XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (I.Kukharchuk, 8) Let AA_1, CC_1 be the altitudes of triangle ABC, and P be an arbitrary point of side BC. Point Q on the line AB is such that $QP = PC_1$, and point R on the line AC is such that $RP = CP$. Prove that QA_1RA is a cyclic quadrilateral.

   Solution. It is clear that A, C, A_1, C_1 are concyclic. Denote the corresponding circle by ω_1. Furthermore the midpoints X and Y of segments QC_1 and RC are the projections of P to AB and AC respectively, thus X, Y and A_1 lie on the circle ω_2 with diameter AP. Let O be symmetric to the center of ω_1 (the midpoint of AC) about the center of ω_2. By Thales theorem, the projections of O to AB and AC are the midpoints of segments AQ and AR respectively, i.e. O is the circumcenter of triangle AQR. Since O lies on the perpendicular bisector to AA_1, the point A_1 also lies on the circle ABC (fig.1).

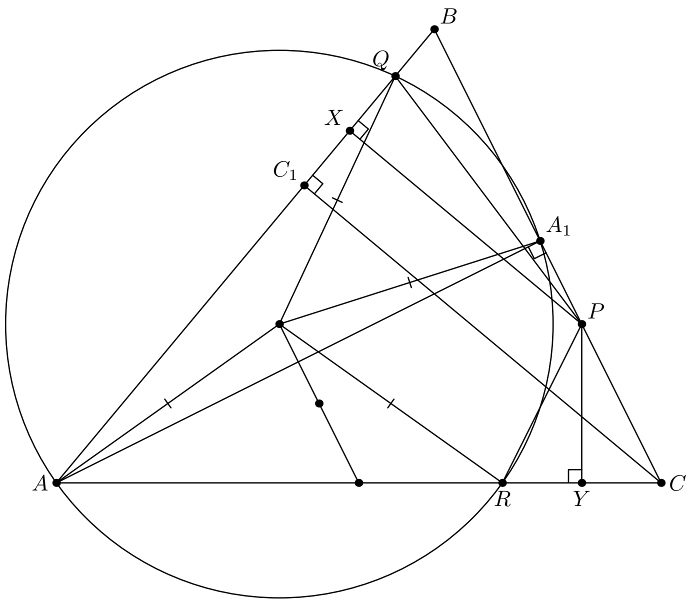

Fig. 1

2. (D.Shvetsov, 8) The circle ω_1 passes through the center O of the circle ω_2 and meets it at points A and B. The circle ω_3 centered at A with radius AB meets ω_1 and ω_2 at points C and D (distinct from B). Prove that C, O, D are collinear.

   Solution. Since the arcs AC and AB of ω_1 are congruent, we obtain that $\angle AOC = 180° - \angle AOB$. But it is clear that $\angle AOD = \angle AOB$ (fig.2), so we obtain the required assertion.

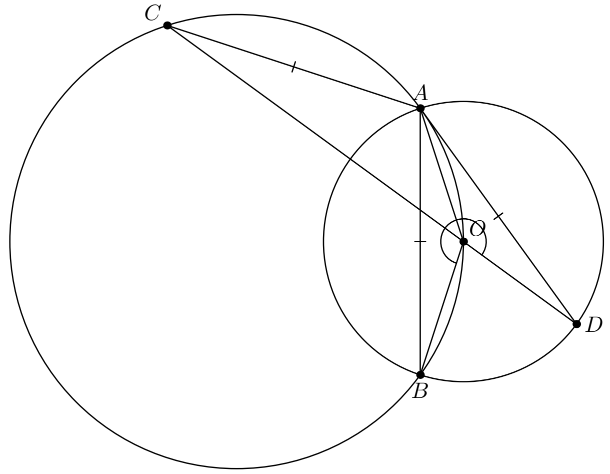

Fig. 2

3. (L.Shteingarts, 8) The rectangle $ABCD$ lies inside a circle. The rays BA and DA meet this circle at points A_1 and A_2. Let A_0 be the midpoint of A_1A_2. Points B_0, C_0, D_0 are defined similarly. Prove that $A_0C_0 = B_0D_0$.

   Solution. Let X, Y be the projections of the center of the circle to AB, CD respectively (fig.3). Then $BB_1 - AA_1 = (XB_1 - XB) - (XA_1 - XA) = AX - BX = DY - CY = CC_1 - DD_1$. Therefore the projection of segment A_0C_0 to the line AB, equal to $(A_1B_1 + C_1D_1 - AA_1 - CC_1)/2$, is congruent to the projection of segment B_0D_0 to the same line. Similarly the projections of these segments to the line AD are congruent, thus the segments are congruent too.

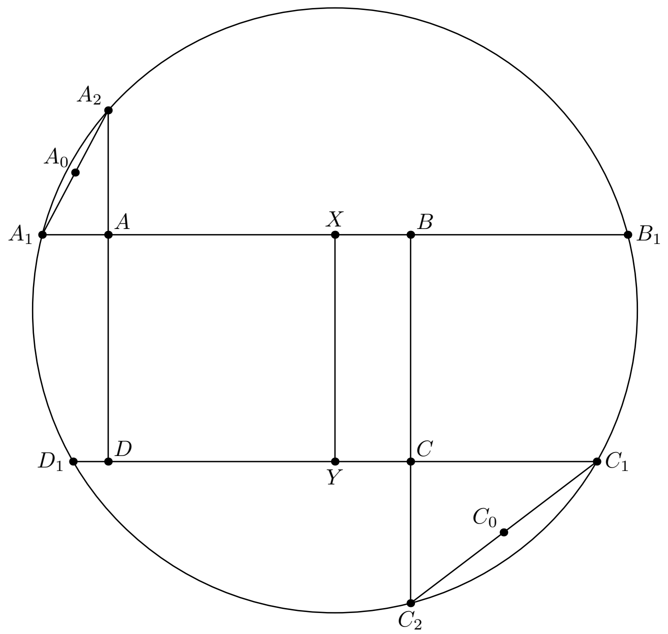

Fig. 3

4. (A.Trigub, 8) The side AB of triangle ABC touches the corresponding excircle at point T. Let J be the center of the excircle inscribed into angle A, and M be the midpoint of AJ. Prove that $MT = MC$.

