## IX Geometrical Olympiad in honour of I.F.Sharygin

## Final round. Ratmino, 2013, August 1

### Solutions

### First day. 8 grade

**8.1.** (*N. Moskvitin*) Let $ABCDE$ be a pentagon with right angles at vertices $B$ and $E$ and such that $AB = AE$ and $BC = CD = DE$. The diagonals $BD$ and $CE$ meet at point $F$. Prove that $FA = AB$.

**First solution.** The problem condition implies that the right-angled triangles $ABC$ and $AED$ are equal, thus the triangle $ACD$ is isosceles (see fig. 8.1a). Then $\angle BCD = \angle BCA + \angle ACD = \angle EDA + \angle ADC = \angle CDE$. Therefore, the isosceles triangles $BCD$ and $CDE$ are equal. Hence $\angle CBD = \angle CDB = \angle ECD = \angle DEC$.

Since the triangle $CFD$ is isosceles and $BD = CE$, we obtain that $BF = FE$. Therefore $\triangle ABF = \triangle AEF$. Then $\angle AFB = \frac{\angle BFE}{2} = \frac{180^\circ - 2\angle FCD}{2} = 90^\circ - \angle ECD = 90^\circ - \angle DBC = \angle ABF$, hence $AB = AF$, QED.

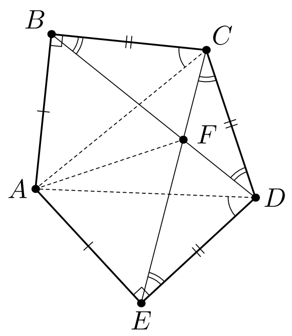



Fig. 8.1a

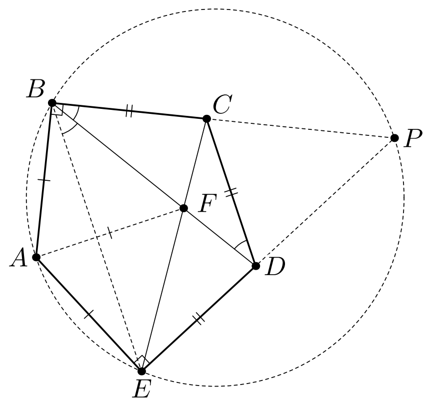



Fig. 8.1b

**Second solution.** Let $BC$ meet $DE$ at point $P$ (see fig. 8.1b). Notice that $\angle CBD = \angle CDB = \angle DBE$, i.e., $BD$ is the bisector of $\angle CBE$. Thus $F$ is the incenter of $\triangle PBE$. Since the quadrilateral $PBAE$ is cyclic and symmetrical, we obtain that $A$ is the midpoint of arc $BE$ of the circle $(PBE)$. Therefore, by the trefoil theorem we get $AF = AB$, QED.

**Remark.** The problem statement holds under the weakened condition of equality of side lengths. It is sufficient to say that $AB = AE$ and $BC = CD = DE$.

**8.2.** (*D. Shvetsov*) Two circles with centers $O_1$ and $O_2$ meet at points $A$ and $B$. The bisector of angle $O_1AO_2$ meets the circles for the second time at points $C$ and $D$. Prove that the distances from the circumcenter of triangle $CBD$ to $O_1$ and to $O_2$ are equal.

**First solution.** Without loss of generality, suppose that $C$ lies on the segment $AD$. Let $P$ be the common point of the lines $O_1C$ and $O_2D$ (see fig. 8.2). The triangle $AO_1C$ is isosceles, thus $\angle O_1CA = \angle O_1AC = \angle CAO_2$, therefore $O_1C \parallel AO_2$. Similarly, we obtain that $O_1A \parallel O_2D$. Hence $O_1AO_2P$ is a parallelogram.

Let us prove that the quadrilateral $BCPD$ is cyclic, and $O_1O_2PB$ is an isosceles trapezoid. Then the assertion of the problem follows. Indeed, then the circumcenter $O$ of $\triangle BCD$ is equidistant from the points $B$ and $P$, therefore $O$ is equidistant from $O_1$ and $O_2$.

Notice that $O_1P = AO_2 = BO_2$ and $O_1B = O_1A = O_2P$, i.e., the triangles $BO_1P$ and $PO_2B$ are equal. Therefore $\angle BO_1P = \angle PO_2B$, and hence the quadrilateral $O_1O_2PB$ is cyclic. Then $\angle O_1O_2B = \angle O_1PB$.

On the other hand, we have $\angle BDA = \frac{1}{2}\angle AO_2B = \angle AO_2O_1 = \angle O_1O_2B$ and $\angle O_2O_1P = \angle AO_2O_1$. Therefore $\angle BDA = \angle O_1PB = \angle O_2O_1P$, i.e., the quadrilateral $BCPD$ is cyclic, and $O_1O_2 \parallel BP$. From $O_1B = O_2P$ we obtain that $O_1O_2PB$ is an isosceles trapezoid.

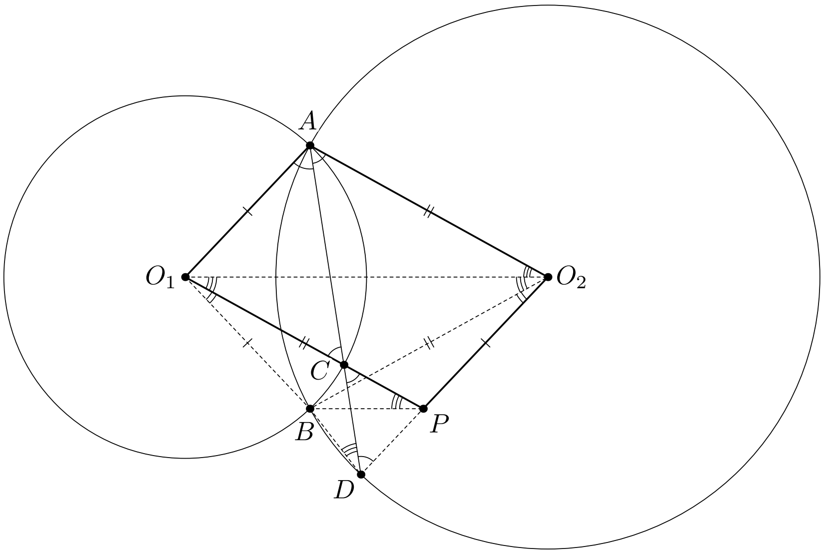



Fig. 8.2

**Second solution.** By $OO_1 \perp BC$ and $O_1O_2 \perp AB$, we get $\angle OO_1O_2 = \angle ABC = \dfrac{\angle AO_1C}{2}$. Similarly, we obtain $\angle OO_2O_1 = \dfrac{\angle AO_2D}{2}$. It remains to notice that $\angle AO_1C = \angle AO_2D$; it can be shown as in the previous solution.

**8.3.** (*B. Frenkin*) Each vertex of a convex polygon is projected to all nonadjacent sidelines. Can it happen that each of these projections lies outside the corresponding side?

**Ответ:** no.

**Solution.** Let $AB$ be the longest side of the polygon (see fig. 8.3). Let us project all the vertices of the polygon different from $A$ and $B$ onto $AB$. Assume that none of the projections lies on the segment $AB$; then the projection of some side $s$ different from $AB$ strictly contains $AB$. However, this implies that $s > AB$, a contradiction.

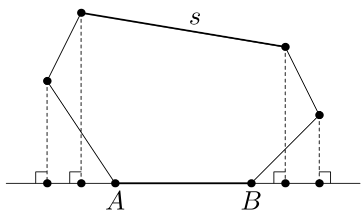



Fig. 8.3

**8.4.** (*A. Zaslavsky*) The diagonals of a convex quadrilateral $ABCD$ meet at point $L$. The orthocenter $H$ of the triangle $LAB$ and the circumcenters $O_1$, $O_2$, and $O_3$ of the triangles $LBC$, $LCD$, and $LDA$ were marked. Then the whole configuration except for points $H$, $O_1$, $O_2$, and $O_3$ was erased. Restore it using a compass and a ruler.

