XVII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Solutions. Final round. First day. 8 form

1. (B.Frenkin) Let $ABCD$ be a convex quadrilateral. The circumcenter and the incenter of triangle ABC coincide with the incenter and the circumcenter of triangle ADC respectively. It is known that $AB = 1$. Find the remaining sidelengths and the angles of $ABCD$.

   Answer. $BC = CD = DA = 1$, $\angle A = \angle C = 72^\circ$, $\angle B = \angle D = 108^\circ$.

   Solution. Since the incenters of triangles ABC and ADC lie on the perpendicular bisector to AC these triangles are isosceles. Also since the circumcenters lie outside these triangles angles B and D are obtuse. Let O be the circumcenter of triangle ABC. Theb $\angle AOC = 360^\circ - 2\angle B$. On the other hand since O is the incenter of triangle ADC, we have $\angle AOC = 90^\circ + \angle D/2$. Similarly we obtain that $360^\circ - 2\angle D = 90^\circ + \angle B/2$, which yields that $\angle B = \angle D = 108^\circ$ and $ABCD$ is a rhombus (fig. 8.1).

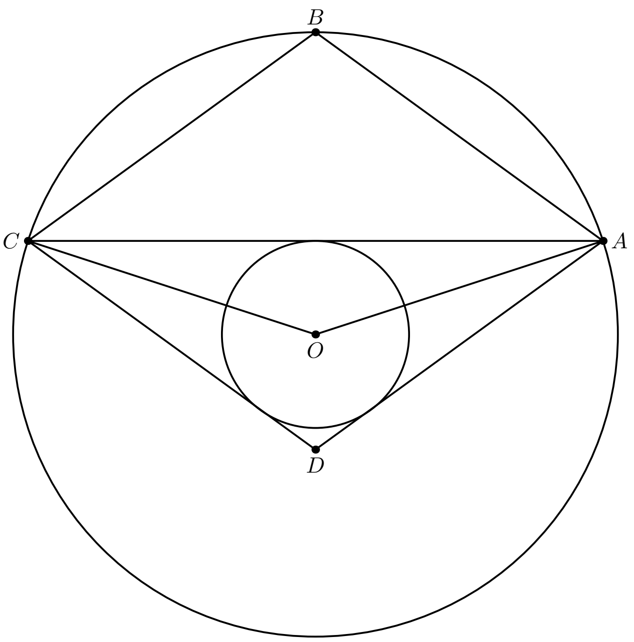

Fig. 8.1.

2. (P.Kozhevnikov) Three parallel lines l_a, l_b, l_c pass through the vertices of triangle ABC. A line a is the reflection of altitude AH_a about l_a. Lines b, c are defined similarly. Prove that a, b, c are concurrent.

   Solution. Since the angle between a and b equals the angle between the altitudes we obtain that these lines meet at the circle which is the reflection of the circle ABH about AB (H is the orthocenter of ABC), i.e. their common point lies on the circumcircle of ABC. The line c meets the circumcircle at the same point (fig. 8.2).

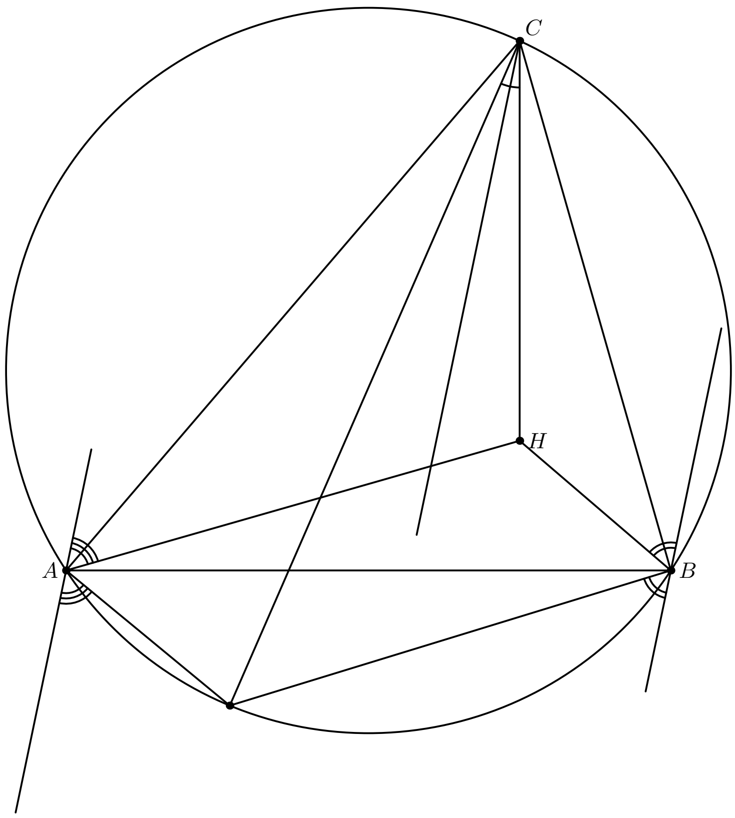

Fig. 8.2.

3. (A.Zaslavsky) Three cockroaches run along a circle in the same direction. They start simultaneously from a point S. Cockroach A runs twice as slow than B, and three times as slow than C. Points X, Y on segment SC are such that $SX = XY = YC$. The lines AX and BY meet at point Z. Find the locus of centroids of triangles ZAB.

   Answer. The center O of the given circle.

   Solution. Let points U, V lie on the line AB in such a way that $UA = AB = BV$. Then lines US and CV pass through Z, and parallel lines passing through A and B respectively meet at the centroid M of triangle ABZ. Since $UA = AS$, $VB = BC$, we obtain that $\angle AUS = \angle ASU = \angle MAB = \angle MBA$ and $\angle AMB = \angle UAS = \angle ASC = 2\angle ASB = \angle AOB$. Thus M coincides with O (fig. 8.3).

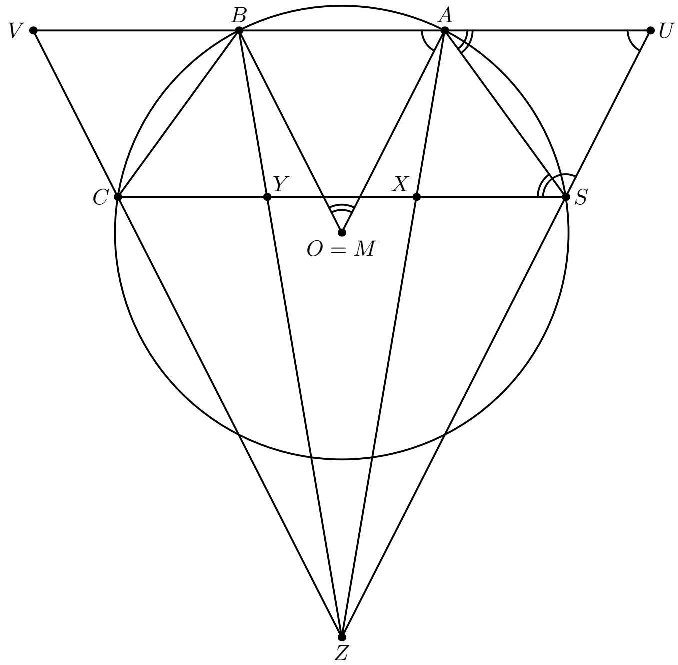

Fig. 8.3.