Solution. Let R be the projection of J to AC. Then $CR = p - AC = AT$. Furthermore $MR = MA$ as a median of the right-angled triangle AJR, and $\angle MRA = \angle MAR = \angle MAT$ (fig.4). Hence the triangles MTA and MCR are congruent and $MT = MC$.

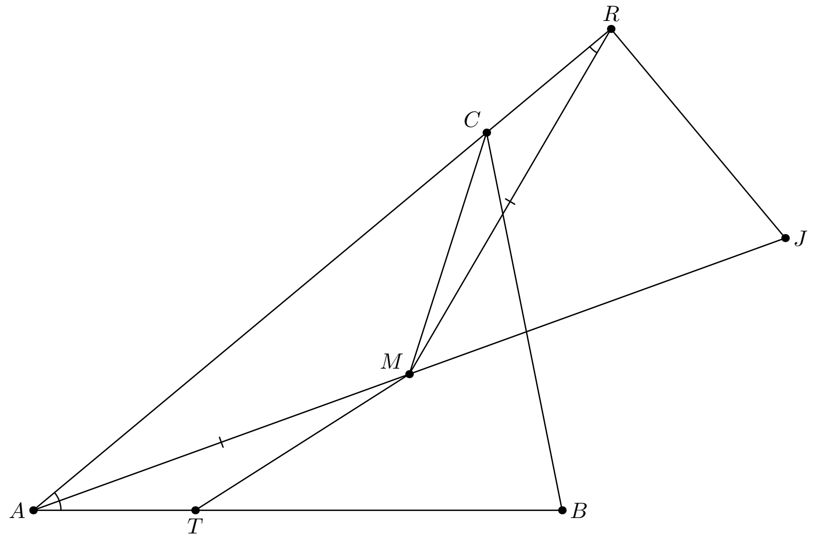

Fig. 4

5. (F.Ivlev, 8–9) Let A, B, C and D be four points in general position, and ω be a circle passing through B and C. A point P moves along ω. Let Q be the common point of circles ABP and PCD distinct from P. Find the locus of points Q.

   Solution. We have that $\angle(QA, QD) = \angle(QA, BA) + \angle(BA, DC) + \angle(DC, DQ) = \angle(QP, PB) + \angle(BA, DC) + \angle(PC, PQ) = \angle(PC, PB) + \angle(BA, DC)$ do not depend on P. Therefore the locus of Q is the circle passing through A and D.

6. (A.Akopyan, (8–9) Two quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are mutually symmetric with respect to the point P. It is known that A_1BCD, AB_1CD and ABC_1D are cyclic quadrilaterals. Prove that the quadrilateral $ABCD_1$ is also cyclic.

   Solution. We have $\angle(AD_1, D_1B) = \angle(AD_1, AB_1) + \angle(A_1B, D_1B) = \angle(A_1D, A_1B) + \angle(AB_1, B_1D) = \angle(AC, CD) + \angle(CD, BC) = \angle(AC, BC)$. Thus A, B, C, D_1 are concyclic.

7. (P.Bibikov, (8–9) Let AH_A, BH_B, CH_C be the altitudes of the acute-angled triangle ABC. Let X be an arbitrary point of segment CH_C, and P be the common point of circles with diameters H_CX and BC, distinct from H_C. The lines CP and AH_A meet at point Q, and the lines XP and AB meet at point R. Prove that A, P, Q, R, H_B are concyclic.

   Solution. Since $BCPH_C$ is a cyclic quadrilateral, we obtain $\angle CPH_C = 180° - \angle B = 180° - \angle AHH_C$, where H is the orthocenter of ABC. Hence $HQPH_C$ is cyclic, i.e. $\angle CQH = \angle HH_CP$. But $\angle HH_CP = \angle H_CRP$ because H_CP is an altitude of the right-angled triangle H_CRX. Thus A, R, P and Q are concyclic. Since H_CPH_BC is cyclic, we obtain $\angle PH_BA = \angle PH_CC = \angle PRB$, therefore H_B lies on the same circle (fig.7).

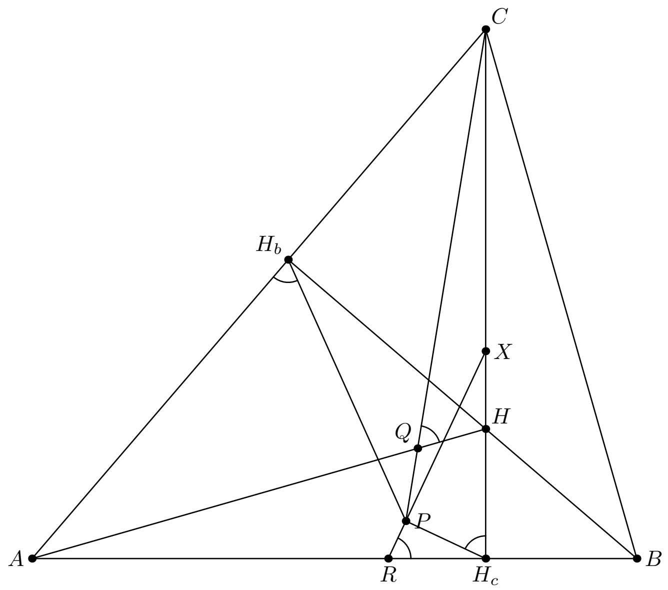

Fig. 7

8. (M.Etesamifard, 8–9) The circle ω_1 passes through the vertex A of the parallelogram $ABCD$ and touches the rays CB, CD. The circle ω_2 touches the rays AB, AD and touches ω_1 externally at point T. Prove that T lies on the diagonal AC.

   Solution. Let T' be the common point of ω_1 and the ray AC. The homothety with center T' mapping C to A maps the rays CB, CD to AD, AB respectively. Thus it maps ω_1 to circle ω' which touches these rays and is tangent to ω_1 at T'. Therefore ω' coincides with ω_2, and T' coincides with T.

9. (E.Bakaev, 8–9) Let A_M be the midpoint of side BC of an acute-angled triangle ABC, and A_H be the foot of the altitude to this side. Points B_M, B_H, C_M, C_H are defined similarly. Prove that one of the ratios $A_MA_H : A_HA$, $B_MB_H : B_HB$, $C_MC_H : C_HC$ is equal to the sum of two remaining ratios.

