IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The final round. Solutions. 8 form. First day

1. (B.Frenkin) Does a convex quadrilateral without parallel sidelines exist such that it can be divided into four equal triangles?

   Answer. Yes. See for example fig.8.1.

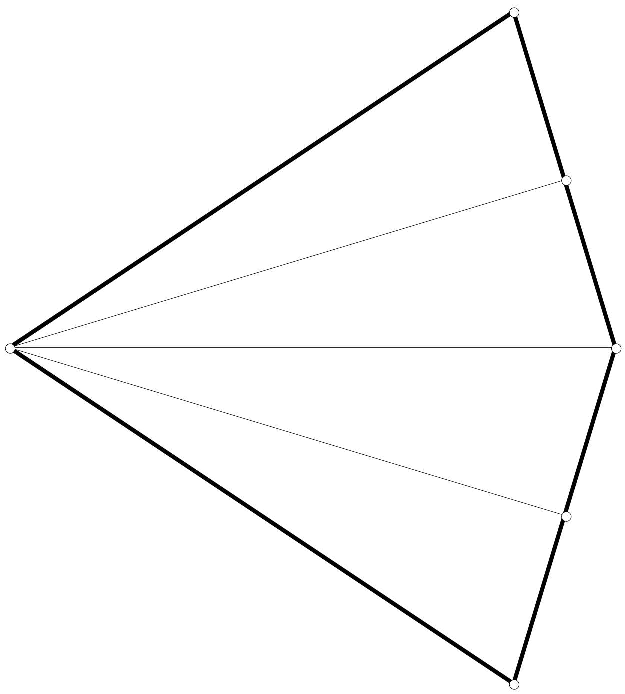

Fig.8.1.

2. (F.Nilov) Given right triangle ABC with hypothenuse AC and $\angle A = 50^\circ$. Points K and L on the cathetus BC are such that $\angle KAC = \angle LAB = 10^\circ$. Determine the ratio CK/LB.

   Answer. 2.

   Solution. Let L' is the reflection of L in AB (fig.8.2). As $\angle L'KA = 50^\circ = \angle KAL'$, we have $L'K = L'A = LA$. On the other hand, $\angle CAL = 40^\circ = \angle ACL$, i.e. $AL = CL$. So $CK = LL' = 2LB$.

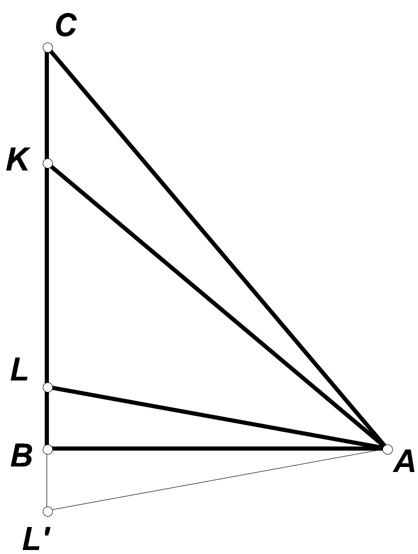

Fig.8.2.

3. (D.Shnol) Two opposite angles of a convex quadrilateral with perpendicular diagonals are equal. Prove that a circle can be inscribed in this quadrilateral.

   Solution. Let O be the common point of the diagonals in quadrilateral $ABCD$ with $\angle B = \angle D$. Suppose that $OB > OD$. Then point D' which is the reflection of D in AC lies on segment OB (fig.8.3). Thus by the property of external angle $\angle AD'O > \angle ABO$, $\angle CD'O > \angle CBO$. But then $\angle D = \angle AD'C > \angle B$ — a contradiction. So $OB = OD$ and AC is the symmetry axis of $ABCD$. Thus the bisectors of angles B, D and AC concur and their common point is the incenter of $ABCD$.