Remark. The solution dots not change if segments AX and BY intersect.

4. (I.Kukharchuk) Let A_1 and C_1 be the feet of altitudes AH and CH of an acute-angled triangle ABC. Points A_2 and C_2 are the reflections of A_1 and C_1 about AC. Prove that the distance between the circumcenters of triangles C_2HA_1 and C_1HA_2 equals AC.

   Solution. Let M be the midpoint of AC, and B_1 be the foot of altitude from B. Then $MA_1 = MC_1 = MA_2 = MC_2 = MA$, and lines A_1C_2, A_2C_1 meet at B_1. Therefore the perpendicular bisector to A_2C_1 coincide with the perpendicular from M to B_1C_1 which is parallel to the radius OA of the circumcircle of triangle ABC. Hence it meets the perpendicular to AC from A at point P such that $AP = OM = BH/2$. The perpendicular bisector to C_1H also passes through this point, thus P is the circumcenter of triangle C_1HA_2 (fig. 8.4). Similarly the circumcenter of triangle A_2HC_1 coincide with point Q lying on the perpendicular to AC from C and such that $CQ = OM$. Since $APQC$ is a rectangle, we obtain that $PQ = AC$.

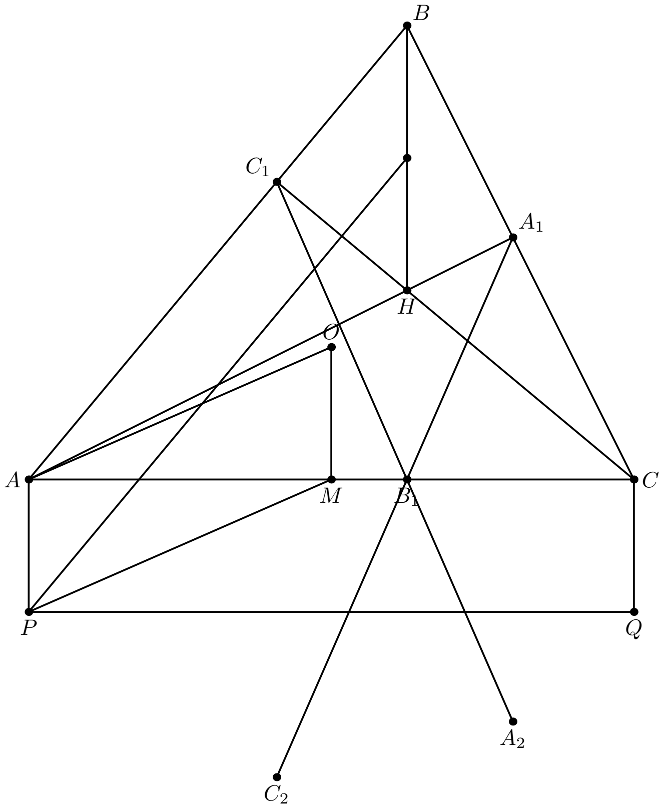

Fig. 8.4.

XVII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Solutions. Final round. Second day. 8 form

5. (M.Saghafian) Points A_1, A_2, A_3, A_4 are not concyclic, the same for points B_1, B_2, B_3, B_4. For all i, j, k the circumradii of triangles $A_iA_jA_k$ and $B_iB_jB_k$ are equal. Can we assert that $A_iA_j = B_iB_j$ for all i, j?

   Answer. No.

   First solution. Let $A_1A_2A_3$, $B_1B_2B_3$ be two non-congruent triangles with equal circumradii R, and A_4, B_4 be their orthocenters. Then the circumradii of all triangles $A_iA_jA_k$ and $B_iB_jB_k$ are equal to R, but several equalities $A_iA_j = B_iB_j$ are not correct.

   Second solution. Let $A_1B_1A_2B_2$ be a rectangle, points A_3, A_4 lie on a line parallel to A_1B_2 and are symmetric with respect the center of this rectangle, B_3 coincide with A_4, B_4 coincide with A_3 and $A_3A_4 \neq A_1A_2$. Then A_1, A_2, A_3, A_4 (B_1, B_2, B_3, B_4) are not concyclic, triangles $A_iA_jA_k$ and $B_iB_jB_k$ are congruent for all i, j, k, but $A_1A_3 \neq B_1B_3$.

6. (M.Didin) Let ABC be an acute-angled triangle. Point P is such that $AP = AB$ and $PB \parallel AC$. Point Q is such that $AQ = AC$ and $CQ \parallel AB$. Segments CP and BQ meet at point X. Prove that the circumcenter of triangle ABC lies on the circle (PXQ).

   Solution. Let D be the vertex of parallelogram $ABDC$. Then $APDC$ and $AQDB$ are isosceles trapezoids. Therefore the perpendicular bisectors to segments PD and QD coincide with the perpendicular bisectors to AC and AB respectively, the circumcenter O of triangle ABC is also the circumcenter of DPQ and $\angle POQ = 2\angle A$. Also since $\angle XPD = \angle ADP$, $\angle XQD = \angle ADQ$ we obtain that $\angle PXQ = 2\angle A$ (fig.8.6). Thus O, P, Q, X are concyclic.

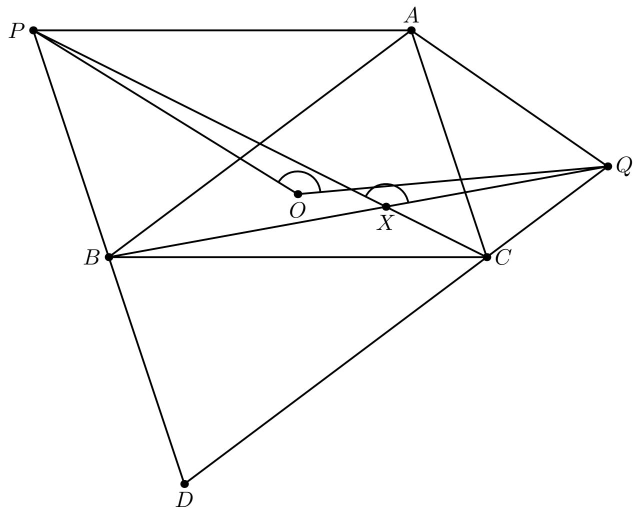

Fig. 8.6.