   Solution. Note that, for example, $C_MC_H = |AC_H - BC_H|/2$. Therefore $C_MC_H/C_HC = |\cot A - \cot B|/2$. The required assertion follows from this and two similar equalities.

10. (A.Trigub, 8–9) Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, and NP, NT be the tangents to the incircle of this triangle. The lines BP and BT meet the circumcircle for the second time at points P_1 and T_1 respectively. Prove that $PP_1 = TT_1$.

    Solution. Let I be the incenter of ABC. Since BN is the external bisector of angle B, we have $\angle IBN = 90^\circ = \angle IPN = \angle ITN$. Thus B, I, N, T, P are concyclic, and since $IT = IP$, we obtain that BI bisects angle PBT. Hence P_1 and T_1 are symmetric with respect to the diameter of the circumcircle passing through N, i.e. $NP_1 = NT_1$. Furthermore $\angle NPB = \angle NTB$ and $NP = NT$, Therefore the triangles NPN_1 and NTT_1 are congruent and we obtain the required assertion (fig.10).

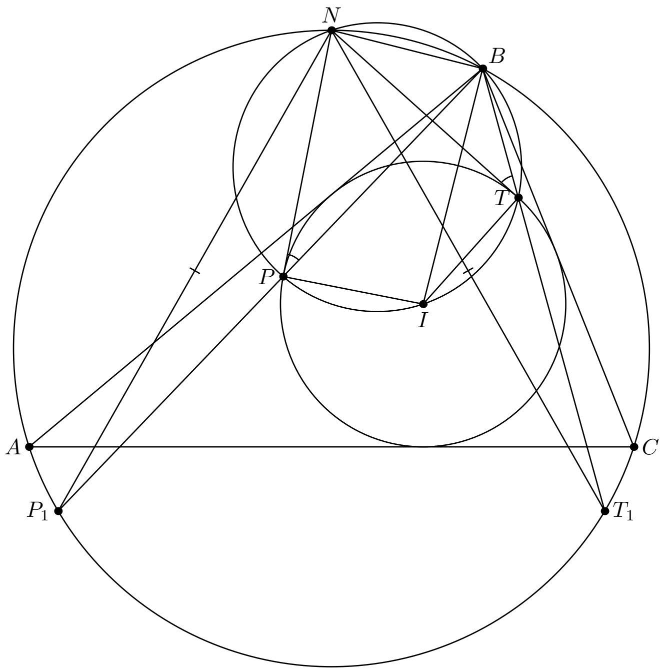

Fig. 10

11. (M.Saghafian, 8–9) Morteza marks six points in the plane. He then calculates and writes down the area of every triangle with vertices in these points (20 numbers). Is it possible that all of these numbers are integers, and that they add up to 2019?

    Answer. No.

    Solution. Consider any four of marked points. If they form a convex quadrilateral $ABCD$, then $S_{ABC} + S_{ACD} = S_{ABD} + S_{BCD}$. And if one point lies inside the triangle formed by three remaining ones, then the area of this triangle is equal to the sum of areas of three inner triangles. In both cases the sum of areas of four triangles formed by these points will be even. If we sum up all such sums then each triangle will be counted three times, therefore the sum of all 20 areas is also even.

12. (B.Frenkin, 8–11) Let $A_1A_2A_3$ be an acute-angled triangle inscribed into a unit circle centered at O. The cevians from A_i passing through O meet the opposite sides at points B_i $(i = 1, 2, 3)$ respectively.

    (a) Find the minimal possible length of the longest of three segments B_iO.

    (b) Find the maximal possible length of the shortest of three segments B_iO.

    Answer. (a), (b) 1/2.

    Solution. Firstly let us show that among two segments B_iO, the longer segment is directed to the shorter side (clearly, equality of sides implies equality of segments). Suppose, for example, that $A_1A_3 < A_2A_3$. Since $\angle OA_1A_2 = \angle OA_2A_1$, we have $\angle OA_2B_1 < \angle OA_1B_2$. For triangles A_1OB_2 and A_2OB_1, we have $A_1O = A_2O$, $\angle A_1OB_2 = \angle A_2OB_1$. Hence $BO_1 < B_2O$ as required.

    (a) Suppose that in an acute-angled triangle $A_1A_2A_3$ with the circumcircle of radius 1 the side A_1A_2 is the shortest. Then the segment B_3O is the longest among B_iO. Since $\angle A_3 \leq 60°$, we have $\angle A_1OA_2 \leq 120°$ and $\angle OA_1A_2 \geq 30°$. From O, draw the perpendicular OP to A_1A_2. Then $1/2 \le OP \le B_3O$. The equality is attained for the equilateral triangle.

b) Suppose that in an acute-angled triangle $A_1A_2A_3$ with the circumcircle of radius 1 the side A_1A_2 is the shortest, and the side A_2A_3 is the longest, so that the segment B_1O is the shortest among B_iO. Let us move point A_1 along the circumcircle towards point A_2. Then the segment B_1O will increase because it will move away from the perpendicular from O to A_2A_3. When the angle $A_1A_3A_2$ will equal $180° - 2\angle A_2A_1A_3$, we will obtain an isosceles triangle with $A_1A_3 = A_2A_3 \ge A_1A_2$.

In triangle $A_1B_1A_3$, the segment A_3O is a bisector, so $B_1O/A_1O = B_1A_3/A_1A_3 = B_1A_3/A_2A_3$. It is easily seen that the last ratio does not exceed $1/2$ for $A_1A_2 \le A_1A_3$. Hence $B_1O \le 1/2$. The equality is attained for the equilateral triangle.

13. (G.Filippovsky, 9–10) Let ABC be an acute-angled triangle with altitude $AT = h$. The line passing through its circumcenter O and incenter I meets the sides AB and AC at points F and N respectively. It is known that $BFNC$ is a cyclic quadrilateral. Find the sum of the distances from the orthocenter of ABC to its vertices.

Answer. $2h$.

