VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. Grade 8. Solutions.

1. (M.Rozhkova, Ukraine) For a nonisosceles triangle ABC three lines are considered: the altitude from vertex A and two bisectrices from remaining vertices. Prove that the circumcircle of the triangle formed by these three lines touches the bisectrix from vertex A.

   Solution. Let I be the incenter of the triangle, B' be the foot of the bisectrix from vertex B, and X be the common point of the bisectrix from C and the altitude from A Then $\angle AIB' = \angle A/2 + \angle C/2 = 90° - \angle B/2 = \angle IXA$, and we obtain the required assertion. The other dispositions of points can be considered similarly.

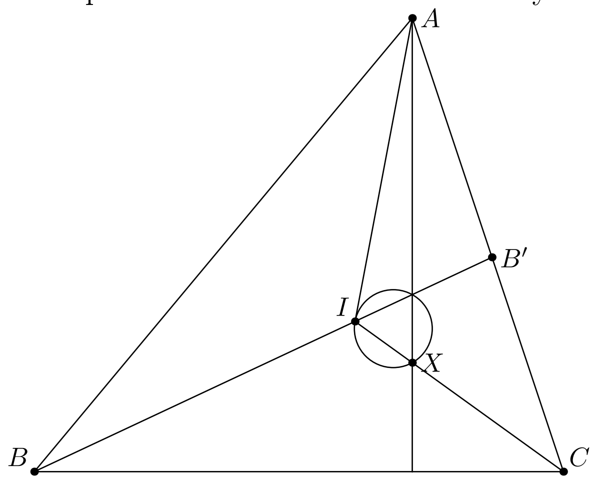

Fig.8.1

2. (A.Akopyan) Two points A and B are given. Find the locus C of points such that triangle ABC can be covered by a circle with radius 1.

   Solution. Evidently the required locus is empty when $AB > 2$, and it is a circle with diameter AB when $AB = 2$. Let $AB < 2$, and let P, Q be the common points of two circles with centers A, B and radii equal to 1. Then the required locus is the union of unit circles with centers in the circular «lens» formed by arcs PQ of these circles. Let P_1, P_2, Q_1, Q_2 be points such that P is the midpoint of segments AP_1, BP_2, and Q is the midpoint of segments AQ_1, BQ_2. Construct four arcs: P_1Q_1 with center A and radius 2, P_2Q_2 with center B and radius 2, P_1P_2 with center P and radius 1, Q_1Q_2 with center Q and radius 1 These arcs bound the required locus.

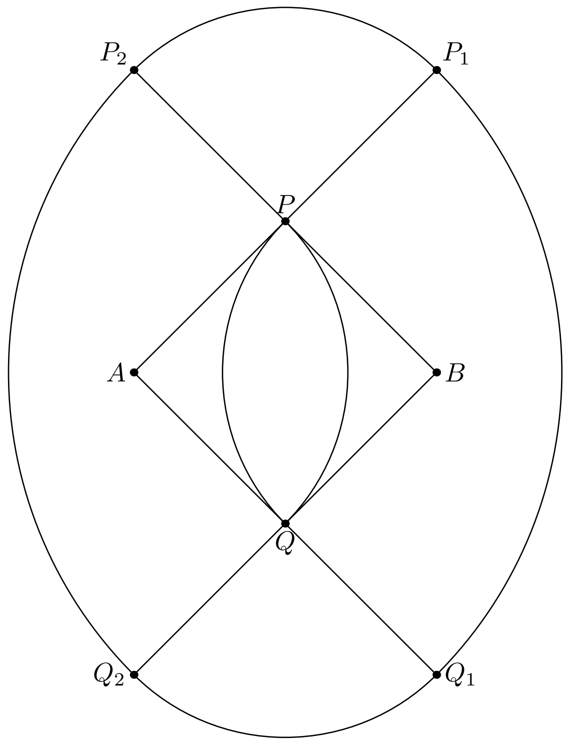

Fig.8.2

3. (S.Berlov, D.Prokopenko) Let $ABCD$ be a convex quadrilateral and K be the common point of rays AB and DC. There exists a point P on the bisectrix of angle AKD such that lines BP and CP bisect segments AC and BD respectively. Prove that $AB = CD$.