7. (I.Kulharchuk) Let $ABCDE$ be a convex pentagon such that angles CAB, BCA, ECD, DEC and AEC are equal. Prove that CE bisects BD.

   Solution. From the assumption we have $CD \parallel AE$. Let the line passing through B and parallel to AE meet AC and CE at points P and Q respectively. Then P and Q divide the bases CA and CE of similar isosceles triangles ABC and CDE in the same ratio. hence $\angle CBQ = \angle CDQ$, $BCDQ$ is a parallelogram, and the midpoints of segments BD and CQ coincide (fig. 8.7).

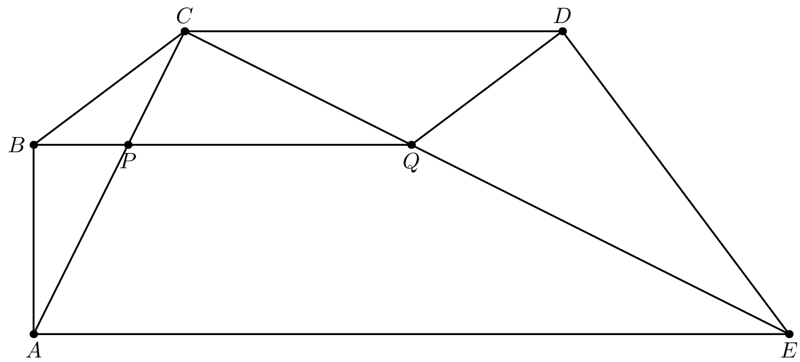

Fig. 8.7.

8. (S.Berlov) Does there exist a convex polygon such that all its sidelengths are equal and all triangle formed by its vertices are obtuse-angled?

   Answer. No.

   Solution. Suppose the opposite. Let the sidelengths of the polygon are equal to 1. Suppose that the side AB is horizontal and the polygon lies above it. Consider the stripe between the perpendiculars to segment AB at its endpoints. Since the angles A and B are obtuse the vertices adjacent with A and B lie on the different sides of this stripe. Hence there is a vertex C lying inside the stripe. Only angle C of triangle ABC may be obtuse, thus the distance from C to AB is less than $1/2$. At least one of two vertices adjacent with C lie below than C, Let this is a right vertex. Consider the most right vertex D of the polygon. Two adjacent vertices lie between AB and parallel line passing through C (fig. 8.8). Since the distance between these lines is less than $1/2$, the angles between the corresponding sides and the vertical are greater than 60°, therefore $\angle D < 60^\circ$ contradiction.

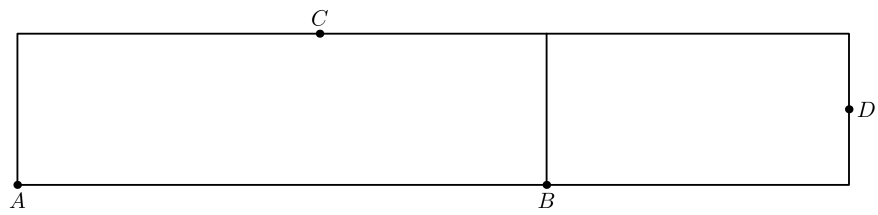

Fig. 8.8.

XVII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Solutions. Final round. First day. 9 form

1. (F.Ivlev, A.Mardanov) Three cevians concur at a point lying inside a triangle. The feet of these cevians divide the sides into six segments, and the lengths of these segments form (in some order) a geometric progression. Prove that the lengths of the cevians also form a geometric progression.

   Solution. Suppose the the minimal length of the segments equals 1. Then the remaining lengths are q, q^2, q^3, q^4 and q^5, where $q \geq 1$ is the denominator of the progression. By the Ceva theorem the product of several three of these numbers equals the product of the remaining ones i.e. $\sqrt{q^{15}}$. This is possible only if $q = 1$. Thus the given triangle is regular and the cevians are its medians, i.e. their lengths are equal.

2. (M.Volchkevich) A cyclic pentagon is given. Prove that the ratio of its square to the sum of the diagonals is not greater than the quarter of the circumradius.

   Solution. Let $A_1A_2A_3A_4A_5$ be a cyclic pentagon with circumcenter O. Then for each $i = 1, \dots, 5$ $S_{OA_{i-1}A_iA_{i+1}} \leq OA_i \cdot A_{i-1}A_{i+1}/2$ (we suppose that $A_{i+5} = A_i$). The sum of these five areas is not less than the doubled area of the pentagon which yields the required inequality.

3. (M.Didin, I.Frolov) Let ABC be an acute-angled scalene triangle and T be a point inside it such that $\angle ATB = \angle BTC = 120°$. A circle centered at point E passes through the midpoints of the sides of ABC. For B, T, E collinear find angle ABC.

   Answer. 30°

   Solution. Let A_0, B_0, C_0 be the midpoints of BC, CA, AB respectively and D be the vertex of a regular triangle ACD lying outside ABC. It is known that T lies on BD. The homothety with center B and coefficient $1/2$ maps the line B_0D to the perpendicular bisector to A_0C_0, therefore E is the midpoint of BD and $\angle C_0EA_0 = 60°$ (fig.9.3). Thus $\angle ABC = \angle A_0B_0C_0 = 30°$.

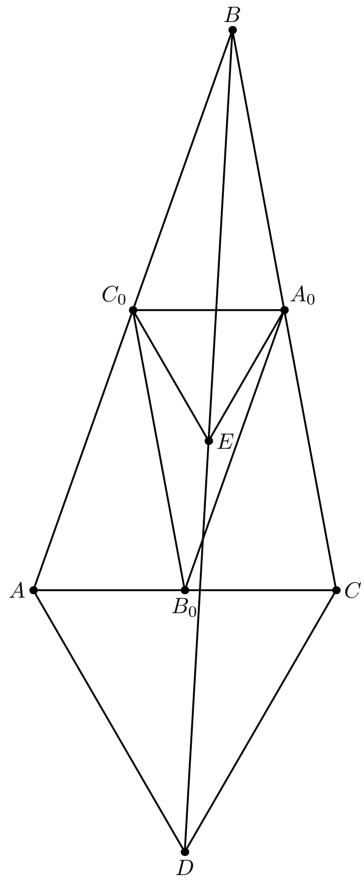

Fig. 9.3.

4. (M.Saghafian) Define the distance between two triangles to be the closest distance between two vertices, one from each triangle. Is it possible to draw five triangles in the plane such that for any two of them, their distance equals the sum of their circumradii?

   Answer. No.

   Solution. Call *a cloud* of triangle the union of three discs centered at its vertices with radii equal to its circumradius. The distance between two triangles equals the sum of their circumradii if and only if the corresponding clouds touche. But five pairwise touching clouds do not exist because the graph K_5 is not planar.