Solution. Since $BNFC$ is cyclic, we have $\angle ONA = \angle B$. On the other hand, $\angle OAC = \pi/2 - \angle B$. Thus $AO \perp OI$. Draw the perpendicular IT to AH. Since AI bisects angle OAH, we obtain that the right-angled triangles AOI and ATI are congruent, i.e. $AT = AO = R$ and $h = AH = R + r$, where R and r are the circumradius and the inradius of triangle ABC (fig.13). It is known that the sum of distances from O to the sidelines of the triangle is equal to $R + r$, and the sum of distances from the orthocenter to the vertices is twice as large, which yields the answer.

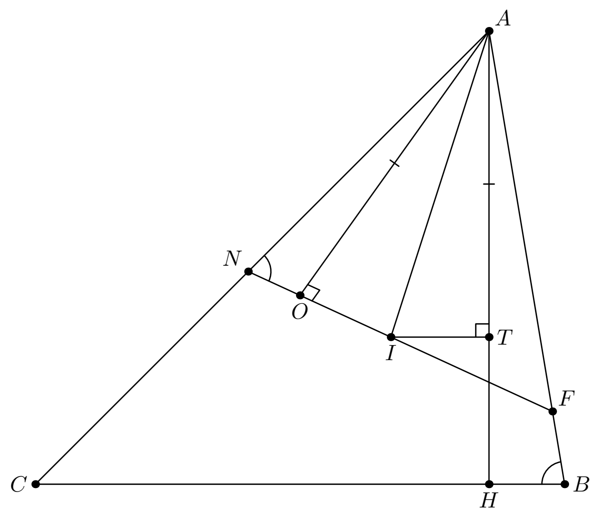

Fig. 13

14. (S.Arutyunyan, 9–11) Let the side AC of triangle ABC touch the incircle and the corresponding excircle at points K and L respectively. Let P be the projection of the incenter onto the perpendicular bisector of AC. It is known that the tangents to the circumcircle of triangle BKL at K and L meet on the circumcircle of ABC. Prove that the lines AB and BC touch the circumcircle of triangle PKL.

Solution. Suppose that $AB > BC$. Let M be the midpoint of AC, N be the midpoint of arc ABC, NW and KD be the diameters of the circumcircle and the incircle respectively. By the assumption, the tangents to the circle BKL at K and L meet at W, i.e. BW is the symedian of triangle BKL. Furthermore B, D, L are collinear and BW bisects segment KD. Hence triangles BKL and BDK are similar, i.e. $\angle BMC = \angle BID = (\angle C - \angle A)/2$. Then $\angle BMN = (\pi - \angle C + \angle A)/2 = \angle BNM$ and $BM = BN$. Let S be a point on the arc AWC such that $\angle SBC = \angle ABM$. Then $\angle SNB = \angle ABM + \angle BAC = \angle BMC = \angle NSB$, i.e. $BS = BN = BM$ (fig.14). By similarity of triangles ABM and SBC we have $AB \cdot BC = BM \cdot BS = BM^2 = (2AB^2 + 2BC^2 - AC^2)/4$. Therefore $AC^2 = 2(AB - BC^2)$, or $AC = \sqrt{2}KL$. Applying the Stewart theorem to triangle AWC and cevian WK we obtain that $WK^2 = WC^2 - AK \cdot KC = WI^2 - (AM^2 - MK^2) = WI^2 - MK^2 = WI^2 - PI^2 = WP^2$ (by the trident theorem, $WC = WI$). Thus P, K, L lie on the circle centered at W.

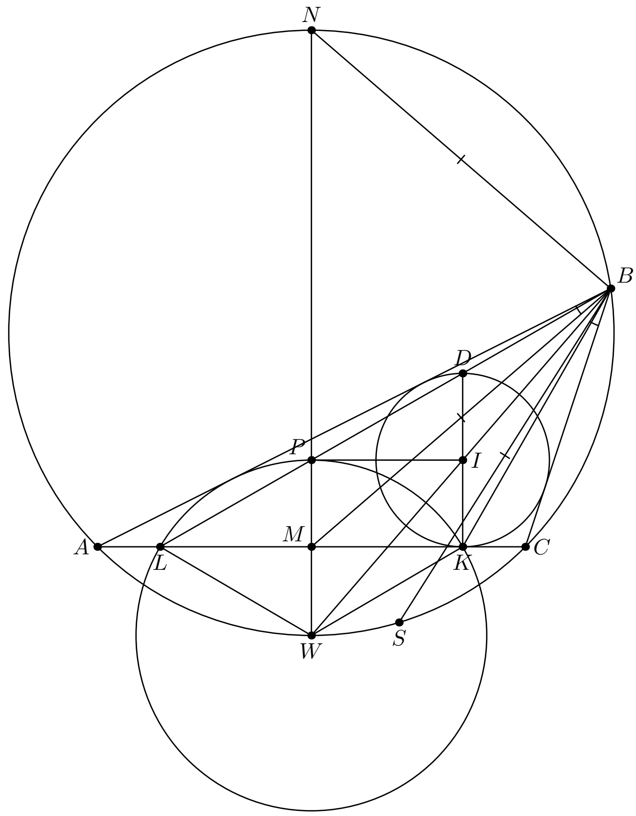

Fig. 14

Let R, r be the circumradius and the inradius of triangle ABC. Then the distance from W to line AB is equal to $BW \sin \frac{\angle B}{2} = 2R \cos \frac{\angle C - \angle A}{2} \sin \frac{\angle B}{2} = R(\sin \angle A + \sin \angle C)$. By the Carnot theorem $R + r = R(\sin \angle A + \sin \angle B + \sin \angle C)$, therefore this distance is equal to $R(1 - \cos \angle B) + r = WM + MP = WP$, which is equivalent to the required assertion.

15. (M.Etesamifard, 9–11) The incircle ω of triangle ABC touches the sides BC, CA and AB at points D, E and F respectively. The perpendicular from E to DF meets BC at point X, and the perpendicular from F to DE meets BC at point Y. The segment AD meets ω for the second time at point Z. Prove that the circumcircle of the triangle XYZ touches ω.