   Solution. Since lines BP and CP contain the medians of triangles ABC and BCD, we have $S_{KAB} = S_{KBC} = S_{KCD}$. Since triangles KAB and KCD have the equal altitudes, their bases are also equal.

4. (I.Bogdanov) Circles ω_1 and ω_2 inscribed into equal angles X_1OY and YOX_2 touch lines OX_1 and OX_2 in points A_1 and A_2 respectively. Also they touch OY in points B_1 and B_2. Let C_1 be the second common point of A_1B_2 and ω_1; C_2 be the second common point of A_2B_1 and ω_2. Prove that C_1C_2 is the common tangent of the two circles.

   Solution. Triangle OA_1B_2 equals triangle OB_1A_2 by two sides and included angle. Thus $A_1B_2 = B_1A_2$. Also from $B_2C_1 \cdot B_2A_1 = B_2B_1^2 = B_1C_2 \cdot B_1A_2$ we obtain $B_2C_1 = B_1C_2$, and since $\angle A_1OB_1 = \angle B_2OA_2$, we have $\angle A_1C_1B_1 + \angle A_2C_2B_2 = 180°$. Therefore quadrilateral $B_2C_2B_1C_1$ is cyclic and so it is an isosceles trapezoid. Then $\angle B_2C_2C_1 = \angle C_2B_2B_1$ and C_2C_1 is the tangent

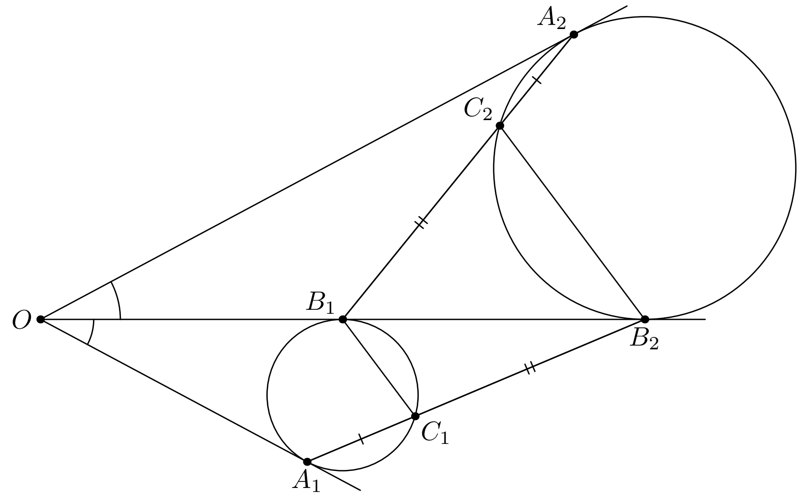

Fig.8.4

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. Grade 8. Solutions.

5. (B.Frenkin) Let AH, BL and CM be the altitude, the bisectrix and the median in triangle ABC. It is known that lines AH and BL are the altitude and the bisectrix of triangle HLM. Prove that line CM is the median of this triangle.

   Solution. Since $AH \perp LM$, we have $LM \parallel BC$, i.e. LM is the medial line of the triangle. Therefore BL is the bisectrix and the median, i.e. $AB = BC$. Now from equality of triangles BLM and BLH we obtain that $BH = BM = AM = CH$. Thus $AB = AC$, and ABC is a regular triangle.

6. (D.Prokopenko) Let E, F be the midpoints of sides BC, CD of square $ABCD$. Lines AE and BF meet in point P. Prove that $\angle PDA = \angle AED$.

   First solution. Let the line passing through A and parallel to BF meet CD in point G. Since $ABFG$ is a parallelogram, we have $FG{=}AB$ and so $FD = DE$. By Thales theorem the line passing through D and parallel to BF is the median of triangle ADP. Evidently $AE \perp BF$, therefore this line is also the altitude. Thus triangle ADP is isosceles as well as triangle AED. Angle EAD is their common angle at base, therefore their angles at apex are equal.