**Solution.** Let $O$ be the circumcenter of the triangle $LAB$ (see fig. 8.4). Then the lines $OO_1$ and $O_2O_3$ are perpendicular to $BD$, while the lines $O_1O_2$ and $O_3O$ are perpendicular to $AC$. Therefore, we can restore the perpendicular bisectors $OO_1$ and $OO_3$ to the sides $LB$ and $LA$ of the triangle $LAB$. The lines $h_a$ and $h_b$ passing through the orthocenter $H$ of this triangle and parallel to $OO_1$ and $OO_3$ coincide with the altitudes of this triangle; i.e., they pass through $A$ and $B$, respectively. Hence the reflections of $h_a$ and $h_b$ in $OO_3$ and $OO_1$, respectively, meet at point $L$. Now the construction is evident.

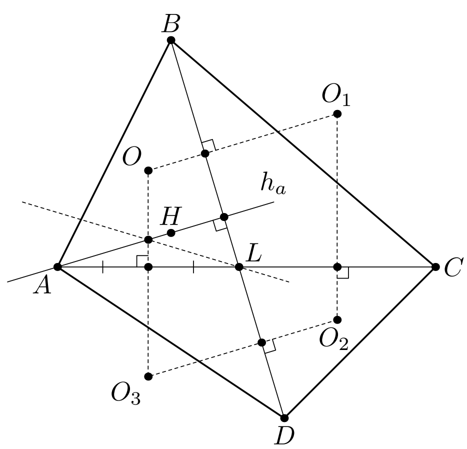



Рис. 8.4

**IX Geometrical Olympiad in honour of I.F.Sharygin**

**Final round. Ratmino, 2013, August 2**

## Solutions

### Second day. 8 grade

**8.5.** (*B. Frenkin*) The altitude $AA'$, the median $BB'$, and the angle bisector $CC'$ of a triangle $ABC$ are concurrent at point $K$. Given that $A'K = B'K$, prove that $C'K = A'K$.

**Solution.** Since the point $K$ lies on the bisector of angle $C$, the distance from $K$ to $AC$ is the same as the distance to $BC$, i.e., this distance is equal to $KA'$ (see fig. 8.5). Since $KA' = KB'$, this yields that $KB' \perp AC$. Thus the median $BB'$ coincides with the altitude from $B$, and hence $AB = BC$. Then $BK$ and $CK$ are the angle bisectors in the triangle $ABC$, therefore $AK$ is also an angle bisector; now, since $AK$ is the altitude we have $AB = AC$. Therefore the triangle $ABC$ is regular, and $A'K = B'K = C'K$.

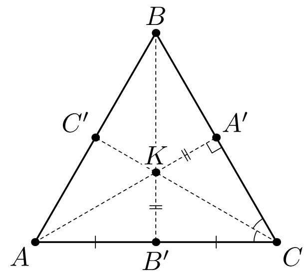
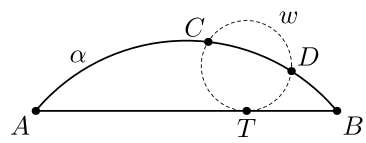



Fig. 8.5

**8.6.** (*F. Nilov*) Let $\alpha$ be an arc with endpoints $A$ and $B$ (see fig.). A circle $\omega$ is tangent to segment $AB$ at point $T$ and meets $\alpha$ at points $C$ and $D$. The rays $AC$ and $TD$ meet at point $E$, while the rays $BD$ and $TC$ meet at point $F$. Prove that $EF$ and $AB$ are parallel.




**Solution.** Let us prove that the quadrilateral $CDEF$ is cyclic (see fig. 8.6); then the assertion of the problem follows. Indeed, then we have $\angle FEC = \angle FDC$ and $\angle FDC = 180^\circ -$ $- \angle BDC = \angle CAB$, i.e., $FE \parallel AB$.

Since $AB$ is tangent to $\omega$, we have $\angle TCD = \angle BTD$. Furthermore, we get $\angle FCE = \angle ACT =$ $= \angle ACD - \angle TCD = (180^\circ - \angle ABD) - \angle BTD = \angle TDB = \angle FDE$. Therefore the quadrilateral $CDEF$ is cyclic, QED.

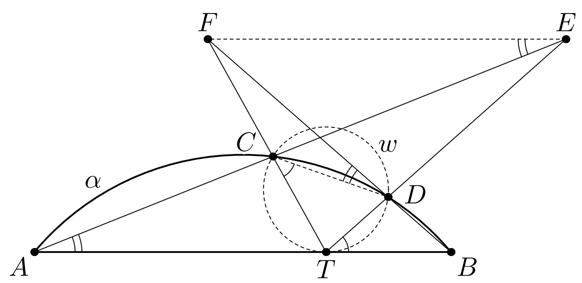



Fig. 8.6

**8.7.** (*B. Frenkin*) In the plane, four points are marked. It is known that these points are the centers of four circles, three of which are pairwise externally tangent, and all these three are internally tangent to the fourth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fourth (the largest) circle.

Prove that these four points are the vertices of a rectangle.

**Solution.** Let $O_0$ and $R_0$ be the center and the radius of the greatest circle, and let $O_1$, $O_2$, $O_3$ and $R_1$, $R_2$, $R_3$ be the centers and the radii of the remaining circles. Then $O_0O_i = R_0 - R_i$ ($i = 1, 2, 3$) and $O_iO_j = R_i + R_j$ ($i, j = 1, 2, 3, i \neq j$). Hence $O_0O_1 - O_2O_3 = O_0O_2 - O_3O_1 =$ $= O_0O_3 - O_1O_2 = R_0 - R_1 - R_2 - R_3 := d$.

If $d > (<)0$, then the distance from $O_0$ to any of points $O_1$, $O_2$, $O_3$ is greater (less) than the distance between two remaining points. This enables us to determine $O_0$ which contradicts the condition. Indeed, if we colour the longer segments in each of the pairs $(O_0O_1, O_2O_3)$, $(O_0O_2, O_1O_3)$, and $(O_0O_3, O_1O_2)$ in red and the shorter ones in blue then $O_0$ is the unique endpoint of three monochromatic segments.

If $d = 0$, then the marked points form a quadrilateral with equal opposite sides and equal diagonals. Such a quadrilateral has to be a rectangle.

**8.8.** (*I. Dmitriev*) Let $P$ be an arbitrary point on the arc $AC$ of the circumcircle of a fixed triangle $ABC$, not containing $B$. The bisector of angle $APB$ meets the bisector of angle $BAC$ at point $P_a$; the bisector of angle $CPB$ meets the bisector of angle $BCA$ at point $P_c$. Prove that for all points $P$, the circumcenters of triangles $PP_aP_c$ are collinear.

**Solution.** Notice first that the lines $PP_a$ and $PP_c$ meet the circumcircle for the second time at the midpoints $C'$ and $A'$ of the arcs $AB$ and $AC$, respectively (see fig. 8.8). Thus $\angle P_aPP_c = (\angle A + \angle C)/2 = 180^\circ - \angle AIC$, where $I$ is the incenter of the triangle. Hence all circles $PP_aP_c$ pass through $I$.

Now let us fix some point $P$ and find the second common point $J$ of circles $PP_aP_c$ and $ABC$. For any other point $P'$ we have $\angle JP'P'_c = \angle JP'A' = 180^\circ - \angle JPA' = 180^\circ - \angle JPP_c = \angle JIP_c = \angle JIP'_c$ (if $P$ and $P'$ lie on arcs $CJ$ and $AJ$, respectively; the remaining cases can be considered similarly). Thus the circle $P'P'_aP'_c$ also passes through $J$.

Therefore the circumcenters of all triangles $PP_aP_c$ lie on the perpendicular bisector of the segment $IJ$.

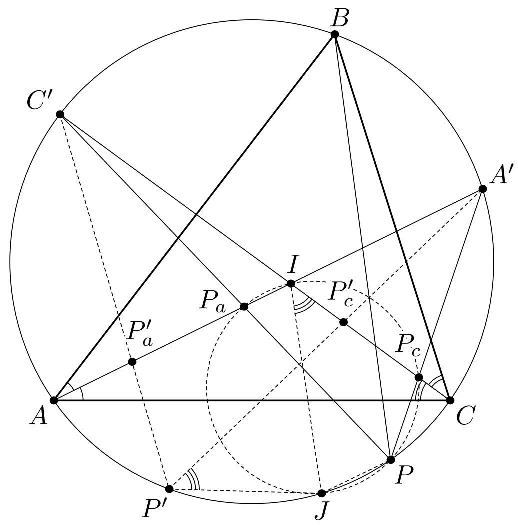



Рис. 8.8

**Remark.** Consider a “semiincircle” $\omega$ which is tangent to the segments $BA$, $BC$ and to the arc $APC$). In a special case when $P$ is the tangent point of $\omega$ and $(ABC)$ we see that $J$ coincides with $P$. Thus we can determine $J$ as a touching point of the circumcircle and the semiincircle. It is known that $J$ lies also on line $IS$, where $S$ is the midpoint of arc $ABC$.