Solution. Let I be the center of ω. Note that $\angle FYX = \angle ICB = \angle FEX$, i.e. $XYEF$ is a cyclic quadrilateral. Now BC, EF and the tangent to ω at Z concur at the pole T of AD with respect to ω. Hence $TZ^2 = TF \cdot TE = TX \cdot TY$, i.e. TZ touches the circle XYZ (fig.15).

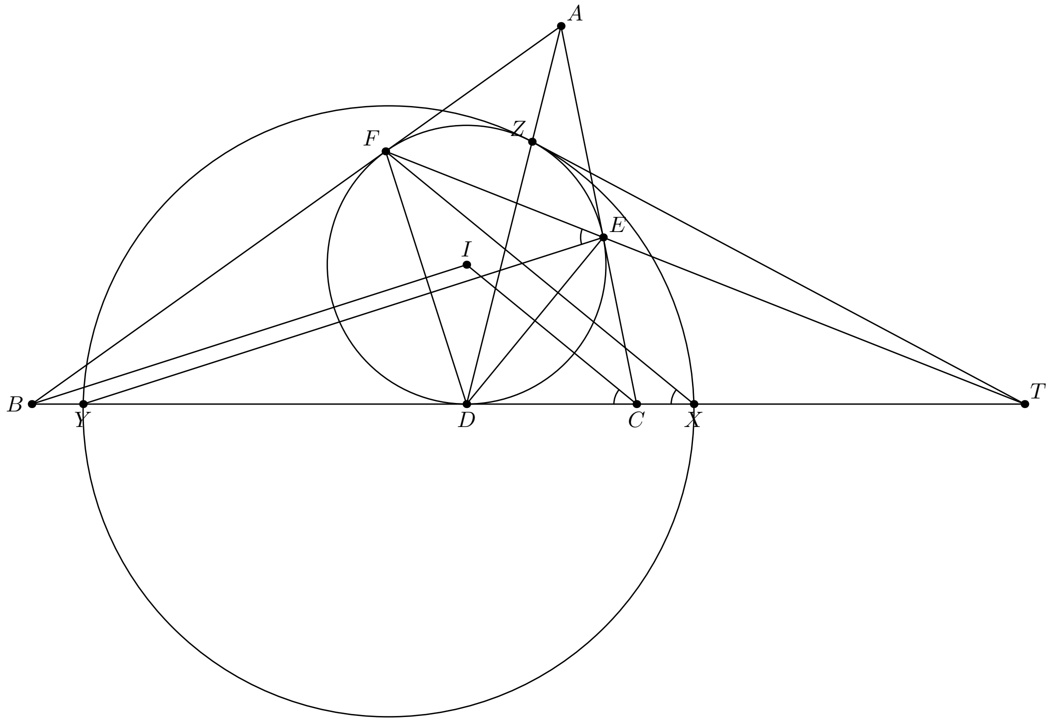

Fig. 15

16. (M.Plotnikov, 9–11) Let AH_1 and BH_2 be the altitudes of triangle ABC; let the tangent to the circumcircle of ABC at A meet BC at point S_1, and the tangent at B meet AC at point S_2; let T_1 and T_2 be the midpoints of AS_1 and BS_2 respectively. Prove that T_1T_2, AB and H_1H_2 concur.

    Solution. It is clear that T_1 lies on the medial line B_0C_0 of triangle ABC, and T_1A touches the circle AB_0C_0. Thus $T_1A^2 = T_1B_0 \cdot T_1C_0$. But B_0, C_0 lie on the nine-points circle (NPC) of triangle ABC, therefore T_1 lies on the radical axis of this circle and the circumcircle. By similar reasoning for T_2 we obtain that T_1T_2 is the radical axis of the circumcircle and the NPC. Since A, B, H_1, H_2 are concyclic, we obtain that the lines AB and H_1H_2 are the radical axes of the corresponding circle with the circumcircle and the NPC respectively. Clearly these three radical axes concur.

17. (E.Bakaev, 10–11) Three circles ω_1, ω_2, ω_3 are given. Let A_0 and A_1 be the common points of ω_1 and ω_2, B_0 and B_1 be the common points of ω_2 and ω_3, C_0 and C_1 be the common points of ω_3 and ω_1. Let $O_{i,j,k}$ be the circumcenter of triangle $A_iB_jC_k$. Prove that the four lines of the form $O_{ijk}O_{1-i,1-j,1-k}$ are concurrent or parallel.

    Solution. Let O be the radical center of the given circles. If O lies outside these circles then there exists a circle centered at O and perpendicular to three given circles. The inversion with respect to this circle saves all given circles. Therefore this inversion transposes A_0 and A_1, B_0 and B_1, C_0 and C_1, thus it transposes the circles $A_iB_jC_k$ and $A_{1-i}B_{1-j}C_{1-k}$. Hence the lines joining the centers of such pairs of circles pass through O.

If O lies inside the given circles then they are saved by the composition of the inversion and the central symmetry with center O. Therefore in this case four lines also pass through O.

18. (N.Beluhov, A.Zaslavsky, 10–11) A quadrilateral $ABCD$ without parallel and without equal sides is circumscribed around a circle centered at I. Let K, L, M and N be the midpoints of AB, BC, CD and DA respectively. It is known that $AB \cdot CD = 4IK \cdot IM$. Prove that $BC \cdot AD = 4IL \cdot IN$.

    Solution. Construct J such that $\triangle AJB \sim \triangle DIC$. Then $AJBI$ is cyclic. Let k be its circumcircle, and let IK meet k for the second time at J'. Then from $KJ : AB = IM : CD$, $IK \cdot KJ' = KA \cdot KB = AB^2/4$, and $4IK \cdot IM = AB \cdot CD$ it follows that $KJ = KJ'$ (fig. 18).