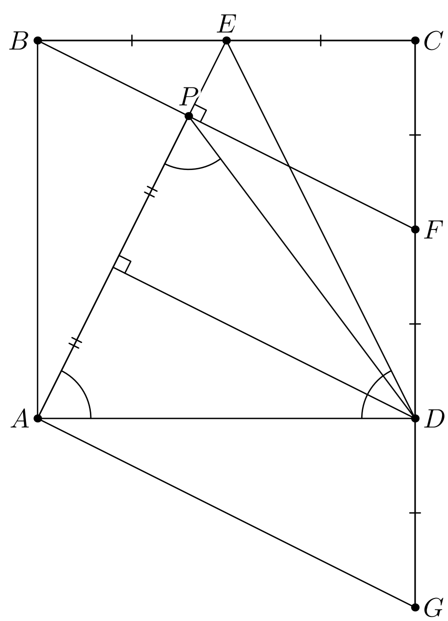

Fig.8.6

   Second solution. Let $AB = 1$. Since BP is the altitude of right-angled triangle with legs 1 and $1/2$, we have $AP : PE = 4 : 1$. Then by Thales theorem the projection of segment DP to CD is equal to $4/5$. Similarly its projection to AD is equal to $3/5$. Therefore by Pythagorean theorem $DP = 1 = AD$ and we can argue as in the previous solution.

7. (B.Frenkin) Each of two regular polygons P and Q was divided by a line into two parts. One part of P and one part of Q were attached along the dividing line to form a regular polygon not equal to P and Q. How many sides can it have?

   Answer. 3 or 4.

Solution. It is evident that the new polygon contains at least one vertex of each of the given polygons. On the other hand it can't have more than one vertex of each of these polygons because it isn't equal to them. Thus it has two vertices which are the vertices of P or Q and one or two vertices on the dividing line, i.e. three or four vertices. Both cases are possible: we can cut off two equal right-angled triangles from two regular triangles or two equal isosceles right-angled triangles from two squares.

8. (A.Zaslavsky) Bisectrices AA_1 and BB_1 of triangle ABC meet in I. Segments A_1I and B_1I are the bases of isosceles triangles with vertices A_2 and B_2 lying on line AB. It is known that line CI bisects segment A_2B_2. Is it true that triangle ABC is isosceles?

   Answer. No, the condition of the problem is true for any triangle with $\angle C = 120°$.

   Solution. Let CC_1 be the bisectrix of angle C. Then CA_1 is the external bisectrix of angle ACC_1, i.e. point A_1 lies on equal distances from lines AC and CC_1. Also this point lies on equal distances from lines AC and AB, thus C_1A_1 is the bisectrix of angle CC_1B. Let J be the common point of lines C_1A_1 and BI. Since C_1A_1 and BI are the bisectrices of triangle BCC_1 with $\angle C = 60°$, we have $\angle IJA_1 = 120°$. Then quadrilateral $CIJA_1$ is cyclic and $IJ = JA_1$. Consider regular triangle IA_1K. Since $JK = JI$ and $\angle C_1JI = \angle C_1JK = 60°$, K lies on C_1B, i.e. coincides with point A_2. Now we have $C_1A_2 = C_1I$. Similarly $C_1B_2 = C_1I$.

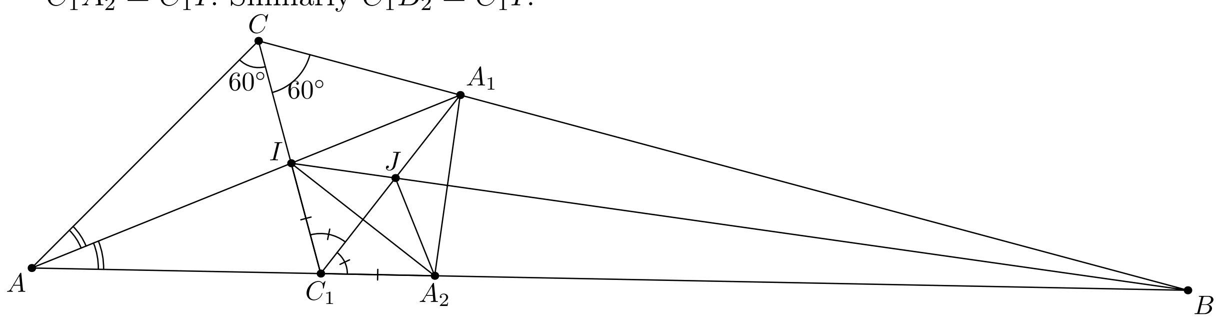

Fig.8.8

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. Grade 9. Solutions.