XVII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Solutions. Final round. Second day. 9 form

5. (P.Kozhevnikov) Let O be the circumcenter of triangle ABC. Points X and Y on side BC are such that $AX = BX$ and $AY = CY$. Prove that the circumcircle of triangle AXY passes through the circumcenters of triangles AOB and AOC.

   Solution. By the assumption we obtain that OX is the perpendicular bisector to AB, i.e. the circumcenter O_1 of triangle AOB lies on OX and $\angle AO_1X = \angle AO_1B/2 = \pi - 2\angle C = \angle AYX$ (fig.9.5). Another dispositions of points are considered similarly.

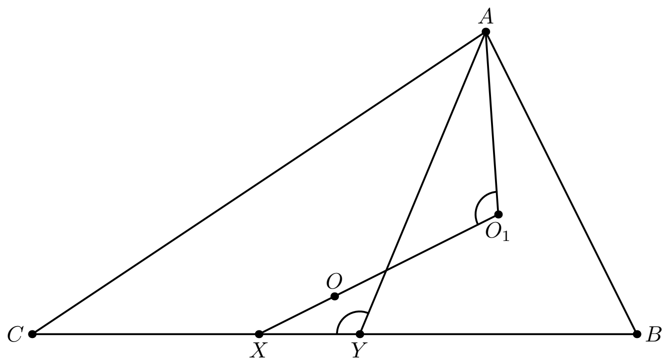

Fig. 9.5.

6. (P.Ryabov) The diagonals of trapezoid $ABCD$ ($BC \parallel AD$) meet at point O. Points M and N lie on the segments BC and AD respectively. The tangent to the circle AMC at C meets the ray NB at point P; the tangent to the circle BND at D meets the ray MA at point R. Prove that $\angle BOP = \angle AOR$.

   Solution. Note that $\angle NBD = \angle ADR$ and $\angle MAC = \angle BCP$ (fig.9.6). Therefore points P and R are isogonally conjugated in similar triangles BOC and AOD, which yields the required equality.

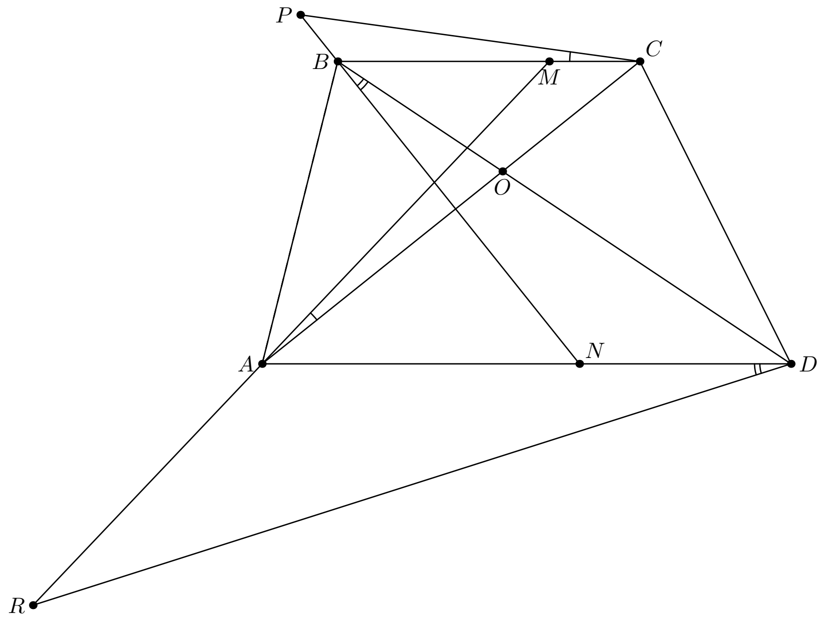

Fig. 9.6.

7. (M.Didin, F.Ivlev, I.Frolov) Three sidelines of on acute-angled triangle are drawn on the plane. Fyodor wants to draw the altitudes of this triangle using a ruler and a compass. Ivan obstructs him using an eraser. For each move Fyodor may draw one line through two marked points or one circle centered at a marked point and passing through another marked point. After this Fyodor may mark an arbitrary number of points (the common points of drawn lines, arbitrary points on the drawn lines or arbitrary points on the plane). For each move Ivan erases at most three of marked point. (Fyodor may not use the erased points in his constructions but he may mark them for the second time). They move by turns, Fydors begins. Initially no points are marked. Can Fyodor draw the altitudes?

   Answer. Yes.

   Solution. Note that Fyodor may mark sufficiently much points on the given line, draw circles centered at these points and mark the common point of these circles, and finally drawing the lines through these common points he obtain sufficiently much perpendicular to the given line. Repeating these operations for a perpendicular line Fyodor may obtain sufficiently much lines parallel to the given one. Thus he may construct many lines parallel to the side AB of given triangle ABC and mark their common points A_i, B_i with BC, AC respectively. Now drawing circles centered at AC and passing through A_i and marking the common points of such circles Fyodor may construct the reflections of A_i about AC and the perpendiculars from A_i

to AC. The perpendiculars from B_i to BC may be constructed similarly. The common points of these perpendiculars — the orthocenters of triangles CA_iB_i lie on the altitude from C.

8. (A.Dadgarnia) A quadrilateral $ABCD$ is circumscribed around a circle ω centered at I. Lines AC and BD meet at point P, lines AB and CD meet at point E, lines AD and BC meet at point F. Point K on the circumcircle of triangle EIF is such that $\angle IKP = 90°$. The ray PK meets ω at point Q. Prove that the circumcircle of triangle EQF touches ω.

   Solution. Let W, X, Y, Z be the touching points of AB, BC, CD, DA with ω. Then P is the common point of diagonals of quadrilateral $WXYZ$. The inversion about ω maps E, F to the midpoints M, N of these diagonals and maps the circle IEF to the Gauss line MN. Since K lies on the circle with diameter IP its image K' lies on the polar of P — the line EF, which is the radical axis of ω and the circle with diameter IP passing through M, N. This inversion maps PK to the circle with diameter IK', therefore $K'Q$ touches ω (fig. 9.8), thus it touches the circle MNQ.