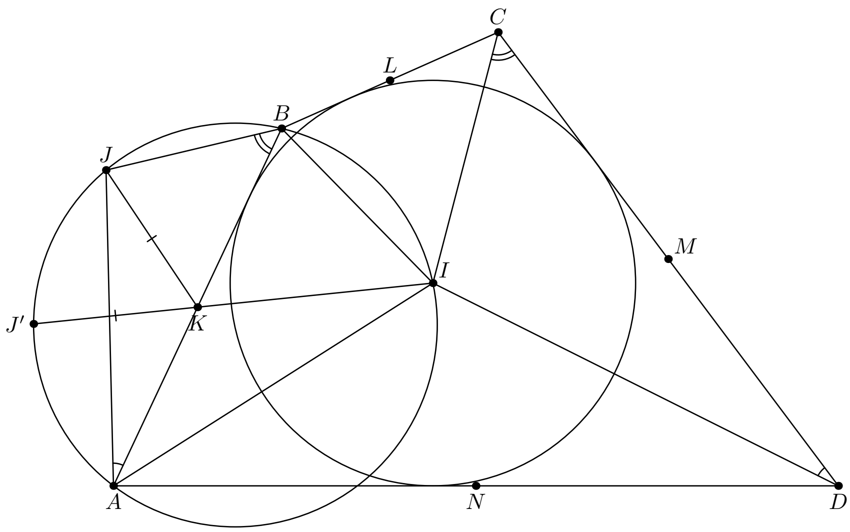

Fig. 18

If AB is a diameter of k, then $\angle AIB = 90^\circ$ and $AD \| BC$ — a contradiction. Therefore, AB is not a diameter of k. If $J = J'$, then $\angle ICB = \angle AIK$, $\angle IDA = \angle BIK$ and $BC = r(\cot \angle IBK + \cot \angle AIK) = r(\cot \angle IAK + \cot \angle BIK) = AD$ — a contradiction. From this and $KJ = KJ'$, it follows that J and J' are symmetric with respect to the perpendicular bisector of AB.

Therefore, $\triangle AIK \sim \triangle J'BK \simeq \triangle JAK \sim \triangle IDM$. From this, together with $\angle IAK = \angle IAD$ and $\angle IDM = \angle IDA$, it follows that $\triangle AIK \sim \triangle ADI \sim \triangle IDM$. Analogously, $\triangle BIK \sim \triangle BCI \sim \triangle ICM$.

Let P and Q be the midpoints of IA and IB. Then $\triangle IND \sim \triangle KPI \simeq \triangle IQK \sim \triangle CLI$. Therefore, $IN : ND = CL : LI$ and $4IL \cdot IN = AD \cdot BC$, as needed.

Note. A circumscribed quadrilateral $ABCD$ that has no parallel or equal sides satisfies the conditions of the problem if and only if its incenter I is the center of gravity of its four vertices A, B, C, and D.

19. (A.Utkin, 10–11) Let AL_a, BL_b, CL_c be the bisectors of triangle ABC. The tangents to the circumcircle of ABC at B and C meet at point K_a, points K_b, K_c are defined similarly. Prove that the lines K_aL_a, K_bL_b and K_cL_c concur.

Solution. Since ABK_c is an isosceles triangle, the sine law applied to triangles AL_cK_c and BL_aK_c implies that $\sin\angle AK_cL_c : \sin\angle BK_cL_c = AL_c : BL_c$. From this and two similar equalities we obtain the required assertion applying the Ceva theorem.

20. (A.Zaslavsky, 10–11) Let O be the circumcenter of triangle ABC, H be its orthocenter, and M be the midpoint of AB. The line MH meets the line passing through O and parallel to AB at point K lying on the circumcircle of ABC. Let P be the projection of K onto AC. Prove that $PH \parallel BC$.

Solution. Let Q be the projection of K to BC. Then PQ is the Simson line of K, therefore PQ bisects segment HK, and the angle between PQ and altitude CH (the Simson line of C) is equal to the half of angle COK. But OK is the perpendicular bisector for segment CL, where L is the second common point of CH with the circumcircle. Hence $\angle HCK = \angle CLK = \angle COK/2$, i.e. $PQ \parallel CK$. Thus PQ bisects segment CH. Also MH meets the circumcircle for the second time at point C' opposite to C, and $C'M = MH$. Therefore $CK \perp KC'$, i.e. the corresponding sides of triangles CPQ and BHC' are perpendicular. Then their medians are perpendicular too, therefore CH bisects segment PQ and $CPHQ$ is a parallelogram (fig.20).

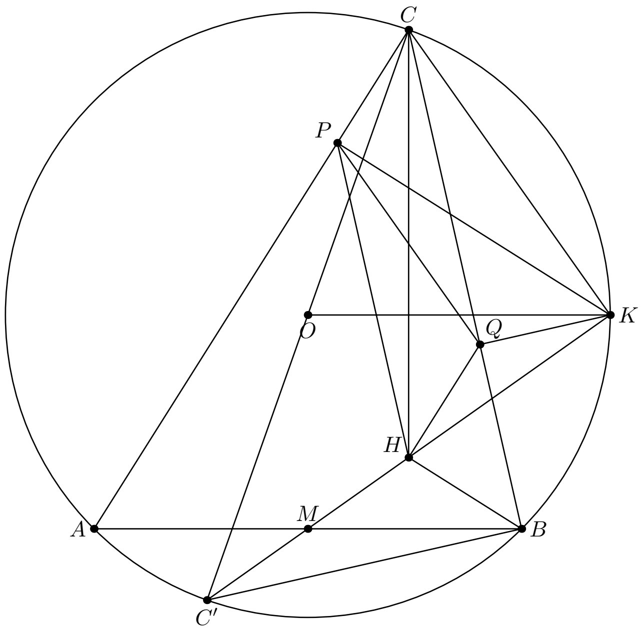

Fig. 20

21. (A.Sgibnev, A.Zaslavsky, 10–11) An ellipse Γ and its chord AB are given. Find the locus of orthocenters of triangles ABC inscribed into Γ.

Solution. Choose a coordinate system such that the line AB is the X-axis. Then the equation for Γ will be $(x-x_a)(x-x_B)+y(ax+by+c) = 0$, where $b > 0$. The coordinates of orthocenter H are (x_C, h), where h satisfies the condition of perpendicularity AH and BC:

$(x_C - x_A)(x_C - x_B) + hy_C = 0$, i.e. $h = -(x_C - x_A)(x_C - x_B)/y_C$. But by the Vieta theorem XH meets Γ for the second time at the point with ordinate $(x_C - x_A)(x_C - x_B)/by_C$. Thus the locus of orthocenters is an ellipse obtained by the contraction of Γ to AB with coefficient $-b$. Since this coefficient is equal to the ratio of squares of two diameters, perpendicular and parallel to AB, we obtain that this ellipse is similar to Γ and their major axes are perpendicular.