1. (B.Frenkin) For each vertex of triangle ABC the angle between the altitude and the bisectrix from this vertex was found. It is known that these angles in vertices A and B are equal and the angle in vertex C is greater than the two other angles. Find angle C of the triangle.

   Answer. 60°.

   Solution. The angle between the altitude and the bisectrix is equal to the half of the absolute difference of angles adjacent to the opposite side of the triangle. Thus if these angles in vertices A and B are equal then $\angle A - \angle C = \angle B - \angle C$ or $\angle A - \angle C = \angle C - \angle B$. In the first case the triangle is isosceles, i.e. the altitude and the bisectrix from C coincide which contradicts the condition of the problem. In the second case $\angle C = (\angle A + \angle B)/2 = (180^\circ - \angle C)/2 = 60^\circ$.

2. (A.Akopyan) Two intersecting triangles are given. Prove that at least one of their vertices lies inside the circumcircle of the other triangle.

   Solution. If one of the circumcircles lies inside the other one then the assertion of the problem is evidently true. If each circumcircle lies outside the other one then the triangles can't intersect. Let the circumcircles intersect in points P and Q, and suppose that the assertion in question isn't true. Then all vertices of each triangle lie on the arc PQ of the respective circle lying outside the second circle. But these arcs lie in the distinct semiplanes wrt line PQ. Thus the triangles also lie in the distinct semiplanes and can't intersect.

3. (V.Yasinsky, Ukraine) Points X, Y, Z lie on the line (in the indicated order). Triangles XAB, YBC, ZCD are regular, the vertices of the first and the third one are oriented counterclockwise and the vertices of the second are oriented oppositely. Prove that AC, BD and XY concur.

   Solution. The rotation around B by 60° maps A and C to X and Y respectively. Thus the angle between AC and XY is equal to 60°. Let P be the common point of these lines. Then since quadrilateral $AXPB$ is cyclic we have $\angle APB = 60^\circ$ and quadrilateral $PYCB$ also is cyclic. Similarly we obtain that BD also passes through P

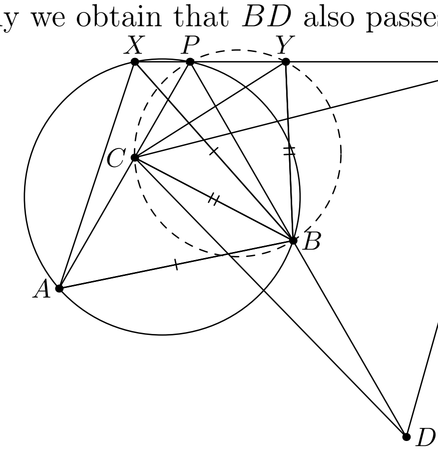

Fig.9.3

4. (A.Zaslavsky) In triangle ABC, touching points A', B' of the incircle with BC, AC and common point G of segments AA' and BB' were marked. After this the triangle was erased. Restore it by ruler and compass.

   Solution. Let C' be the touching point of the incircle with AB; A_1, B_1, C_1 be the projections of G to the sidelines of $A'B'C'$. Then G as the Lemoine point of $A'B'C'$ is the centroid of triangle $A_1B_1C_1$. Therefore we have the following construction. Find point C_1 and its image C_2 under the homothety with center G and coefficient $-1/2$. Now construct the circles with diameters GA', GB' and find the common point of one of them with the reflection of the remaining one in C_2. This point lies on one side of $A'B'C'$ and the symmetric point on the other side. In general case the problem has two solutions.

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. Grade 9. Solutions.

5. (D.Shvetsov) The incircle of right-angled triangle ABC ($\angle ABC = 90^\circ$) touches AB, BC, AC in points C_1, A_1, B_1 respectively. One of the excircles touches the side BC in point A_2. Point A_0 is the circumcenter of triangle $A_1A_2B_1$; point C_0 is defined similarly. Find angle A_0BC_0.

   Solution. Points A_1 and A_2 are symmetric wrt the midpoint of BC, thus $A_0B = A_0C$. On the other hand A_0 lies on the bisector of segment A_1B_1 coinciding with the bisectrix of angle C. Therefore $\angle CBA_0 = \angle A_0CB = \angle C/2$. Similarly $ABC_0 = \angle A/2$, thus $\angle A_0BC_0 = 45^\circ$.