## IX Geometrical Olympiad in honour of I.F.Sharygin

## Final round. Ratmino, 2013, August 1

### Solutions

### First day. 9 grade

**9.1.** (*D. Shvetsov*) All angles of a cyclic pentagon $ABCDE$ are obtuse. The sidelines $AB$ and $CD$ meet at point $E_1$; the sidelines $BC$ and $DE$ meet at point $A_1$. The tangent at $B$ to the circumcircle of the triangle $BE_1C$ meets the circumcircle $\omega$ of the pentagon for the second time at point $B_1$. The tangent at $D$ to the circumcircle of the triangle $DA_1C$ meets $\omega$ for the second time at point $D_1$. Prove that $B_1D_1 \parallel AE$.

**Solution.** Let us take any points $M$ and $N$ lying outside $\omega$ on the rays $B_1B$ and $D_1D$, respectively (see fig. 9.1). The angle $MBE_1$ is equal to the angle $BCE_1$ as an angle between a tangent line and a chord. Similarly, we get $\angle NDA_1 = \angle DCA_1$. Using the equality of vertical angles we obtain $\angle ABB_1 = \angle MBE_1 = \angle BCE_1 = \angle DCA_1 = \angle NDA_1 = \angle EDD_1$. Therefore, the arcs $AD_1$ and $EB_1$ are equal, and the claim follows.

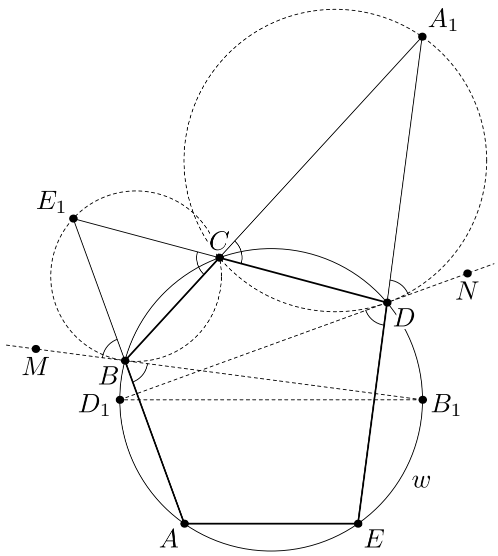



Fig. 9.1

**9.2.** (*F. Nilov*) Two circles $\omega_1$ and $\omega_2$ with centers $O_1$ and $O_2$ meet at points $A$ and $B$. Points $C$ and $D$ on $\omega_1$ and $\omega_2$, respectively, lie on the opposite sides of the line $AB$ and are equidistant from this line. Prove that $C$ and $D$ are equidistant from the midpoint of $O_1O_2$.

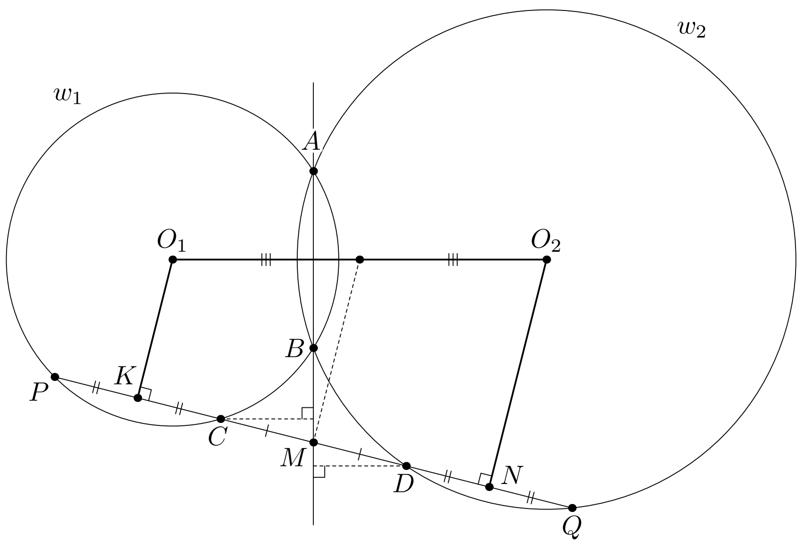



Fig. 9.2

**Solution.** Since the points $C$ and $D$ are equidistant from $AB$, the midpoint $M$ of $CD$ lies on $AB$ (see fig. 9.2). Let $P$ and $Q$ be the second common points of the line $CD$ with $\omega_1$ and $\omega_2$, respectively. Then $MC \cdot MP = MB \cdot MA = MD \cdot MQ$. Since $MC = MD$, we obtain that $MP = MQ$ and $PC = DQ$. Let $K$ and $N$ be the midpoints of $PC$ and $DQ$, respectively. Then $M$ is the midpoint of $KN$. Hence the midline of the right-angled trapezoid $O_1KNO_2$ is the perpendicular bisector of segment $CD$. Therefore the points $C$ and $D$ are equidistant from the midpoint of $O_1O_2$.

**9.3.** (*I. Bogdanov*) Each sidelength of a convex quadrilateral $ABCD$ is not less than 1 and not greater than 2. The diagonals of this quadrilateral meet at point $O$. Prove that $S_{AOB} + S_{COD} \leqslant 2(S_{AOD} + S_{BOC})$.

**Solution.** It suffices to prove that one of the ratios $\frac{AO}{OC}$ and $\frac{BO}{OD}$ is at most 2 and at least $\frac{1}{2}$. Indeed, assuming that $\frac{1}{2} \leqslant \frac{AO}{OC} \leqslant 2$ we get $S_{AOB} \leqslant 2S_{BOC}$ and $S_{COD} \leqslant 2S_{AOD}$; the claim follows. Thus, let us prove this fact.

Without loss of generality, we have $AO \leqslant OC$ and $BO \leqslant OD$. Assume, to the contrary, that $AO < \frac{OC}{2}$ and $BO < \frac{OD}{2}$. Let $A'$ and $B'$ be the points on segments $OC$ and $OD$, respectively, such that $OA' = 2OA$ and $OB' = 2OB$ (see fig. 9.3). Then we have $A'B' = 2AB \geq 2$. Moreover, the points $A'$ and $B'$ lie on the sides of triangle $COD$ and do not coincide with its vertices; hence the length of the segment $A'B'$ is less than one of the side lengths of this triangle. Let us now estimate the side lengths of $COD$.

The problem condition yields $CD \leqslant 2$. Since $O$ lies between $B$ and $D$, the length of the segment $CO$ does not exceed the length of one of the sides $CB$ and $CD$, therefore $CO \leqslant 2$. Similarly, $DO \leqslant 2$. Now, the length of $A'B'$ has to be less than one of these side lengths, which contradicts the fact that $A'B' \geq 2$.

**Remark.** The equality is achieved for the following degenerate quadrilateral. Consider a triangle $ABC$ with $1 \leqslant AB, BC \leqslant 2$ and $AC = 3$, and take a point $D$ on the segment $AC$ such that $CD = 1$, $DA = 2$.

It is easy to see that the inequality is strict for any non-degenerate quadrilateral.