22. (P.Kozhevnikov, 10–11) Let AA_0 be the altitude of the isosceles triangle ABC ($AB = AC$). A circle γ centered at the midpoint of AA_0 touches AB and AC. Let X be an arbitrary point of line BC. Prove that the tangents from X to γ cut congruent segments on lines AB and AC.

First solution. For simplicity, we consider only the case when X lies inside segment BA_0. All other cases are similar.

Let B_0 and C_0 be the midpoints of segments AC and AB, respectively. Let one tangent meet segment AC_0 at P and let the other tangent meet segment CB_0 at Q.

By Newton's theorem for circumscribed quadrilateral $APXQ$, the midpoints of segments AA_0, AX, and PQ are collinear. Therefore, the midpoint R of segment PQ lies on the midline of triangle ABC opposite to vertex A.

Let S be the reflection of point A about point R. Then S lies on line BC, and quadrilateral $APSQ$ is a parallelogram (fig. 22.1). Therefore, $C_0P : A_0S = B_0Q : A_0S$ and $C_0P = B_0Q$.

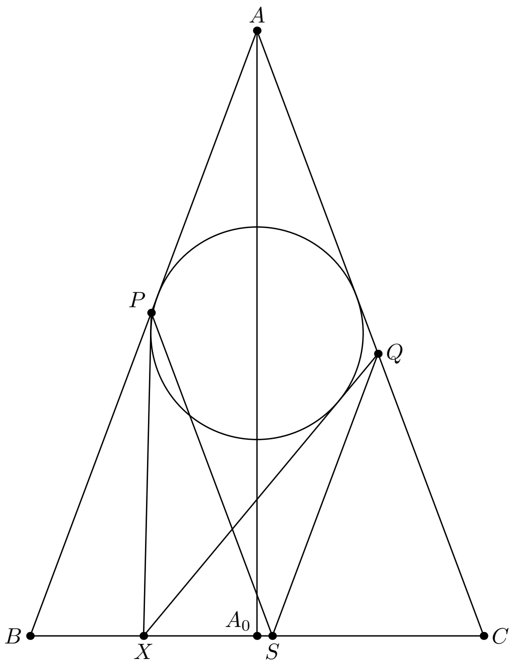

Fig. 22.1

Let one tangent meet ray C_0B at P', and let the other tangent meet ray B_0A at Q'. Similarly, $C_0P' = B_0Q'$. Therefore, $PP' = QQ'$, as needed.

Second solution. Let one of two tangents meet AB and AC at points Y_1 and Y_2, and the second one meet them at Z_1 and Z_2 respectively. Since the relation between these points is projective, it is sufficient to prove that $Y_1Z_1 = Y_2Z_2$ for three positions of X, i.e. by symmetry for some point distinct from the midpoint of BC. When X tends to B then one of points Z_1, Y_1 also tends to B, and the second one tends to the touching point P of γ with AB. Let Q be distinct from A point of AC such that BQ touches γ, and let B_0, C_0 be the midpoints of AC, AB respectively. Then we have $(B, C_0, P, \infty) = (Q, \infty, A, B_0)$, i.e. $AQ = BP$ (fig.22.2).

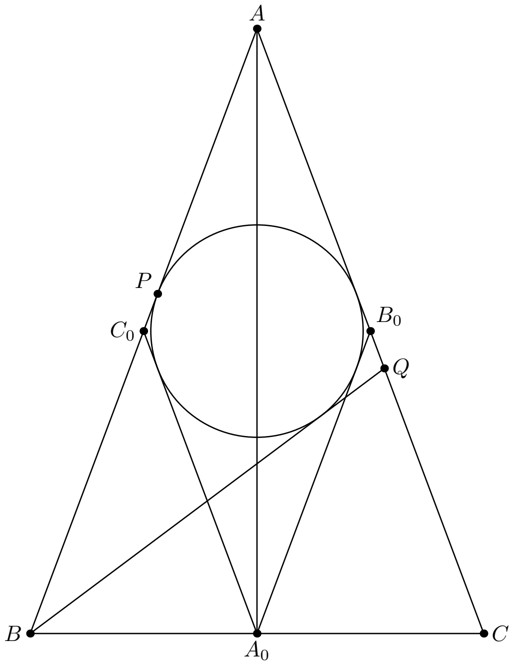

Fig. 22.2

23. (A.Skopenkov, (10–11) In the plane, let a, b be two closed broken lines (possibly self-intersecting), and K, L, M, N be four points. The vertices of a, b and the points K L, M, N are in general position (i.e. no three of these points are collinear, and no three segments between them concur at an interior point). Each of segments KL and MN meets a at an even number of points, and each of segments LM and NK meets a at an odd number of points. Conversely, each of segments KL and MN meets b at an odd number of points, and each of segments LM and NK meets b at an even number of points. Prove that a and b intersect.

First solution. Since the vertices of a are in general position, this broken line divides the plane into several parts which can be colored black and white regularly (i.e. in such a way that the colors of adjacent parts are different). See the proof, for example, in [Sk, §1.3, §2.2], [Sk18, §1.3, §2.2]. Let the "external" part be white. Consider an arbitrary point O of self-intersection of a and take segments $OA = OB = OC = OD = \varepsilon$ on its links passing through, such that $ABCD$ is a rectangle. If ε is sufficiently small then all common points of a with b and segments KL, LM, MN, NK lie outside this rectangle. If we construct such rectangles for all points of self-intersection and color them white, then the black part of the plane will be the union of several not intersecting polygons. Now recolor several rectangles to obtain a black polygon restricted by not self-intersecting broken line a'. Construct not self-intersecting broken line b' in the similar way. By the construction the broken lines a', b' intersect one another and meet segments KL, LM, MN, NK at the same points as a, b. Suppose that a' and b' do not intersect. Then they divide the plane into three parts, therefore two of points K, L, M, N lie in the same part. But this is impossible because a' separates K and L from M and N, and b' separates K and N from M and L. Thus a' and b' intersect, and hence the given broken lines intersect too.