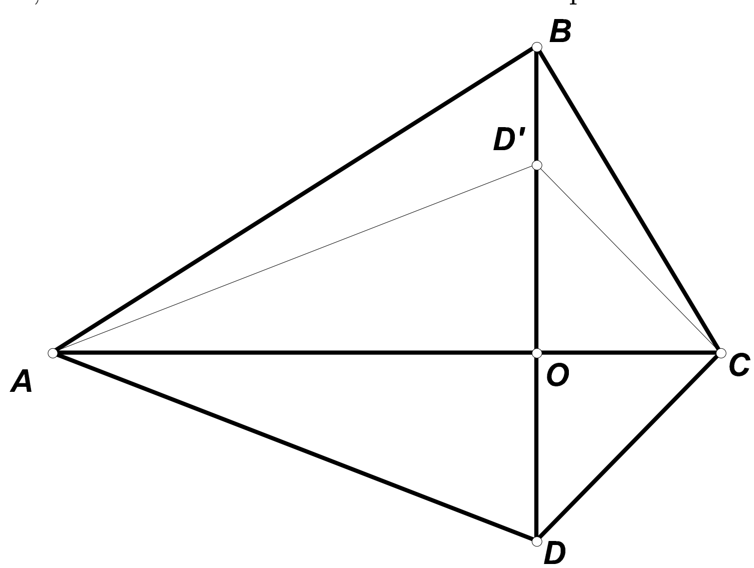

Fig.8.3.

4. (F.Nilov, A.Zaslavsky) Let CC_0 be a median of triangle ABC; the medial perpendiculars to AC and BC intersect CC_0 in points A', B'; C_1 is the meet of lines AA' and BB'. Prove that $\angle C_1CA = \angle C_0CB$.

Solution. Since triangles CAA', CBB' are isosceles, we have $\angle CAA' = \angle C_0CA$, $\angle CBB' = \angle C_0CB$. So the distances from C to the lines AC_1 and BC_1 are equal respectively to the distances from A and B to the line CC_0. But these distances are equal because CC_0 is a median. Thus C is equidistant from C_1A and C_1B. So $\angle CC_1A = \angle CC_1B$, and $\angle C_1CA - \angle C_1CB = \angle C_1BC - \angle C_1AC = \angle C_0CB - \angle C_0CA$ (fig.8.4). This is equivalent to the required assertion.

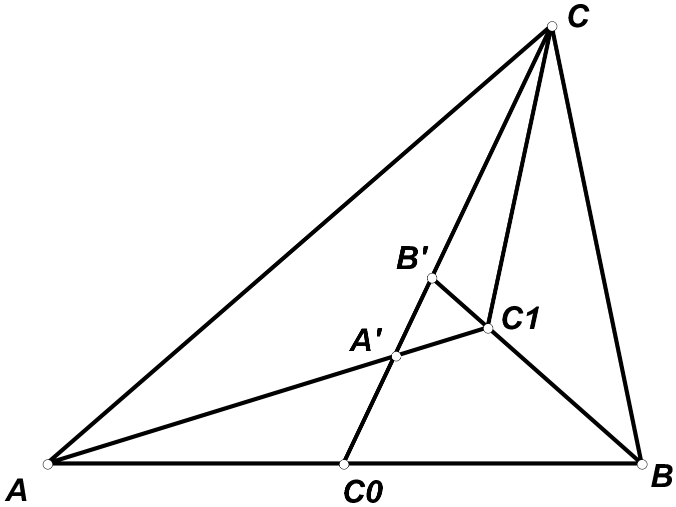

Fig.8.4.

5. (A.Zaslavsky) Given two triangles ABC, $A'B'C'$. Denote by α the angle between the altitude and the median from vertex A of triangle ABC. Angles β, γ, α', β', γ' are defined similarly. It is known that $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$. Can we conclude that the triangles are similar?

   Answer. No.

   Solution. Let the sidelines of $A'B'C'$ be parallel to the medians of ABC. Then the sidelines of ABC are parallel to the medians of $A'B'C'$ and the angles between the medians and the respective altitudes are the same for both triangles. But in general case these triangles aren't similar.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions. 8 form. Second day

6. (B.Frenkin) Consider the triangles such that all their vertices are vertices of a given regular 2008-gon. What triangles are more numerous among them: acute-angled or obtuse-angled?

   Answer. Obtuse-angled.

   Solution. Fix two vertices A and B of one of given triangles. If they are the opposite vertices of the 2008-gon then for any third vertex C triangle ABC is right-angled. Otherwise denote by A', B' the vertices of the 2008-gon opposite to A, B respectively. Triangle ABC is acute-angled iff C lies on the smallest of two arcs of circumcircle, bounded by A', B'. So for any fixed A, B the number of acute-angled triangles having these two vertices is less than that of obtuse-angled. Thus the total number of obtuse-angled triangles is greater as well.

7. (F.Nilov) Given isosceles triangle ABC with base AC and $\angle B = \alpha$. The arc AC constructed outside the triangle has angular measure equal to β. Two lines passing through B divide the segment and the arc AC into three equal parts. Find the relation α/β.

   Answer. 1/3.