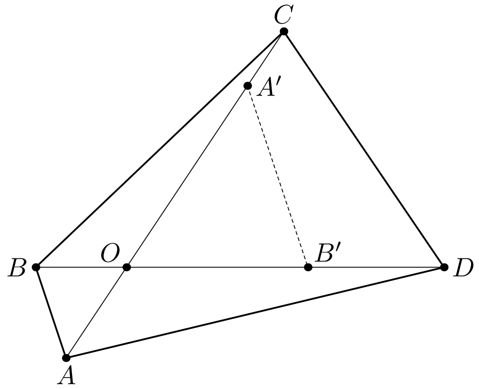



Fig. 9.3

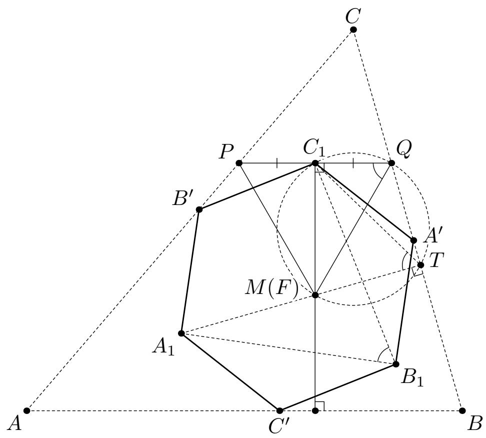



Fig. 9.4a

**9.4.** (*N. Beluhov*) A point $F$ inside a triangle $ABC$ is chosen so that $\angle AFB = \angle BFC = \angle CFA$. The line passing through $F$ and perpendicular to $BC$ meets the median from $A$ at point $A_1$. Points $B_1$ and $C_1$ are defined similarly. Prove that the points $A_1$, $B_1$, and $C_1$ are three vertices of some regular hexagon, and that the three remaining vertices of that hexagon lie on the sidelines of $ABC$.

**First solution.** We will reconstruct the whole picture from the other end. Let us start with some regular hexagon $A_1B'C_1A'B_1C'$ (see fig. 9.4a). Next, let $M$ be a point inside $\triangle A_1B_1C_1$ such that $\angle B_1MC_1 = 180^\circ - \alpha$, $\angle C_1MA_1 = 180^\circ - \beta$, and $\angle A_1MB_1 = 180^\circ - \gamma$ (this point lies inside the triangle $A_1B_1C_1$ since $F$ lies inside the triangle $ABC$). Let us draw the lines through $A'$, $B'$, and $C'$ perpendicular to $A_1M$, $B_1M$, and $C_1M$, respectively. Consider a triangle $ABC$ formed by them. This triangle is similar to the initial triangle from the problem statement, so we may assume that it is exactly that triangle.

Thus we are only left to show that the lines $AA_1, BB_1$ and $CC_1$ are the medians of $\triangle ABC$, and $M$ is its Fermat point (i.e., $M \equiv F$). Let the line parallel to $AB$ through $C_1$ meet $CA$ and $CB$ at points $P$ and $Q$, respectively. Construct $T = A_1M \cap CA'B$. Since $\angle A_1TA' = 90°$, point $T$ belongs to the circumcircle of $A_1B'C_1A'B_1C'$, and the quadrilateral $MC_1QT$ is cyclic. Therefore $\angle C_1QM = \angle C_1TM = \angle C_1TA_1 = \angle C_1B_1A_1 = 60°$. Similarly we get $\angle QPM = 60°$; thus $\triangle MPQ$ is equilateral, and $C_1$ the midpoint of $PQ$. Now, a homothety with center $C$ shows that $CC_1$ is a median of $\triangle ABC$, and that $CM$ passes through the third vertex of the equilateral triangle with base $AB$ constructed outside $ABC$ (this is a well-known construction for the Fermat point). By means of symmetry, the claim follows.

**Second solution.** Let $A_p$ be a first Apollonius point (see fig. 9.4b). It is known that the pedal triangle $A_0B_0C_0$ of $A_p$ is regular. Next, the Apollonius and the Torricelli point are isogonally conjugate. Therefore their pedal triangles have a common circumcircle $\omega$.

Let us describe the point $A_1$ in a different way. Let $E$ be the projection of $F$ to $BC$. Then $E$ lies on $\omega$, and the line $EF$ meets $\omega$ for the second time at point $A_1$. Notice that $\angle A_0EA_1 = 90°$; therefore $A_0A_1$ is a diameter of $\omega$. Similarly we may define the points $B_1$ and $C_1$. Thus, the triangles $A_1B_1C_1$ and $A_0B_0C_0$ are symmetric with respect to the center of $\omega$. Therefore, the hexagon $A_1B_0C_1A_0B_1C_0$ is regular. Now it remains to prove that the points $A_1$, $B_1$, and $C_1$ lie on the corresponding medians. This can be shown as in the previous solution.

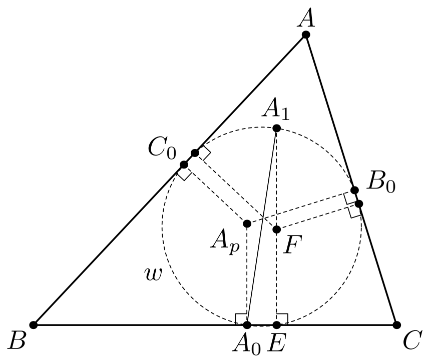



Fig. 9.4б

## Solutions
### Second day. 9 grade

**9.5** (*V. Yassinsky*) Points $E$ and $F$ lie on the sides $AB$ and $AC$ of a triangle $ABC$. Lines $EF$ and $BC$ meet at point $S$. Let $M$ and $N$ be the midpoints of $BC$ and $EF$, respectively. The line passing through $A$ and parallel to $MN$ meets $BC$ at point $K$. Prove that $\frac{BK}{CK} = \frac{FS}{ES}$.

**Solution.** Let the lines passing through $F$ and $E$ and parallel to $AK$ meet $BC$ at points $P$ and $Q$, respectively (see fig. 9.5). Since $N$ is the midpoint of $EF$, we have $PM = MQ$, therefore $CP = BQ$ and

$$\frac{BK}{CK} = \frac{CP}{CK} \cdot \frac{BK}{BQ} = \frac{CF}{CA} \cdot \frac{BA}{BE}.$$

Applying now the Menelaus theorem to triangle $AFE$ and line $CB$ we obtain

$$\frac{CF}{CA} \cdot \frac{BA}{BE} \cdot \frac{ES}{FS} = 1,$$

QED.

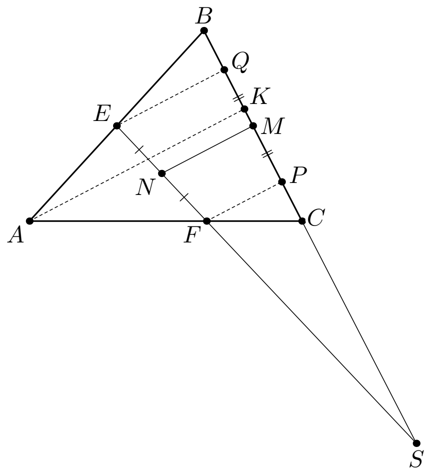



Fig. 9.5

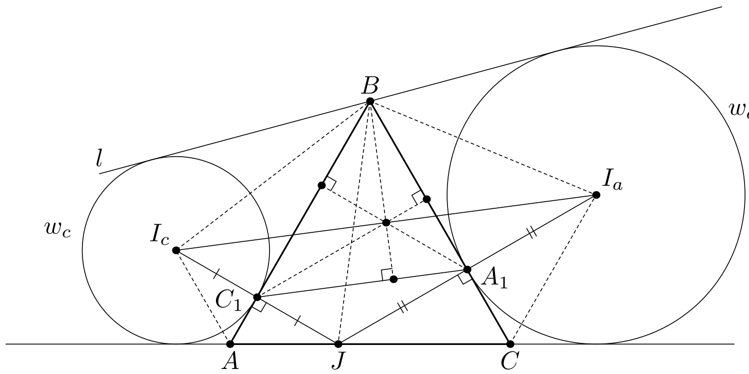



Fig. 9.6

**9.6** (*D. Shvetsov, J. Zaytseva, A. Sokolov*) A line $\ell$ passes through the vertex $B$ of a regular triangle $ABC$. A circle $\omega_a$ centered at $I_a$ is tangent to $BC$ at point $A_1$, and is also tangent to the lines $\ell$ and $AC$. A circle $\omega_c$ centered at $I_c$ is tangent to $BA$ at point $C_1$, and is also tangent to the lines $\ell$ and $AC$.

Prove that the orthocenter of triangle $A_1BC_1$ lies on the line $I_aI_c$.