6. (Y.Blinkov) An arbitrary line passing through vertex B of triangle ABC meets side AC in point K and the circumcircle in point M. Find the locus of circumcenters of triangles AMK.

   Solution. Let O be the circumcenter of AMK. Since $\angle AMK = \angle AMB = \angle C$, we have $\angle AOK = 2\angle C$ and $\angle OAC = 90^\circ - \angle C$, i.e. this angle doesn't depend on K, M . Therefore all circumcenters are collinear. When K moves from A to C, they fill the lateral side of an isosceles triangle with base AC and the angle at base equal to $90^\circ - \angle C$. (If angle C is obtuse then the respective angle is equal to $\angle C - 90^\circ$ and the locus in question lies on the same side wrt AC as point B).

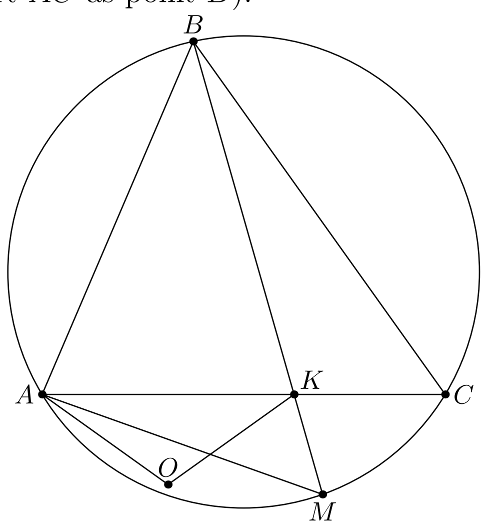

Fig.9.6

7. (N.Beluhov, Bulgaria) Given triangle ABC. Let AL_a and AM_a be the internal and the external bisectrix of angle A. Let ω_a be the reflection of the circumcircle of $\triangle AL_aM_a$ in the midpoint of BC. Circle ω_b is defined similarly. Prove that ω_a and ω_b touch iff $\triangle ABC$ is right-angled.

   Solution. The circumcircle of triangle AL_aM_a is orthogonal to the circumcircle of triangle ABC. Also it is the locus of points X for which $BX : CX = AB : AC$. Therefore ω_a also is orthogonal to the circumcircle of ABC and is the locus of points X for which $BX : CX = AC : AB$. From similar reasoning for ω_b we obtain that when these circles touch, the touching point X lies on the circumcircle of ABC and $AX : BX : CX = BC :$

$CA : AB$. By Ptolomeus theorem one of products $AX \cdot BC$, $BX \cdot CA$, $CX \cdot AB$ is equal to the sum of two remaining ones. These products are proportional to the squares of sides of triangle ABC, thus it is right-angled. The converse assertion can be obtained similarly.

8. (V.Gurovits) Given is a regular polygon. Volodya wants to mark k points on its perimeter in such a way that any other regular polygon (maybe having another number of sides) doesn't contain all marked points on its perimeter. Find the minimal k sufficient for any given polygon.

   Answer. $k = 5$.

   Solution. Firstly let us prove that five points are sufficient. Let A, B, C, D be four successive vertices of the given polygon (the case of a triangle is evident). Mark points A, B, point X on AB, point Y on BC close to B and point Z on CD close to C. Then line AB must contain the side of the polygon. The angle between this line and the side passing through Y is equal to some angle from the finite set. It is clear that the respective common point can't lie on ray BA, and if it lies on the opposite ray then point Z is outside the polygon. Thus we restore line BC. Line CD and so the polygon are restored similarly.

   Now let us prove that four points don't suffice wnen the number of sides is big. Consider firstly the case when three of marked points lie on the line ℓ. The regular triangle based on the respective side lies inside the polygon, thus the remaining marked point lies outside this triangle. Constructing two lines passing through this point and forming angles with ℓ equal to $60°$ we obtain the regular triangle containing all marked points on its perimeter.

   Now let marked points form a convex quadrilateral. Suppose that two of its opposite angles are less than $60°$. On diagonal AB passing through two remaining vertices, construct an arc equal to $240°$ and lying in the other semiplane than the center of the polygon. This arc meets the circumcircle of the polygon in two points close to A и B. The sides containing A and B form with AB the angles less than $120°$, thus respective part of the perimeter lies inside the arc which contradicts our supposition. Therefore there exist two adjacent angles of quadrilateral greater than $60°$. Constructing two lines which form angles equal to $60°$ with the respective side of the quadrilateral and passing two remaining vertices we obtain the regular triangle containing all marked points on its perimeter.