   Solution. Let points X, Y divide the segment AC into three equal parts ($AX = XY = YC$); U, V be the common points of rays BX, BY with arc AC; Z be the common point of BC and UV (fig.8.7). Since $UV \parallel AC$, we have $VZ = UV = VC$. So $\angle UCZ = 90°$. On the other hand, $\angle ACU = \angle UCV = \beta/6$, and $\angle BCA = 90° - \alpha/2$. Consequently $\beta = 3\alpha$.

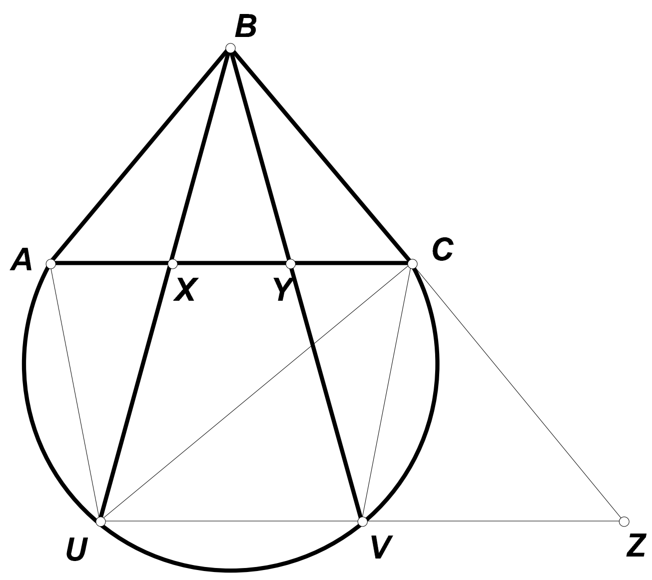

Fig.8.7.

8. (B.Frenkin, A.Zaslavsky) A convex quadrilateral was drawn on the blackboard. Boris marked the centers of four excircles each touching one side of the quadrilateral

and the extensions of two adjacent sides. After this, Alexey erased the quadrilateral. Can Boris define its perimeter?

Answer. Yes.

Solution. Let $ABCD$ be the quadrilateral formed by the excenters, and the vertex X of the original quadrilateral lies on AB. The sidelines of $ABCD$ are the external bisectors of the angles of original quadrilateral. So a billiard ball moving from X along a side of the original quadrilateral, will after the reflections in the sides of $ABCD$ continue to move along the sides. "Straighten"the trajectory of ball constructing quadrilaterals: A_1BCD_1 — the reflection of $ABCD$ in BC, $A_2B_1CD_1$ — the reflection of A_1BCD_1 in CD_1, and $A_2B_2C_1D_1$ — the reflection of $A_2B_1CD_1$ in D_1A_2. Then the trajectory of the ball transforms to segment XX', where X' lies on A_2B_2 and $A_2X' = AX$ (fig.8.8). Since $\angle X'XB = \angle XX'A_2$, we have $A_2B_2 \parallel AB$. Thus joining any other point of segment AB with the respective point of segment A_2B_2, after inverse reflections we obtain another quadrilateral satisfying the conditions of the problem. So there exists an infinite set of quadrilaterals having points A, B, C, D as excenters. But the perimeters of all these quadrilaterals are equal to $XX' = AA_2$ and so don't depend on X.