**Solution.** By $\angle BAI_c = \angle BCI_a = 60^\circ$, the reflections of $I_c$ and $I_a$ in $BA$ and $BC$ respectively lie on $AC$. On the other hand, from $\angle ABI_c + \angle CBI_a = 60^\circ = \angle ABC$ we get that the reflections of $BI_c$ and $BI_a$ in $AB$ and $BC$ respectively meet $AC$ at the same point $J$ (see fig. 9.6). Hence $A_1C_1$ is the midline of triangle $JI_aI_c$. Then the altitudes of $\triangle A_1BC_1$ from $A_1$ and $C_1$ (which are parallel to the radii $I_cC_1$ and $I_aA_1$, respectively) are also the midlines of this triangle, thus meet at the midpoint of $I_aI_c$.

**9.7** (*A. Karlyuchenko*) Two fixed circles $\omega_1$ and $\omega_2$ pass through point $O$. A circle of an arbitrary radius $R$ centered at $O$ meets $\omega_1$ at points $A$ and $B$, and meets $\omega_2$ at points $C$ and $D$. Let $X$ be the common point of lines $AC$ and $BD$. Prove that all the points $X$ are collinear as $R$ changes.

**First solution.** Let $K$ be the second common point of $\omega_1$ and $\omega_2$ (see fig. 9.7). It suffices to prove that $\angle OKX = 90^\circ$.

We know that $OA = OB = OC = OD$. Therefore, the triangles $AOB$ and $COD$ are isosceles. Let $\alpha$ and $\beta$ be the angles at their bases, respectively. Then we have $\angle BKC = \angle BKO + \angle CKO = \angle BAO + \angle CDO = \alpha + \beta$. Since the quadrilateral $ACBD$ is cyclic, we obtain that $\angle BXC = 180^\circ - \angle XBC - \angle XCB = 180^\circ - \angle CAD - \angle ADB = 180^\circ - \frac{1}{2}(\smile{AB} + \smile{CD})$, where $\smile{AB}$ and $\smile{CD}$ are the arcs of the circle with center $O$. We have $\smile{AB} = 180^\circ - 2\alpha$ and $\smile{CD} = 180^\circ - 2\beta$; thus $\angle BXC = \angle BKC$, i.e., the quadrilateral $BXKC$ is cyclic. Hence $\angle XKB = \angle XCB = 180^\circ - \angle ACB = 90^\circ - \alpha$. Therefore $\angle OKX = \angle BKX + \angle BKO = 90^\circ$, QED.

**Second solution.** Let $OP$ and $OQ$ be diameters of $\omega_1$ and $\omega_2$, respectively. Then $X \in PQ$; one may easily prove this by means of an inversion with center $O$. Indeed, let $S$ be the common point of $AB$ and $CD$, let $M$ and $N$ be the midpoints of $AB$ and $CD$, respectively, and let $Y$ be the second common point of the circles $(ACS)$ and $(BDS)$. Since the figures $AYBM$ and $CYDN$ are similar, we have $Y \in (OMSN)$, and the claim follows as $Y$ and $(OMSN)$ are the images of $X$ and $PQ$.

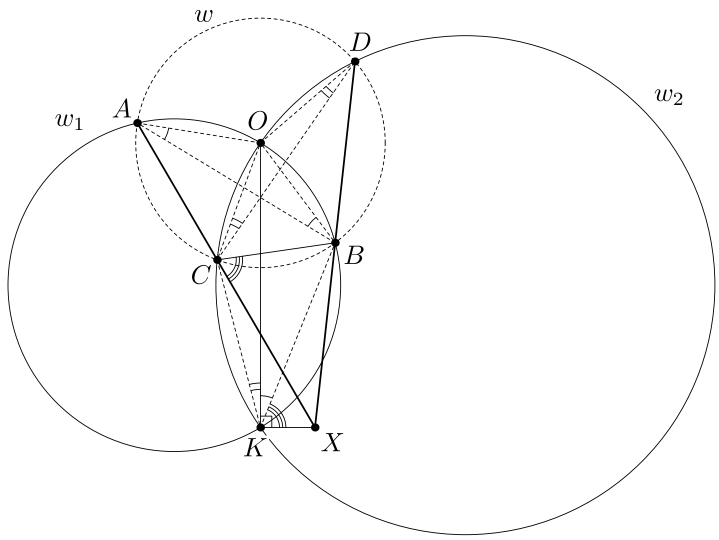



Fig. 9.7

**9.8** (*V. Protasov*) Three cyclists ride along a circular road with radius 1 km counterclockwise. Their velocities are constant and different. Does there necessarily exist (in a sufficiently long time) a moment when all the three distances between cyclists are greater than 1 km?

**Answer:** no.

**Solution. Solution.** If one changes the velocities of cyclists by the same value, then the distances between them stay the same. Hence, it can be assumed that the first cyclist stays at a point $A$ all the time.

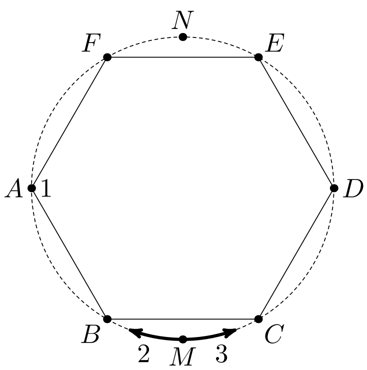



Fig. 9.8a

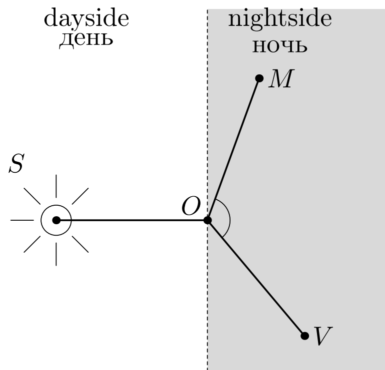



Fig. 9.8б

Let us inscribe a regular hexagon $ABCDEF$ in the circle. Let $M$ and $N$ be the midpoints of arcs $BC$ and $EF$ respectively. Suppose the second and the third cyclists start at the point $M$ with equal velocities and go to opposite directions: the second does towards $B$, the third does towards $C$. The distance between them is less than 1 km, until they reach those points. Then the second one is located less than 1 km away from the first, i.e., from the point $A$, until he reaches the point $F$. Simultaneously, the third one reaches $E$, and the distance between the second and the third becomes 1 km. Then this distance is reduced monotone until they meet at the point $N$. We obtain a configuration symmetric to the initial one with respect to the axis $AD$, with the interchange of the second and the third cyclists. Then the process is repeated all over again.

**Remark.** It can be shown that this is the only possible example, up to a shift of velocities of cyclists. It corresponds to the case when the three velocities form an arithmetic progression. In all other cases there exists a moment when the distances between cyclists exceed not only 1 km, but $\sqrt{2}$ km! This is equivalent to the following theorem, whose proof is left to the reader:

**Theorem.** *If, under the assumptions of Problem 9.8, the velocities of cyclists do not form an arithmetic progression, then there exists a moment when the three radii to the cyclists form obtuse angles.*

By applying this fact, ancient astronomers could have rigorously shown the impossibility of geocentric model of the Universe. To this end, it suffices to consider the orbits of three objects: the Sun, Mercury, and Venus. Let us denote them by points $S$, $V$, $M$ respectively and assume they move around the Earth (point $O$) along circular orbits. We suppose that they move on one plane (actually the planes of their orbits almost coincide). Their angular velocities are known to be different and not forming an arithmetic progression. Then there exists a moment when all the three angles between the rays $OS$, $OM$ and $OV$ are obtuse. Suppose an observer stands on the surface of the Earth at the point opposite to the direction of the ray $OS$. He is located on the nightside of the Earth and sees Mercury and Venus, since the angles $SOM$ and $SOV$ are both obtuse. The angular distance between those two planets, the angle $MOV$, is greater than $90^\circ$. However, the results of long-term observations available for ancient astronomers showed that the angular distance between Mercury and Venus never exceeds $76^o$. This contradiction shows the impossibility of the geocentric model with circular orbits.

## IX Geometrical Olympiad in honour of I.F.Sharygin

## Final round. Ratmino, 2013, August 1

### Solutions

### First day. 10 grade

**10.1** (*V. Yassinsky*) A circle $k$ passes through the vertices $B$ and $C$ of a triangle $ABC$ with $AB > AC$. This circle meets the extensions of sides $AB$ and $AC$ beyond $B$ and $C$ at points $P$ and $Q$, respectively. Let $AA_1$ be the altitude of $ABC$. Given that $A_1P = A_1Q$, prove that $\angle PA_1Q = 2\angle BAC$.