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. Grade 10. Solutions.

1. (A.Zaslavsky) Let O, I be the circumcenter and the incenter of a right-angled triangle; R, r be the radii of respective circles; J be the reflection of the vertex of the right angle in I. Find OJ.

   Answer. $R - 2r$.

   Solution. Let ABC be the given triangle, $\angle C = 90°$. Clearly the circle with center J and radius $2r$ touches AC, BC. Let us prove that it also touches the circumcircle of ABC.

   Consider the circle touching AC and BC in points P, Q and touching the circumcircle internally in point T. Since T is the homothety center of this circle and the circumcircle, lines TP, TQ secondly meet the circumcircle in midpoints B', A' of arcs AC, BC. Therefore lines AA' and BB' meet in point I, and applying Pascal theorem to hexagon $CAA'TB'B$ we obtain that P, I, Q are collinear. The respective line is perpendicular to the bisectrix of angle C, thus P, Q are the projections of J to AC and BC .

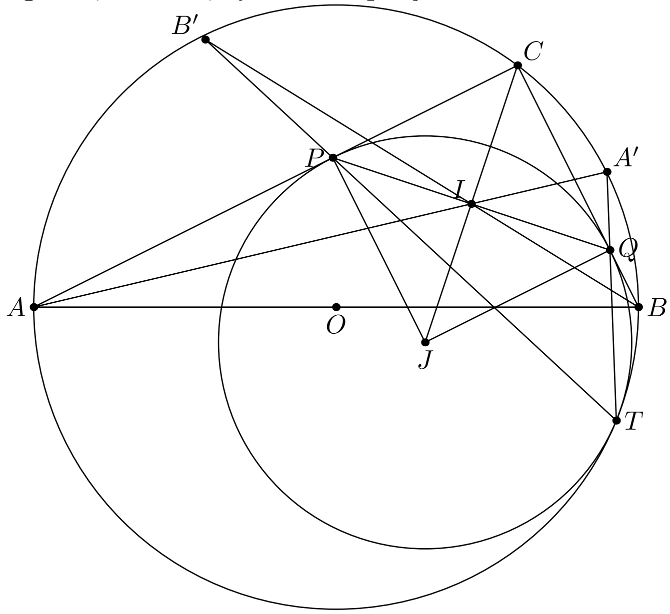

Fig.10.1

2. (P.Kozhevnikov) Each of two equal circles ω_1 and ω_2 passes through the center of the other. Triangle ABC is inscribed into ω_1, and lines AC, BC touch ω_2. Prove that $\cos A + \cos B = 1$.

   Solution. Let O be the center of ω_2, P be the point of ω_1 opposite to O. Since CO is the bisectrix of angle ACB, points A and B are symmetric wrt line OP. Transform the sum of cosines to the product: $\cos A + \cos B = 2\sin\frac{C}{2}\cos\frac{A-B}{2}$. From indicated symmetry we obtain that $|A - B|/2 = \angle COP$, i.e. $OP\cos\frac{A-B}{2} = CO$, and since $C/2$ is the angle between CO and the tangent, $CO\sin\frac{C}{2}$ is equal to the radius of ω_2, i.e. $OP/2$.

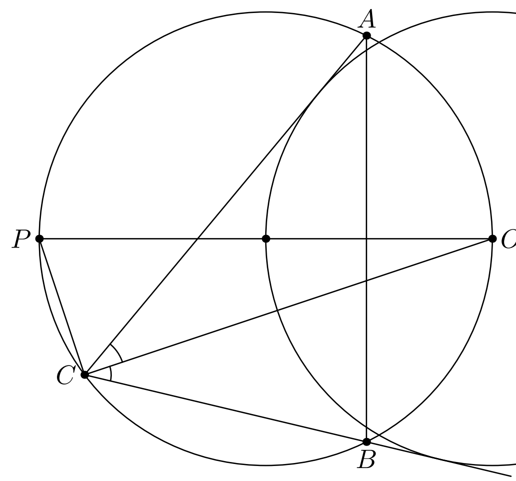

Fig.10.2

3. (A.Akopyan) All sides of a convex polygon were decreased in such a way that they formed a new convex polygon. Is it possible that all diagonals were increased?