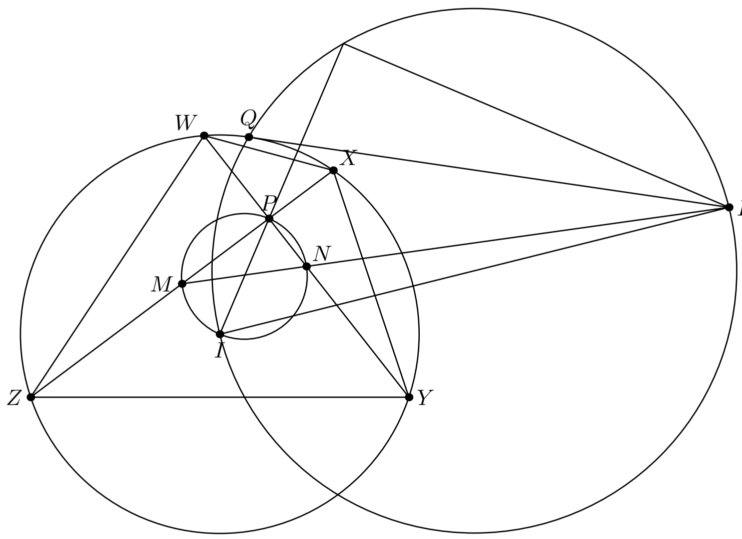

Fig. 9.8.

XVII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Solutions. Final round. First day. 10–11 form

1. (D.Shvetsov) Let CH be an altitude of right-angled triangle ABC ($\angle C = 90°$), HA_1, HB_1 be the bisectors of angles CHB, AHC respectively, and E, F be the midpoints of HB_1 and HA_1 respectively. Prove that the lines AE and BF meet on the bisector of angle ACB.

   First solution. Let M be the midpoint of bisector CL of triangle ABC. Then from the similarity of triangles ABC, ACH and CHB we obtain that $\angle BAF = \angle MAC$, $\angle ABE = \angle MBC$. Therefore the common point of lines AE and BF is isogonally conjugated to M, i.e. it lies on CL (fig. 10.1).

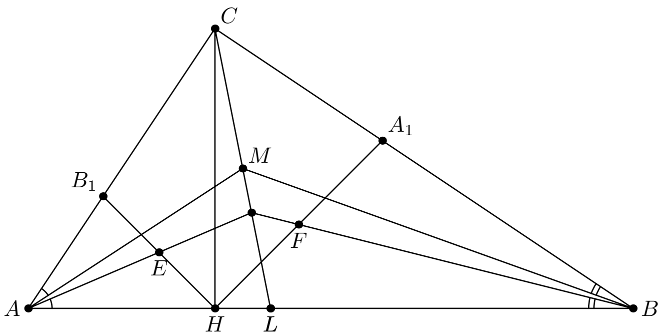

Fig. 10.1.

   Second solution. Since AE is a median of triangle AHB_1 we have $\sin\angle B_1AE : \sin\angle HAE = AH : AB_1 = (AH + CH) : AC$ (the second equality follows from the bisector property). Similarly $\sin\angle A_2BF : \sin\angle HBF = (CH + HB) : BC$. From the similarity of triangles AHC and CHB we obtain that these ratios are equal and by the Ceva theorem AE and BF meet on the bisector.

2. Let ABC be a scalene triangle, and A_0, B_0, C_0 be the midpoints of BC, CA, AB respectively. The bisector of angle C meets A_0C_0 and B_0C_0 at points B_1 and A_1 respectively. Prove that the lines AB_1, BA_1 and A_0B_0 concur.

   Solution. Points A_1, B_1 are the projections of A, B to the bisector CL, i.e. AB_1BA_1 is a trapezoid. Hence the common point T of AB_1 and A_1B, L, and the midpoints of segments AA_1, BB_1 are collinear and form a harmonic quadruple (fig. 10.2). Projecting these points to CL by the lines parallel to AB and using the homothety with center L and coefficient 2 we obtain the harmonic quadruple C, L, A_1, B_1. Thus T lies on A_0B_0.

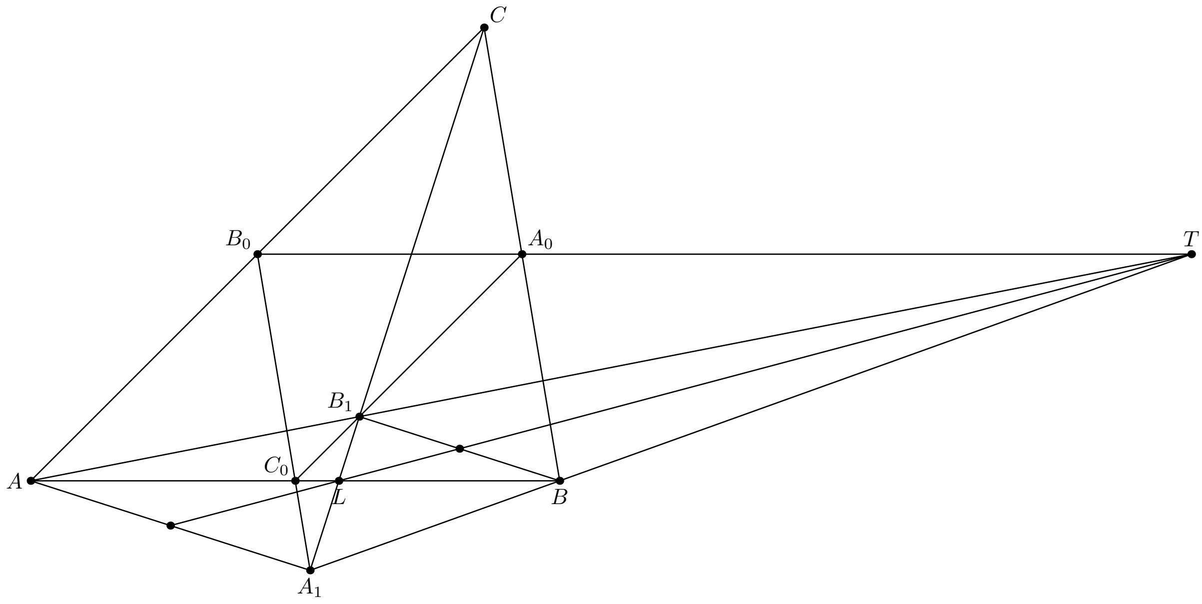

Fig. 10.2.

3. K.Knop, G.Chelnokov) The bisector of angle A of triangle ABC ($AB > AC$) meets its circumcircle at point P. The perpendicular to AC from C meets the bisector of angle A at point K. A circle with center P and radius PK meets the minor arc PA of the circumcircle at point D. Prove that the quadrilateral $ABDC$ is circumscribed.

   Solution. The arc AB not containing C is greater than the arc AC not containing B because $AB > AC$. The minor arc BP equals the minor arc CP because AP is a bisector. therefore the arc ACP is less than $180°$ and K lies on the chord EC where E is opposite to A. From this we obtain that $PK < PC$ and so D lie on the minor arc PC.

   We have to prove that $AB + DC = AC + BD$.

   Let the circle centered at P with radius PD meet for the second time DB at point M. Construct the perpendicular PH from P to BD. By the Archimedes lemma H bisects the length of the broken line BDC, i.e. $BH = HD + DC$, but since $MH = HD$ we obtain that $BM = DC$.