**Solution.** Since $\angle A_1AP = 90^\circ - \angle ABC = 90^\circ - \angle AQP$, the ray $AA_1$ passes through the circumcenter $O$ of the triangle $APQ$ (see fig. 10.1). This circumcenter also lies on the perpendicular bisector $\ell$ of the segment $PQ$. Since $AB \neq AC$, the lines $AO$ and $\ell$ are not parallel, so they have exactly one common point. But both $O$ and $A_1$ are their common points, so $A_1 = O$. Therefore, the inscribed angle $PAQ$ is the half of the central angle $PA_1Q$.

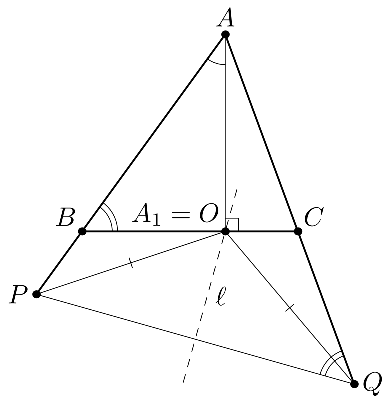



Рис. 10.1

**10.2** (*A. Polyansky*) Let $ABCD$ be a circumscribed quadrilateral with $AB = CD \neq BC$. The diagonals of the quadrilateral meet at point $L$. Prove that the angle $ALB$ is acute.

**Solution.** Assume to the contrary that $\angle ALB \geqslant 90^\circ$. Then we get $AB^2 \geqslant AL^2 + BL^2$ and $CD^2 \geqslant CL^2 + DL^2$; similarly, $AD^2 \leqslant AL^2 + DL^2$ and $BC^2 \leqslant BL^2 + CL^2$. Thus, $2AB^2 = AB^2 + CD^2 \geqslant AD^2 + BC^2$.

On the other hand, since the quadrilateral is circumscribed, we have $2AB = AB + CD = BC + AD$. This yields $AD \neq BC$ and

$$2(AD^2 + BC^2) = (AD + BC)^2 + (AD - BC)^2 > (2AB)^2 = 4AB^2.$$

A contradiction.

**10.3** (*A. Karlyuchenko*) Let $X$ be a point inside a triangle $ABC$ such that $XA \cdot BC = XB \times AC = XC \cdot AB$. Let $I_1$, $I_2$, and $I_3$ be the incenters of the triangles $XBC$, $XCA$, and $XAB$, respectively. Prove that the lines $AI_1$, $BI_2$, and $CI_3$ are concurrent.

**Solution 1.** Consider a tetrahedron $ABCX'$ with

$$AB \cdot CX' = BC \cdot AX' = CA \cdot BX'. \qquad (*)$$

Denote by $I'_a$, $I'_b$ and $I'_c$ the incenters of the triangles $BCX'$, $ACX'$, and $ABX'$. Then $(*)$ implies that the bisectors $AI'_b$ and $BI'_a$ of the angles $X'AC$ and $X'BC$ meet the segment $X'C$ at the same point. This implies that the segments $AI'_a$ and $BI'_b$ have a common point. Similarly, each of them has a common point with the segment $CI'_c$. Since these three segments are not coplanar, all three of them have a common point.

Now, tending $X'$ to $X$ along the intersection circle of the three corresponding Apollonius spheres for the pairs $(A,B)$, $(B,C)$, and $(A,C)$, we come to the problem statement.

**Solution 2.** Let $I$ be the incenter of the triangle $ABC$, and let $A_1$, $B_1$, and $C_1$ be the feet of the respective bisectors in this triangle. Let $T_c$ be the common point of the lines $CI_3$ and $XI$; define the points $T_a$ and $T_b$ similarly. We will prove that $T_a = T_b = T_c$.

Since $XB/XA = BC/AC$, the bisector $XI_3$ of the angle $BXA$ passes through $C_1$. Applying the Menelaus theorem to the triangle $\triangle XIC_1$ and the line $CI_3$, and using the properties of the bisector $AI_3$ of the angle $XAC_1$, we obtain

$$\frac{XT_c}{T_cI} = \frac{XI_3}{I_cC_1}\cdot\frac{C_1C}{CI} = \frac{XA}{AC_1}\cdot\frac{C_1C}{CI} = \frac{XA}{CI}\cdot\frac{C_1C}{AC_1} = \frac{XA}{CI}\cdot\frac{\sin A}{\sin(C/2)}.$$

Similarly we get

$$\frac{XT_b}{T_bI} = \frac{XA}{BI}\cdot\frac{\sin A}{\sin(B/2)}.$$

But $\dfrac{BI}{CI} = \dfrac{\sin(C/2)}{\sin(B/2)}$, so $\dfrac{XT_c}{T_cI} = \dfrac{XT_b}{T_bI}$, as desired.

**10.4** (*N. Beluhov*) We are given a cardboard square of area $1/4$ and a paper triangle of area $1/2$ such that all the squares of the side lengths of the triangle are integers. Prove that the square can be completely wrapped with the triangle. (In other words, prove that the triangle can be folded along several straight lines and the square can be placed inside the folded figure so that both faces of the square are completely covered with paper.)

**Solution. 1.** We say that a triangle is *elementary* if its area equals $\frac{1}{2}$, and the squares of its side lengths are all integral. Denote by $\Delta$ the elementary triangle with side lengths 1, 1, and $\sqrt{2}$.

Now we define the operation of *reshaping* as follows. Take a triangle $ABC$; let $AM$ be one of its medians. Let us cut it along $AM$, and glue the pieces $\triangle ABM$ and $\triangle ACM$ along the equal segments $BM$ and $CM$ to obtain a new triangle with the side lengths $AB$, $AC$, and $2AM$.

**2.** We claim that for every elementary triangle $\delta$, one may apply to it a series of reshapings resulting in $\Delta$.

To this end, notice that a reshaping always turns an elementary triangle into an elementary triangle: indeed, reshaping preserves the area, and, by the median formula $4m_a^2 = 2b^2 + 2c^2 - a^2$, it also preserves the property that the side lengths are integral.

Now let us take an arbitrary elementary triangle $\delta$. If its angle at some vertex is obtuse, then let us reshape it by cutting along the median from this vertex; the maximum side length of the new triangle will be strictly smaller than that of the initial one. Let us proceed on this way. Since all the squares of the side lengths are integral, we will eventually stop on some triangle $\delta'$ which is right- or acute-angled. The sine of the maximal angle of $\delta'$ is not less than $\sqrt{3}/2$, so the product of the lengths of the sides adjacent to this angle is at most $2/\sqrt{3}$. Hence both of them are unit, and the angle between them is right. Thus $\delta' = \Delta$, as desired.

**3.** Conversely, if $\delta'$ is obtained from $\delta$ by a series of reshapings, then $\delta$ can also be obtained from $\delta'$. Therefore, each elementary triangle $\delta$ can be obtained from $\Delta$.

**4.** Now, let us say that a triangle $\delta$ forms a *proper wrapping* if our cardboard square can be wrapped up completely with $\delta$ in such a way that each pair of points on the same side of $\delta$ equidistant from its midpoint comes to the same point on the same face of the folded figure. The triangle $\Delta$ forms a proper wrapping when folded along two its shorter midlines.

Suppose that a triangle $\delta = ABC$ forms a proper wrapping, and let $AM$ be one of its medians. Consider the corresponding folding of this triangle. In it, let us glue together the segments $BM$ and $CM$ (it is possible by the definition of a proper wrapping), and cut our triangle along $AM$. We will obtain a folding of the reshaping of $\delta$ along $AM$; thus, this reshaping is also a proper wrapping. Together with the statement from part 3, this implies the problem statement.

**Remark 1.** From this solution, one may see that the following three conditions are equivalent:

(a) the triangle $ABC$ is elementary;

(b) there exists a copy of $\triangle ABC$ such that all its vertices are integer points;

(c) there exist six integers $p$, $q$, $r$, $s$, $t$, $u$ such that $p+q+r=s+t+u=0$ and $p^2+s^2=AB$, $q^2+t^2=BC$, $r^2+u^2=CA$.