   Answer. No, it isn't.

   Solution. Firstly let us prove the following lemma.

   Let ABC, ABC' be two triangles such that $AC > AC'$, $BC > BC'$. Then for any point K of segment AB we have $CK > C'K$.

   Indeed, points A, B, C' lie on the same side of the bisector of segment CC'. Thus K also lies on this side which is equivalent to the required inequality.

   Let us prove now that the indicated situation is impossible for a quadrilateral. Indeed, otherwise we can suppose that one of the diagonals wasn't changed and the other one was increased. Joining equal diagonals we obtain quadrilaterals $ABCD$, $AB'CD'$ with $AB > AB'$, $BC > B'C$, $CD > CD'$, $DA > D'A$. Let E be the common point of diagonals of $ABCD$. Then by lemma we have $BE > B'E$, $DE > D'E$ and $B'D' \le B'E + D'E < BD$, a contradiction.

   The impossibility for an arbitrary polygon can be proved by induction. As in the previous case suppose that one diagonal wasn't changed and all remaining ones weren't decreased. Considering the parts of the polygon cut by the restored diagonal we reduce the problem to the polygon with smaller number of sides.

4. (F.Nilov) Projections of two points to the sidelines of a quadrilateral lie on two concentric circles (projections of each point form a cyclic quadrilateral and the radii of circles are different). Prove that this quadrilateral is a parallelogram.

   Solution. Let the projections of point P to the sidelines lie on the circle with center O, and P' be the reflection of P in O. Then the projection of P' lies on the same circle and P, P' are the foci of some inconic. From the condition we obtain that the sidelines of the quadrilateral are the common tangents to two concentric conics. Therefore they form a parallelogram.

VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. Grade 10. Solutions.

5. (D.Shvetsov) Let BH be the altitude of right-angled triangle ABC ($\angle B = 90°$). The incircle of triangle ABH touches AB, AH in points H_1, B_1; the incircle of triangle CBH touches CB, CH in points H_2, B_2; point O is the circumcenter of triangle H_1BH_2. Prove that $OB_2 = OB_1$.

   Solution. Let I_1, I_2 be the incenters of triangles ABH, CBH. From similarity of these triangles $I_1H_1 : I_2H_2 = AB : BC$. Since segments I_1H_1 and I_2H_2 are perpendicular to AB and BC respectively, their projections to AC are equal. Since O is the midpoint of H_1H_2, the projection of O to AC coincides with the midpoint of B_1B_2, and this yields the assertion of the problem .

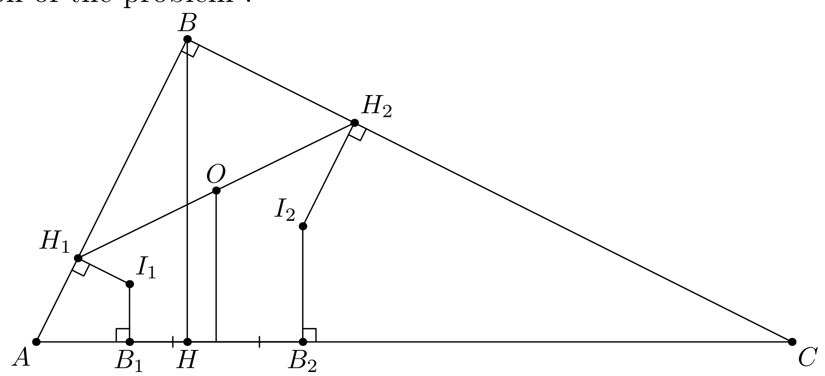

Fig.10.5

6. (F.Nilov) The incircle of triangle ABC touches its sides in points A', B', C'. It is known that the orthocenters of triangles ABC and $A'B'C'$ coincide. Is triangle ABC regular?

   Answer. Yes, it is.