   Consider the reflection N of C about the bisector AP. We have $AC = AN$, also $PC = PN$, i.e C, N, B lie on a circle centered at P. Now wt have to prove that $BN = DM$.

   Let PL be the perpendicular to EC and PR be the perpendicular to BA. Then $BN = 2BR = 2PL = 2DH = DM$ (fig.10.3).

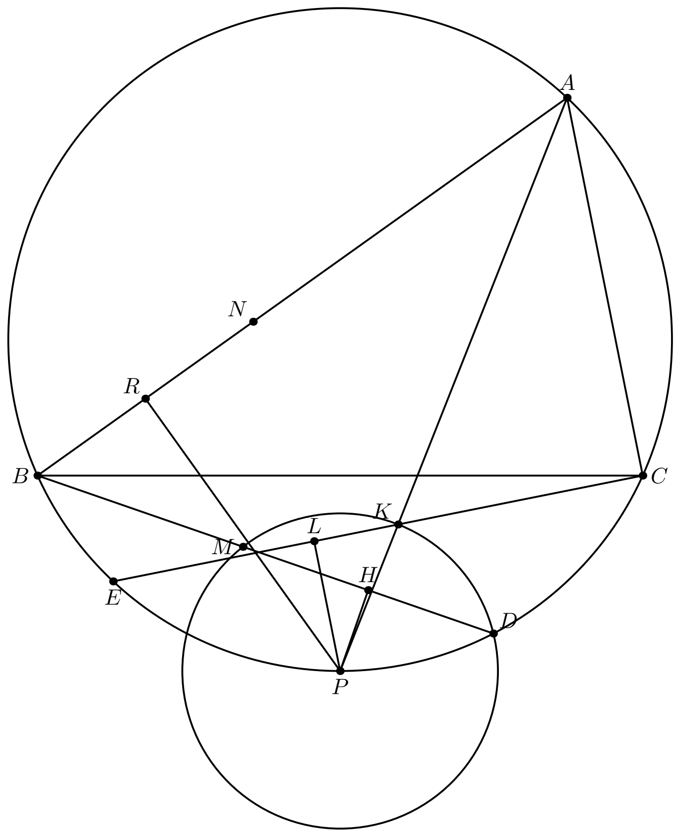

Fig. 10.3.

The second equality is correct because the triangles PRB and CLP are congruent ($\angle BPR = 90^\circ - \angle PBA = \angle PCE = \angle PCL$). The third equality is correct because the triangles PLK and DHP are congruent ($\angle LPK = \angle PAC = \angle PAB = \angle PDH$).

4. (T.Korchyomkina) Can a triangle be a development of a quadrangular pyramid?

   Answer. Yes.

   Solution. Take a triangle SAB with $SA = SB > AB$. Let S'be the midpoint of AB, A', B' be the points on the sides SA, SB such that $SA' = SB' = S'A$, and C, D be the midpoints of BB', AA' respectively. Then we can bend the triangle by segments $A'B'$ and CD in such a way that S will be joined with S'. Now the triangles SAD, $SA'D$, SBC and $SB'C$ are congruent, hence we can bending by SC and SD join A with A' and B with B'. As result we obtain the pyramid $SABCD$ (fig.10.4).

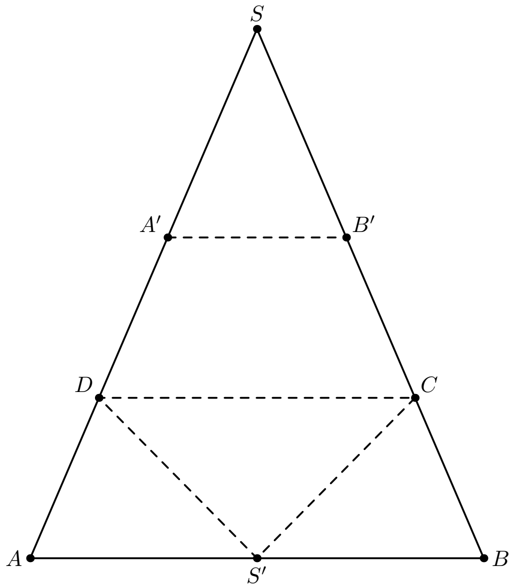

Fig. 10.4.

XVII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10–11 form

5. (P.Kozhevnikov) A secant meets one circle at points A_1, B_1, this secant meets a second circle at points A_2, B_2. Another secant meets the first circle at points C_1, D_1 and meets the second circle at points C_2, D_2. Prove that points $A_1C_1 \cap B_2D_2$, $A_1C_1 \cap A_2C_2$, $A_2C_2 \cap B_1D_1$, $B_2D_2 \cap B_1D_1$ lie on a circle coaxial with two given circles.

   Solution. Let X be the common point of A_1C_1 and A_2C_2. Then the degree of X with respect to the first circle equals to $XA_1 \cdot XC_1$, and its degree with respect to the second circle equals $XA_2 \cdot XC_2$. The ratio of these degrees equals (fig. 10.5)

$$\frac{XA_1}{XA_2}\frac{XC_1}{XC_2} = \frac{\sin\angle B_2A_2C_2}{\sin\angle B_1A_1C_1}\frac{\sin\angle D_2C_2A_2}{\sin\angle D_1C_1A_1} = \frac{A_2D_2 \cdot B_2C_2}{R_2^2} : \frac{A_1D_1 \cdot B_1C_1}{R_1^2}.$$

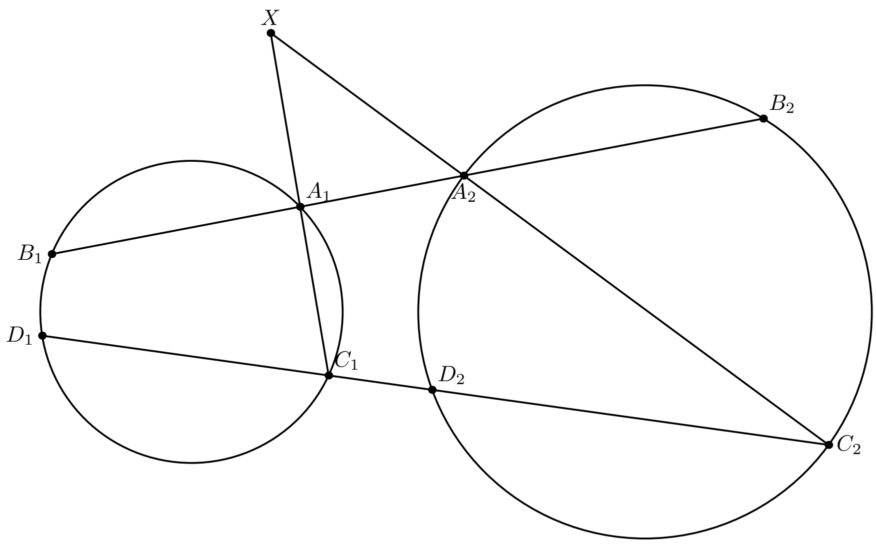

Fig. 10.5.