**Remark 2.** The equivalence of the conditions (b) and (c) is obvious. The fact that (a) is also equivalent to them can be proved in different ways. E.g., one may start from Heron's formula; for the elementary triangle with side lengths $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ it asserts $2(ab+bc+ca)-(a^2+b^2+c^2)=1$. One may show — for instance, by the descent method — that all integral solutions of this equation satisfy (c).

Another approach is the following one. Consider an elementary triangle $ABC$ and let it generate a lattice (that is, take all the endpoints $X$ such that $\overrightarrow{AX}=k\overrightarrow{AB}+\ell\overrightarrow{AC}$ with integral $k$ and $\ell$). Using the cosine law, one easily gets that all the distances between the points of this lattice are roots of integers. Now, from the condition on the area, we have that the minimal area of a parallelogram with vertices in the lattice points is 1. Taking such a parallelogram with the minimal diameter, one may show that it is a unit square[1].

This lattice also helps in a different solution to our problem. For convenience, let us scale the whole picture with coefficient 2; after that, the vertices of the triangle have even coordinates, and its area is 2, and we need to wrap a unit square. Now, let us paint our checkered plane chess-like and draw on it the lattice of the triangles equal to $ABC$; their vertices are all the points with even coordinates. Notice that all the triangles are partitioned into two classes: the translations of $ABC$ and the symmetric images of it.

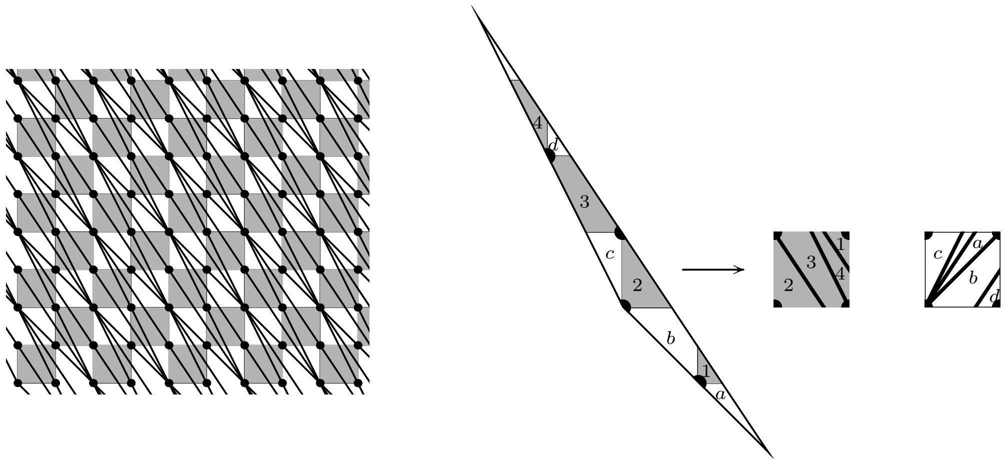


Рис. 10.4

Now, let us wrap a black square with vertex $A$ with the triangle $ABC$, folding it by the sides of the cells. Then a black face of the square will get all black parts of the triangle; the parts from the black squares in even rows will be shifted, while those from the other black squares will be reflected at some points.

On the other hand, all the black squares in the even rows are partitioned by the triangles in the same manner; the partition of any other black square again can be obtained by a reflection of that first partition. Such a reflection interchanges the two classes of triangles. Finally, now it is easy to see that our black square will be completely covered: those its parts which are in the triangles of the first class — by the translations of the parts of $ABC$, and the others — by the reflections of the other black parts of $ABC$. The same applies to the other face of the square.

[1]Cf. problem 10.7 from the Final round of the 5th olympiad in honour of I.F.Sharygin, 2009.

## Solutions
## Second day. 10 grade

**10.5** (*D. Shvetsov*) Let $O$ be the circumcenter of a cyclic quadrilateral $ABCD$. Points $E$ and $F$ are the midpoints of arcs $AB$ and $CD$ not containing the other vertices of the quadrilateral. The lines passing through $E$ and $F$ and parallel to the diagonals of $ABCD$ meet at points $E$, $F$, $K$, and $L$. Prove that line $KL$ passes through $O$.

**Solution.** For concreteness, let $K$ lie on the line parallel to $AC$ through $E$, as well as on the line parallel to $BD$ through $F$ (see fig. 10.5). Notice that

$$\measuredangle(KE, EF) = \measuredangle(AC, EF) = \frac{\smile{CF} + \smile{AE}}{2} = \frac{\smile{FD} + \smile{EB}}{2} = \measuredangle(BD, EF) = \measuredangle(KF, EF).$$

This means that the triangle $KEF$ is isosceles, $KE = KF$. Hence the parallelogram $EKFL$ is in fact a rhombus, and $KL$ is the perpendicular bisector of $EF$, thus it contains $O$.

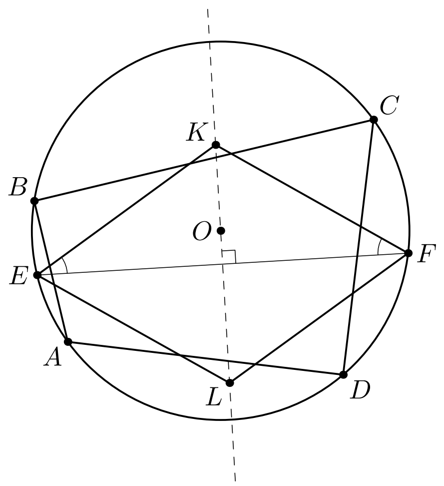



Рис. 10.5

**10.6** (*D. Prokopenko*) The altitudes $AA_1$, $BB_1$, and $CC_1$ of an acute-angled triangle $ABC$ meet at point $H$. The perpendiculars from $H$ to $B_1C_1$ and $A_1C_1$ meet the rays $CA$ and $CB$ at points $P$ and $Q$, respectively. Prove that the perpendicular from $C$ to $A_1B_1$ passes through the midpoint of $PQ$.

**Solution 1.** Let $N$ be the projection of $C$ to $A_1B_1$. Consider a homothety $h$ centered at $C$ and mapping $H$ to $C_1$; thus $h(P) = P_1$ and $h(Q(= Q_1$. We have $C_1P_1 \perp C_1B_1$ and $C_1Q_1 \perp C_1A_1$; it suffices to prove now that the line $CN$ bisects $P_1Q_1$.

Let $K$ and $L$ be the projections of $P_1$ and $Q_1$, respectively, to the line $A_1B_1$. It is well known that $\angle CB_1A_1 = \angle AB_1C_1$; so, $\angle P_1B_1K = \angle P_1B_1C_1$, and the right-angled triangles $P_1B_1K$ and $P_1B_1C_1$ are congruent due to equal hypothenuses and acute angles. Hence $B_1K = B_1C_1$. Similarly, $A_1L = A_1C_1$, so the length of $KL$ equals the perimeter of $\triangle A_1B_1C_1$.

Since $C$ is an excenter of the triangle $A_1B_1C_1$, the point $N$ is the tangency point of the corresponding excircle with $A_1B_1$, so $B_1N = p - B_1C_1$. Then we have $KN = B_1C_1 + p - B_1C_1 = p$, thus $N$ is the midpoint of $KL$. Finally, by the parallel lines $P_1K$, $CN$, and $Q_1L$ we conclude that the line $CN$ bisects $P_1Q_1$, as required.

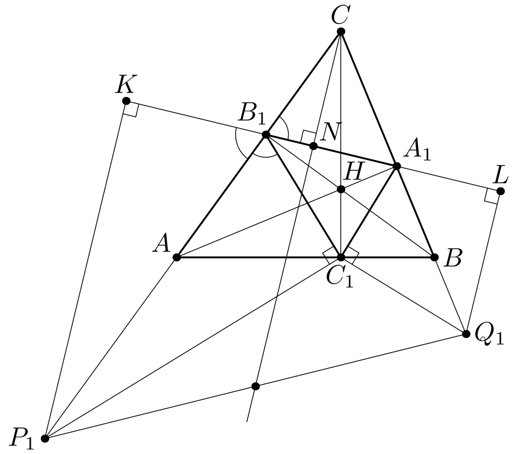



Рис. 10.6а

**Solution 2.** Denote $\angle BAC = \alpha$ and $\angle ABC = \beta$; then we also have $\angle ACC_1 = 90^\circ - \alpha$ and $\angle BCC_1 = 90^\circ - \beta$. By $\triangle AB_1C_1 \sim \triangle A_1BC_1 \sim \triangle ABC$ we get $\angle HPC = 90^\circ - \angle AB_1C_1 = 90^\circ - \beta$; similarly, $\angle HQC = 90^\circ - \alpha$. Next, let the perpendicular from $C$ to $A_1B_1$ meet $PQ$ at $X$. Then $\angle PCX = 90^\circ - \beta$ and $\angle QCX = 90^\circ - \alpha$.