   Solution. Let O, I be the circumcenter and the incenter of ABC; H' be the orhocenter of $A'B'C'$; A'', B'', C'' be the second common point of lines $A'H'$, $B'H'$, $C'H'$ with the incircle. Then $\angle A''C''C' = \angle A''A'C' = \angle B''B'C' = \angle B''C''C'$, i.e. $A''B'' \parallel AB$. Therefore triangles ABC and $A''B''C''$ are homothetic. This homothety transforms O to I, and I to H'. Thus H' lies on OI. Then by condition the orthocenter H of triangle ABC lies on OI and $OI : IH = R : r$

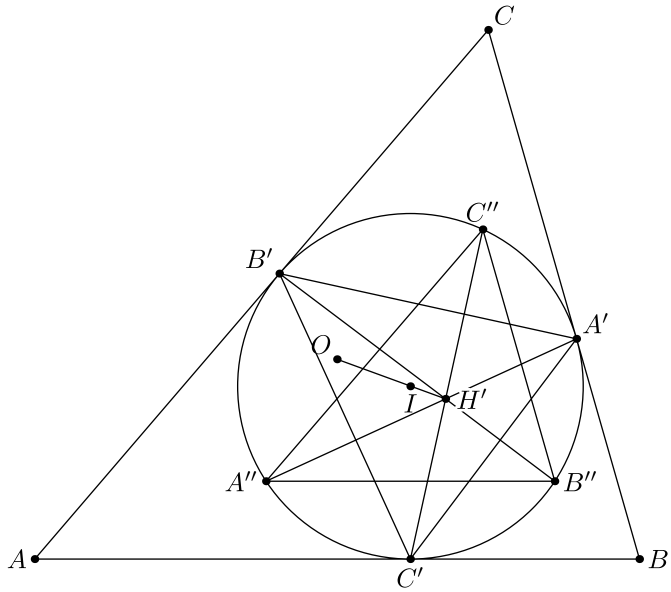

Fig.10.6

Suppose that triangle ABC isn't regular. Then two of its vertices, for example A, B, don't lie on OI. Since AI, BI are the bisectrices of angles OAH, OBH, we have $OI : IH = AO : OH = BO : BH$. Therefore $AH = BH = r$ which clearly isn't possible. Thus ABC is regular.

7. (B.Frenkin) Each of two regular polyhedrons P and Q was divided by a plane into two parts. One part of P and one part of Q were attached along the dividing plane and formed a regular polyhedron not equal to P and Q. How many faces can it have?

   Answer. 4 or 8.

   Solution. One of polyhedral angles of obtained polyhedron is equal to a polyhedral angle of P, thus these polyhedrons have the same number of faces. Similarly Q has the same number of faces. On the other hand, the new polyhedron can contain only one vertex of each of polyhedrons P, Q, since otherwise it is equal to one of these polyhedrons. Therefore each part of P, Q is a pyramid and the number of its lateral faces isn't less than the half of the number of faces of P and of Q. This inequality is correct only for the tetrahedron and the octahedron. Both cases are possible if we cut out two equal pyramids from P and Q by the planes parallel to their symmetry planes.

8. (N.Beluhov, Bulgaria) Triangle ABC is inscribed into circle k. Points A_1, B_1, C_1 on its sides were marked, and after that the triangle was erased. Prove that it can be restored uniquely iff AA_1, BB_1 and CC_1 concur.

   Solution. Fix triangle ABC and points A_1, B_1. Let A_2, B_2 be the second common points of AA_1, BB_1 with k; C' be an arbitrary point of arc A_2CB_2; A', B' be the second common points of $C'A_1$, $C'B_1$ with k; C_1 be the common point of AB and $A'B'$. If C' moves from A_2 to B_2 then point C_1 moves from A to B, thus there exist two triangles with the sides passing through A_1, B_1, C_1. The unique exclusion is the limiting position of C_1 when C' tends to C.

   Let us prove now that if AA_1, BB_1, CC_1 concur then the triangle can be restored uniquely. Consider the projective map restoring k and transforming the common point of these lines to the centroid of ABC, such that A_1, B_1, C_1 are transformed to the midpoints of its sides. Suppose that there exists another triangle $A'B'C'$ with sides passing through A_1, B_1, C_1. Let for example quadrilateral AB_1C_1A' be convex. Then quadrilaterals BC_1A_1B' and CA_1B_1C' are also convex. Since point A' lies outside the circumcircle of triangle AB_1C_1, we have $\angle B_1C'A_1 < \angle B_1AC_1$ and $\angle BC_1B' = \angle AC_1A' > \angle AB_1A'$. Similarly $\angle CA_1C' > \angle BC_1B'$ and $\angle AB_1A' > \angle CA_1C'$, a contradiction.