For the remaining three points the ratios of degrees are the same. Buy the locus of points with fixed ratio of degrees is a circle coaxial with two given ones.

6. (D.Brodsky) The lateral sidelines AB and CD of trapezoid $ABCD$ meet at point S. The bisector of angle ASC meets the bases of the trapezoid at points K and L (K lies inside segment SL). Point X is chosen on segment

SK, and point Y is selected on the extension of SL beyond L in such a way that $\angle AXC - \angle AYC = \angle ASC$. Prove that $\angle BXD - \angle BYD = \angle BSD$.

Solution. Let C' be the reflection of C about SX and Y' be such point on ray CX that $SX \cdot SY' = SB \cdot SD = SA \cdot SC$. Then $SX \cdot SY' = SC' \cdot SA$, i.e. X, Y', A, C' are concyclic (fig. 10.6). Therefore $\angle AY'S = \angle SC'X = \angle SCX$. Similarly $\angle XY'C = \angle SAX$, thus $\angle AXC = \angle SAX + \angle SCX + \angle ASC = \angle AY'C + \angle ASC$ and Y' coincides with Y. Similarly we obtain that $\angle ASD = \angle BXD - \angle BYD$.

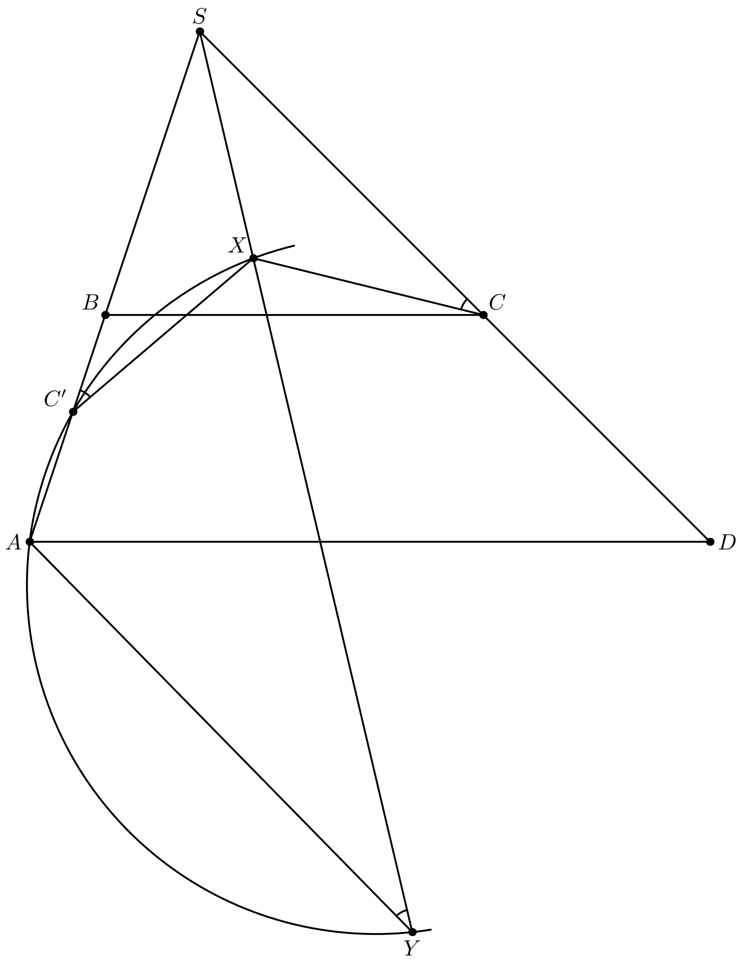

Fig. 10.6.

Remark. We can also prove that $\angle XAY = \angle XCY$, $\angle XBY = \angle XDY$.

7. (M.Etesamifard) Let I be the incenter of a right-angled triangle ABC, and M be the midpoint of hypothenuse AB. The tangent to the circumcircle of ABC at C meets the line passing through I and parallel to AB at point P. Let H be the orthocenter of triangle PAB. Prove that lines CH and PM meet at the incircle of triangle ABC.

Solution. Since the quadruple A, B, H, P is orthocentric we obtain that H lies on the polar of P with respect to the circle with diameter AB, i.e. the circumcircle of ABC. It is clear that C also lies on this polar, therefore $MP \perp CH$. Prove that CH passes through the homothety center Q of the incircle and the circumcircle of ABC.

Let S be the midpoint of arc AB of the circumcircle, T be the projection of M to PI, and T' be the reflection of T about the circumcircle. Since T and C lie on the circle with diameter MP, we obtain that T' lies on the inversion image of this circle — the line CH. Note that T', C, Q lie on lines MS, SI, MI respectively and $T'M : T'S = R^2/r : (R^2/r + R) = R : (R + r)$, $CS : CI = (R+r) : r$ (because $SI = SA = SB = R\sqrt{2}$), $QI : QM = r : R$. By the Menelaos theorem Q, T', C are collinear.

Now let F be the Feuerbach point of triangle ABC. Since MC is a diameter of nine-points circle, points C, Q, F are the pairwise homothety centers of three circles: the nine-points circle, the circumcircle, and the incircle. Therefore F lies on CH and since $\angle CFM = 90^\circ$, we obtain that F is the common point of CH and MP (fig. 10.7).