We need to show that $CX$ is a median in $\triangle CPQ$; since $\angle PCX = \angle QCH$, this is equivalent to the fact that $CH$ is its symmedian. Therefore we have reduced the problem to the following known fact (see, for instance, A. Akopyan, “Geometry in figures”, problem 4.4.6).

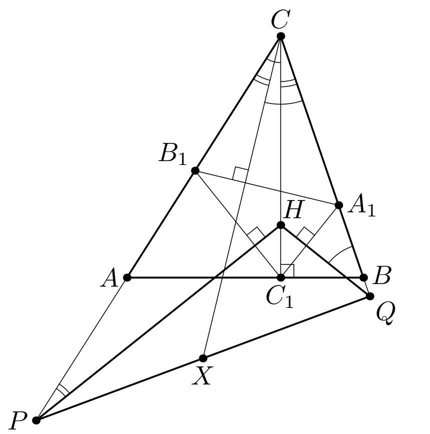



Рис. 10.6б

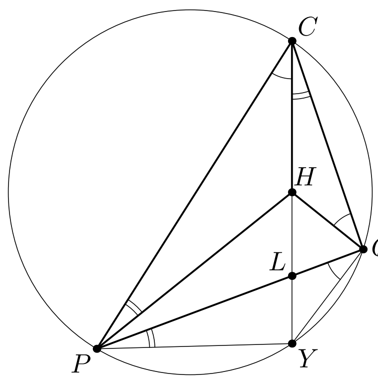



Рис. 10.6в

**Lemma.** *Assume that a point $H$ inside a triangle $CPQ$ is chosen so that $\angle CPH = \angle QCH$ and $\angle CQH = \angle PCH$. Then $CH$ is a symmedian in this triangle.*

*Proof.* The triangles $PHC$ and $CHQ$ are similar due to two pairs of equal angles. Now, let $Y$ be the second intersection point of the circumcircle of $\triangle CPQ$ with $CH$. Then $\angle YPH = \angle YPC - \angle CPH = (180^\circ - \angle YQC) - \angle YCQ = \angle HYQ$, and hence the triangles $PHY$ and $YHQ$ are also similar. From these similarities one gets

$$\left(\frac{PY}{YQ}\right)^2 = \frac{PH}{HY} \cdot \frac{HY}{HQ} = \frac{PH}{HQ} = \left(\frac{PC}{CQ}\right)^2,$$

so $CPYQ$ is a harmonic quadrilateral. This is equivalent to the statement of the Lemma.

**Remark.** One may easily obtain from the proof of the Lemma that $H$ is a midpoint of $CY$.

Another proof of the Lemma (and even of the problem statement) may be obtained as follows. After noticing that the triangles $PHC$ and $CHQ$ are similar, it is easy to obtain the equality $\dfrac{CP}{CQ} = \dfrac{PH}{HC} = \dfrac{\sin\angle PCH}{\sin\angle QCH} = \dfrac{\sin\angle QCX}{\sin\angle PCX}$.

**10.7** (*B. Frenkin*) In the space, five points are marked. It is known that these points are the centers of five spheres, four of which are pairwise externally tangent, and all these four are internally tangent to the fifth one. It turns out, however, that it is impossible to determine which of the marked points is the center of the fifth (the largest) sphere. Find the ratio of the greatest and the smallest radii of the spheres.

**Answer.** $\dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}} = \dfrac{5+\sqrt{21}}{2}$.

**Solution.** Denote by $O$ and $O'$ two possible positions of the center of the largest sphere (among the five marked points). and denote by $A$, $B$, and $C$ the other three marked points.

Consider the points $O$, $O'$, $A$, and $B$. In the configuration of spheres where $O$ is the center of the largest sphere, denote by $R$, $r'$, $r_a$, and $r_b$ the radii of the spheres centered at $O$, $O'$, $A$, and $B$, respectively. Then we have $OO' = R - r'$, $OA = R - r_a$, $OB = R - r_b$, $O'A = r' + r_a$, $O'B = r' + r_b$, and $AB = r_a + r_b$, which yields $OO' - AB = OA - O'B = OB - O'A$; denote this common difference by $d$. Similarly, from the configuration with $O'$ being the center of the largest sphere we obtain $d = OO' - AB = O'A - OB = O'B - OA = -d$. Thus $d = 0$, and therefore $OO' = AB$, $OA = O'B$, and $OB = O'A$.

Applying similar arguments to the tuples $(O, O', A, C)$ and $(O, O', B, C)$ we learn $OO' = AB = AC = BC$ and $OA = O'B = OC = O'A = OB = O'C$. So, the triangle $ABC$ is equilateral (let its side length be $2\sqrt{3}$), and the regular pyramids $OABC$ and $O'ABC$ are congruent. Thus the points $O$ and $O'$ are symmetrical to each other about $(ABC)$. Moreover, we have $OO' = 2\sqrt{3}$, so the altitude of each pyramid has the length $\sqrt{3}$. Let $H$ be the common foot of these altitudes, then $HO = HO' = \sqrt{3}$ and $HA = HB = HC = 2$, thus $OA = O'A = \sqrt{7}$. So the radii of the spheres centered at $A$, $B$, and $C$ are equal to $\sqrt{3}$, while the radii of the other two spheres are equal to $\sqrt{7} - \sqrt{3}$ and $\sqrt{7} + \sqrt{3}$, whence the answer.

**10.8** (*A. Zaslavsky*) In the plane, two fixed circles are given, one of them lies inside the other one. For an arbitrary point $C$ of the external circle, let $CA$ and $CB$ be two chords of this circle which are tangent to the internal one. Find the locus of the incenters of triangles $ABC$.

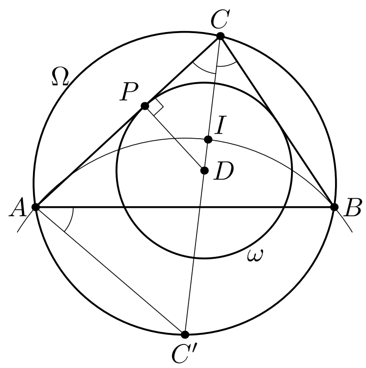


Рис. 10.8

**Solution.** Denote by $\Omega$ and $\omega$ the larger and the smaller circle, and by $R$ and $r$ their radii, respectively (see fig. 10.8). Denote by $D$ the center of $\omega$. Let $C'$ be the midpoint of the arc $AB$ of $\Omega$ not containing $C$, and let $I$ be the incenter of $\triangle ABC$. Then the points $I$ and $D$ lie on $CC'$; next, it is well known that $C'I = C'A = 2R\sin\angle ACC'$.

On the other hand, denoting by $P$ the tangency point of $AC$ and $\omega$, we have $\sin\angle ACC' = PD/CD = r/CD$. Next, the product $d = CD \cdot C'D$ is negated power of the point $D$ with respect to $\Omega$, thus it is constant. So we get $C'I = 2Rr/CD = C'D \cdot 2Rr/d$, whence

$$\overrightarrow{ID} = \overrightarrow{C'D} - \overrightarrow{C'I} = \overrightarrow{C'D} \cdot \left(1 - \frac{2Rr}{d}\right).$$

Thus, the point $I$ lies on the circle obtained from $\Omega$ by scaling at $D$ with the coefficient $\dfrac{2Rr}{d} - 1$.

Conversely, from every point $I$ of this circle, one may find the points $C$ and $C'$ as the points of intersection of $ID$ and $\Omega$; the point $C'$ is chosen as the image of $I$ under the inverse scaling. For the obtained point $C$, the point $I$ is the desired incenter; hence our locus is the whole obtained circle.

**Remark.** If $2Rr = d$, the obtained locus degenerates to the point $D$. In this case, one may obtain from our solution the *Euler formula* for the distance between the circumcenter and the incenter of a triangle.