \perp AO$.

3. (8) Altitudes AA_1, CC_1 of acute-angled triangle ABC meet at point H; B_0 is the midpoint of AC. A line passing through B and parallel to AC meets B_0A_1, B_0C_1 at points A', C' respectively. Prove that AA', CC', and BH concur.

4. (8) Let $ABCD$ be a square with center O, and P be a point on the minor arc CD of its circumcicle. The tangents from P to the incircle of the square meet CD at points M and N. The lines PM and PN meet segments BC and AD respectively at points Q and R. Prove that the median of triangle OMN from O is perpendicular to the segment QR end equals to its half.

5. (8–9) Five points are given in the plane. Find the maximum number of similar triangles whose vertices are among these five points.

6. (8–9) Three circles Γ_1, Γ_2, Γ_3 are inscribed into an angle (the radius of Γ_1 is the minimal, and the radius of Γ_3 is the maximal) in such a way that Γ_2 touches Γ_1 and Γ_3 at points A and B respectively. Let l be a tangent at A to Γ_1. Consider circles ω touching Γ_1 and l. Find the locus of meeting points of common internal tangents to ω and Γ_3.

7. (8–9) The incircle of triangle ABC centered at I touches CA, AB at points E, F respectively. Let points M, N of line EF be such that $CM = CE$ and $BN = BF$. Lines BM and CN meet at point P. Prove that PI bisects segment MN.

8. (8–9) Let ABC be an isosceles triangle ($AB = BC$), and l be a ray from B. Points P and Q of l lie inside the triangle in such a way that $\angle BAP = \angle QCA$. Prove that $\angle PAQ = \angle PCQ$.

9. (8–9) Points E and F lying on sides BC and AD respectively of a parallelogram $ABCD$ are such that $EF = ED = DC$. Let M be the midpoint of BE, and MD meet EF at G. Prove that $\angle EAC = \angle GBD$.

10. (8–9) Prove that two isotomic lines of a triangle cannot meet inside its medial triangle. (*Two lines are isotomic lines of triangle ABC if their common points with BC, CA, AB are symmetric with respect to the midpoints of the corresponding sides.*)

11. (8–9) The midpoints of four sides of a cyclic pentagon were marked, after this the pentagon was erased. Restore it.

12. (8–10) Suppose we have ten coins with radii $1, 2, 3, \ldots, 10$ cm. We can put two of them on the table in such a way that they touch each other, after that we can add the coins in such a way that each new coin touches at least two of previous ones. The new coin cannot cover a previous one. Can we put several coins in such a way that the centers of some three coins are collinear?

13. (9–11) In triangle ABC with circumcircle Ω and incenter I, point M bisects arc BAC and line $\overline{AI}$ meets Ω at $N \neq A$. The excircle opposite to A touches side $\overline{BC}$ at point E. Point $Q \neq I$ on the circumcircle of $\triangle MIN$ is such that $\overline{QI} \parallel \overline{BC}$. Prove that the lines $\overline{AE}$ and $\overline{QN}$ meet on Ω.

14. (9–11) Let γ_A, γ_B, γ_C be excircles of triangle ABC, touching the sides BC, CA, AB respectively. Let l_A denote the common external tangent to γ_B and γ_C distinct from BC. Define l_B, l_C similarly. The tangent from a point P of l_A to γ_B distinct from l_A meets l_C at point X. Similarly the tangent from P to γ_C meets l_B at Y. Prove that XY touches γ_A.

15. (9–11) Let $APBCQ$ be a cyclic pentagon. A point M inside triangle ABC is such that $\angle MAB = \angle MCA$, $\angle MAC = \angle MBA$ and $\angle PMB = \angle QMC = 90°$. Prove that AM, BP, and CQ concur.

16. (9–11) Let circles Ω and ω touch internally at point A. A chord BC of Ω touches ω at point K. Let O be the center of ω. Prove that the circle BOC bisects segment AK.

17. (9–11) Let ABC be an acute-angled triangle. Points A_0 and C_0 are the midpoints of minor arcs BC and AB respectively. A circle passing through A_0 and C_0 meets AB and BC at points P and S, Q and R respectively (all these points are distinct). It is known that $PQ \parallel AC$. Prove that $A_0P + C_0S = C_0Q + A_0R$.

18. (10–11) Let ABC be a scalene triangle, AM be the median through A, and ω be the incircle. Let ω touch side BC at point T, and segment AT meet ω for the second time at point S. Let δ be the triangle formed by lines AM and BC and the tangent to ω at S. Prove that the incircle of triangle δ is tangent to the circumcircle of triangle ABC.

19. (10–11) A point P lies inside a convex quadrilateral $ABCD$. Common internal tangents to the incircles of triangles PAB and PCD meet at point Q, and common internal tangents to the incircles of triangles PBC and PAD meet at point R. Prove that P, Q, R are collinear.

20. (10–11) The mapping f assigns a circle to every triangle in the plane so that the following conditions hold. (We consider only nondegenerate triangles and circles of nonzero radius.)

    (a) Let σ be any similarity in the plane and let σ map triangle Δ_1 onto triangle Δ_2. Then σ also maps circle $f(\Delta_1)$ onto circle $f(\Delta_2)$.

    (b) Let A, B, C, and D be any four points in general position. Then circles $f(ABC)$, $f(BCD)$, $f(CDA)$, and $f(DAB)$ have a common point.

    Prove that for any triangle Δ, the circle $f(\Delta)$ is the Euler circle of Δ.

21. (10–11) A trapezoid $ABCD$ is bicentral. The vertex A, the incenter I, the circumcircle ω and its center O are given and the trapezoid is erased. Restore it using only a ruler.

22. (10–11) A convex polyhedron and a point K outside it are given. For each point M of a polyhedron construct a ball with diameter MK. Prove that there exists a unique point on a polyhedron which belongs to all such balls.

23. (10–11) Six points in general position are given in the space. For each two of them color red the common points (if they exist) of the segment between these points and the surface of the tetrahedron formed by four remaining points. Prove that the number of red points is even.

24. (11) A truncated trigonal pyramid is circumsribed around a sphere touching its bases at points T_1, T_2. Let h be the altitude of the pyramid, R_1, R_2 be the circumradii of its bases, and O_1, O_2 be the circumcenters of the bases. Prove that

$$R_1 R_2 h^2 = (R_1^2 - O_1 T_1^2)(R_2^2 - O_2 T_2^2).$$