Second solution. Let point C be in general position related to the vertices of a, b and points K, L, M, N. Denote the union of segments $CK \cup CL \cup CM \cup CN$ by γ.

As in the first solution color regularly black and white the parts into which a divides the plane. Denote the union of the black parts by α. Construct similarly the set β corresponding to b.

If a and b do not intersect then $a \cap \beta$ is a or $\emptyset$, and $\alpha \cap b$ is b or $\emptyset$. Then the following chain of comparisons modulo 2 yields a contradiction:

$$0 \underset{(1)}{=} |\partial(\gamma\cap\alpha\cap\beta)| \underset{(2)}{=} |\underbrace{\partial\gamma}_{=\{K,L,M,N\}}\cap\alpha\cap\beta| + |\gamma\cap\underbrace{\partial\alpha}_{=a}\cap\beta| + |\gamma\cap\alpha\cap\underbrace{\partial\beta}_{=b}| \underset{(3)}{=} 1+0+0 = 1.$$

Here (1) is true because $\gamma \cap \alpha \cap \beta$ is the union of a finite number of unclosed broken lines with even number of endpoints. The proof of (2) is not difficult (this is the «Leibnitz formula»).

Let us prove (3). We have

$$\{K, L, M, N\}\cap\alpha\cap\beta = (\{K, L, M, N\}\cap\alpha)\cap(\{K, L, M, N\}\cap\beta) = \{K, L\}\cap\{K, N\} = \{K\}.$$

If $a \cap \beta = \emptyset$ then $\gamma \cap a \cap \beta = \emptyset$. And if $a \cap \beta = a$ then

$$|\gamma \cap a \cap \beta| = |\gamma \cap a| = |KN \cap a| + |LM \cap a| = 1 + 1 = 0.$$

Thus in both cases $|\gamma \cap a \cap \beta| = 0$. Similarly $|\gamma \cap \alpha \cap b| = 0$.

Remarks. Similar reasoning about triple intersections demonstrates that the Borromeo rings cannot be uncoupled. See [Sk, §4].

The multidimensional version of the problem, the Borromeo rings lemma, can be proved similarly, see [AMS+]. This lemma is significant for the study of complexity of realizability of hypergraphs in multidimensional spaces, see [MTW11, ST17].

References

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24. (N.Beluhov, 11) Two unit cubes have a common center. Is it always possible to number the vertices of each cube from 1 to 8 so that the distance between each pair of identically numbered vertices would be at most 4/5? What about at most 13/16?

Solution. Let $\kappa = A_1A_2\dots A_8$ be one of the two cubes (with $A_1A_2A_3A_4$ a unit square and A_i adjacent to A_{i+4} for all i), d_1, d_2, d_3, and d_4 be the space diagonals of κ, λ be the second cube, and e_1, e_2, e_3, and e_4 be the space diagonals of λ. Let O be the common center of the two cubes, s be their common circumscribed sphere, μ be a positive real which does not exceed the diameter of s, and α be the central angle of a chord of length μ in s.

Let S_i be the set of all e_j such that the angle between d_i and e_j does not exceed α. Suppose that, for all $1 \le k \le 4$, the union of any k of the sets S_i contains at least k elements. Then, by Hall's representatives theorem, we can select a single representative e'_i from each S_i in such a way that all four representatives are distinct, and pair up the endpoints of each d_i with the endpoints of its corresponding e'_i in such a way that the distance between the two vertices in each pair is at most μ.

Let us then look at the possible values of k and the bounds on μ that they impose.

$k = 4$: Let P be the center of the spherical cap cut off from s by the plane $A_1A_2A_3A_4$. (That is, P lies on s, $PA_1 = PA_2 = PA_3 = PA_4$, and $A_1A_2A_3A_4$ separates O and P.)

We need to ensure that the union of the eight spherical caps with centers A_i and radii α contains all vertices of λ, i.e., that it covers s. This is true just if $\mu \ge PA_1$; denote the length of PA_1 by μ_4.

$k = 3$: Let Q be a point on the shorter great-circle arc $\smile A_1A_3$ of s such that $A_2Q = A_4Q = \mu$ and $A_1Q \le QA_3$. Choose R and S similarly on the shorter great-circle arcs $\smile A_1A_6$ and $\smile A_1A_8$.

Without loss of generality, we need to ensure that the union of the six spherical caps with centers A_2, A_4, A_5, A_3, A_6, and A_8 and radii α contains at least six vertices of λ. Equivalently, we need to ensure that the complement of this union to s contains at most two vertices of λ. This complement consists of two connected components which are symmetric with respect to O; therefore, it is necessary and sufficient to ensure that each component contains at most one vertex of λ. Since each component is contained within the equilateral spherical triangle QRS but contains points arbitrarily close to Q, R, and S, it is necessary and sufficient to have $QR \le 1$ – or, equivalently, $\mu \ge \mu_3$, where μ_3 is the value of μ for which equality is attained. It is easy to see that $\mu_3 > \mu_4$.

For $k = 2$ and $k = 1$, let T_i be the set of all d_j such that the angle between e_i and d_j does not exceed α.

$k = 2$: Without loss of generality, suppose that $S_1 \cup S_2$ does not contain e_1, e_2, and e_3. Then $T_1 \cup T_2 \cup T_3$ does not contain d_1 and d_2. From the case $k = 3$ we know that this is avoided just if $\mu \ge \mu_3$.

$k = 1$: Suppose that, without loss of generality, S_1 does not contain any e_i. Then the union of all T_i does not contain d_1. From the case $k = 4$ we know that this is avoided just if $\mu \geq \mu_4$.

Thus $\mu \geq \mu_3$ always works. In order to see that no $\mu < \mu_3$ works, let λ be the cube with center O and edge QR as in the case $k = 3$. At most one of the vertices Q and R of λ is paired with A_1; whatever vertex of κ we pair with the other one, the distance between them will be at least μ_3. Therefore, the shortest distance μ that satisfies the conditions of the problem is $\mu_3 = \sqrt{\frac{9-2\sqrt{2}-\sqrt{5}}{6}}$. Since $4/5 < \mu_3 < 13/16$, the answer to the first part of the problem is negative and the answer to the second part of the problem is positive.