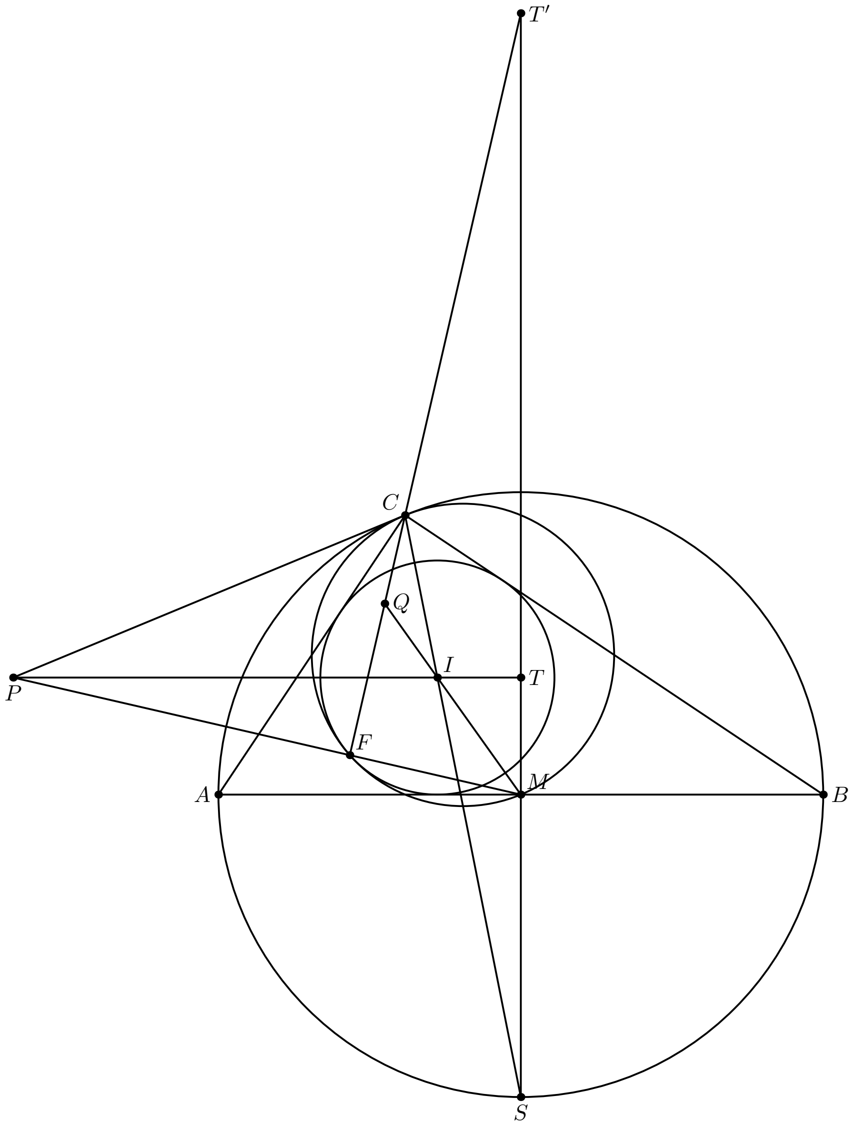

Fig. 10.7.

8. (M.Didin) On the attraction "Merry parking" the auto has only two positions of a steering wheel: "right" and "strongly right" . So the auto can move along an arc with radius r_1 or r_2. The auto started from a point A to the Nord, it covered the distance l and rotated to the angle $\alpha < 2\pi$. Find the locus of its possible endpoints.

   Solution. Since the length of the trajectory and the rotation angle are known we can find the sum of arc of each radius. Hence reformlate the problem.

   A point A and a ray ℓ with origin A are given on the plane. Also two numbers $r_1 > r_2$ and two angles α_1, α_2, $\alpha_1 + \alpha_2 < \pi$ are given. Find the locus of the endpoints B of the following trajectories Γ:

   - Γ has the begin point A and touches ℓ at A;

- Γ is the union of arcs with radii r_1 and r_2, and the sums of angle measures of these arcs are equal to $2\alpha_1$ and $2\alpha_2$ respectively;

- two adjacent arcs have the common tangent at their common endpoint and lie on the same side of this tangent.

Answer. Let O_1, O_2 be the endpoints of arcs satisfying to the first condition with radii r_1, r_2 respectively and angle measures $2(\alpha_1+\alpha_2)$. Consider two discs W_1, W_2 centered at O_1, O_2 with radii $2(r_1-r_2)\sin\alpha_2$, $2(r_1-r_2)\sin\alpha_1$ respectively. The required locus is the intersection of these discs.

Proof. Let PQ be an arbitrary arc of Γ with radius r_1, and R be a point of segment PQ such that $PR : PQ = r_2 : r_1$. An arc PR touching Γ at P has radius r_2 and the tangent to it at R is parallel to the tangent to Γ at Q. Replace the arc PQ to PR and translate the part QB of Γ to $\vec{QR}$. Repeating this operation for each arc with radius r_1 we translate B to O_2, and $\vec{O_2B} = \frac{r_1-r_2}{r_1}\sum \vec{P_iQ_i}$. Similarly $\vec{O_1B} = \frac{r_2-r_1}{r_2}\sum \vec{Q_iP}_{i+1}$ (we suppose that $Q_0 = A$, $P_{n+1} = B$).

Put off all arcs of Γ on the unit circle starting from point X. The endpoint of the last arc is a point Y such that $\smile XY = 2(\alpha_1+\alpha_2)$ and the homothety H with coefficient $1/(r_1-r_2)$ maps vector O_1O_2 to XY. Color all arcs obtained from the arcs with radius r_1 red, and color blue the remaining arcs. Compare to each arc the vector from its begin point to the endpoint. The homothety H maps vector O_2B to the sum of red vectors and maps BO_1 to the sum of blue vectors. Let Z be a point on arc XY such that $\smile XZ = 2\alpha_1$. The homothety H maps the boundary circles of W_2 and W_1 to the circles centered at X, Y respectively and passing through Z.

Prove that the length of O_2B is maximal if the red vector is unique. Let the tangent m to the unit circle parallel to O_2B toche the circle at E. Consider an arc T with length $2\alpha_1$ and midpoint E. If the endpoints of T lie on red arcs divide these arcs into two parts. The projection to m of a red vector with length φ lying outside T is less than $\varphi\cos\alpha_1$, therefore the sum of projections to m of all red vectors is less than the projection of T. Similarly the length of O_1B is maximal if the blue vector is unique. Thus B lies inside the intersection of two discs.

Clearly the common points of boundary circles of W_1 and W_2 correspond to the trajectories containing exactly one arc of each radius, and the points of these circles correspond to the trajectories containing one arc of some radius and two arc of the remaining one. Prove that each point B inside both discs correspond to the unique trajectory containing four arc and starting from the arc with radius r_1.

Let D be the image of B in homothety H. Find on arc XY such points E, F, G that $\vec{XE} + \vec{FG} = \vec{XD}$. Let E, G be the common points of the perpendicular bisector to DZ with arcs XZ, YZ, and F be the second common point of arc XY with the line passing through Z and parallel to EG. Then $DEFG$ is a parallelogram which is equivalent to the required equality. It is easy to see that E, F, G are uniquely defined by D (fig. 10.8).

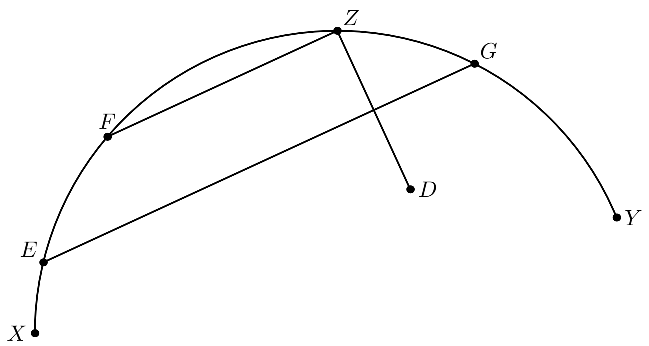

Fig. 10.8.

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