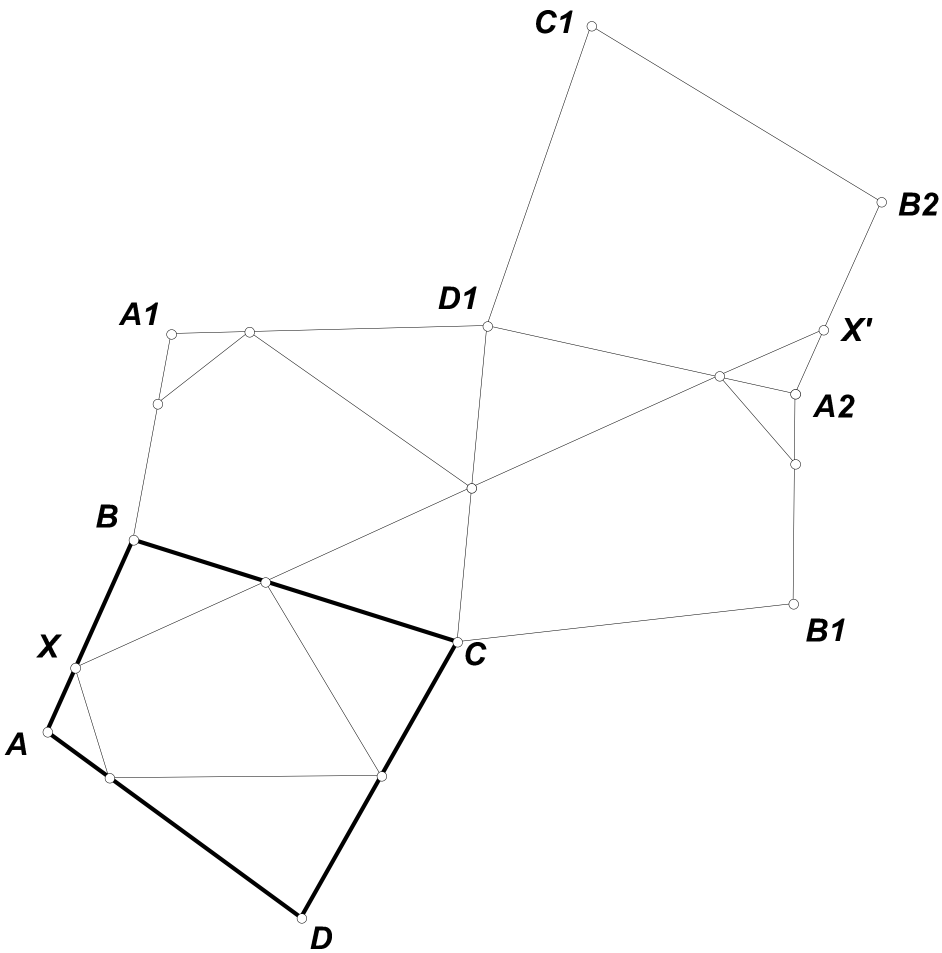

Fig.8.8.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The final round. Solutions. 9 form. First day

1. (A.Zaslavsky) A convex polygon can be divided into 2008 equal quadrilaterals. Is it true that this polygon has a center or an axis of symmetry?

   Answer. No. Take for example the trapezoid with bases equal to 1 and 2, and lateral sides equal to 1 and $\sqrt{2}$. Using the construction of fig.9.1, we can compose from such trapezoids a hexagon which isn't symmetric.

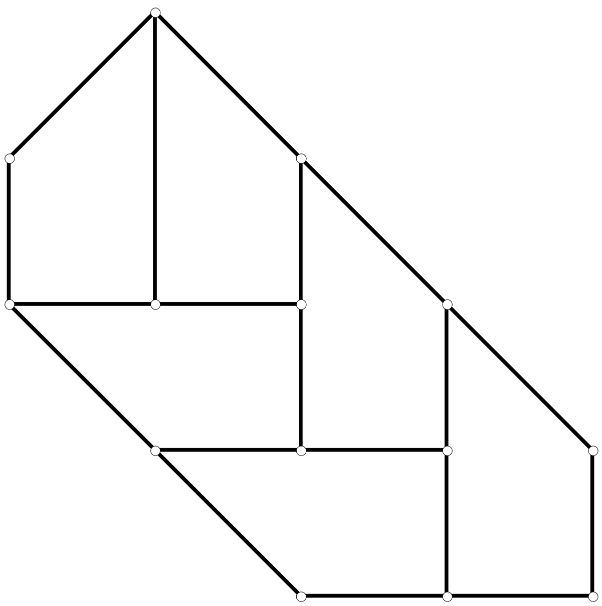

Fig.9.1.

2. (F.Nilov) Given quadrilateral $ABCD$. Find the locus of points such that their projections to the lines AB, BC, CD, DA form a quadrilateral with perpendicular diagonals.

   Solution. If the original quadrilateral is a trapezoid then the projections of a point in the locus to the lateral sidelines lie on a line parallel to the bases. It is evident that the set of such points is the line passing through the common point of lateral sidelines. Also it is clear that the required locus for a rectangle is the whole plane, and that for a parallelogram distinct from a rectangle such points don't exist.

   Let X be the common point of lines AB and CD, Y be a common point of lines BC и DA. Denote the projections of point P to AB, BC, CD, DA by K, L, M, N, and let O be the common point of KM and LN (fig.9.2). Since quadrilaterals $YLPN$ and $XKPM$ are cyclic, we have $\angle PLN = \angle PYA$ and $\angle PMK = \angle PXA$. So $\angle MOL = \pi - \angle C - \angle PLN - \angle PMK = \pi - \angle C - (\angle A - \angle XPY)$. Thus an equality $\angle MOL = \pi/2$ is equivalent to $\angle XPY = \angle A + \angle C - \pi/2$. So the required locus is a circle passing through X and Y.

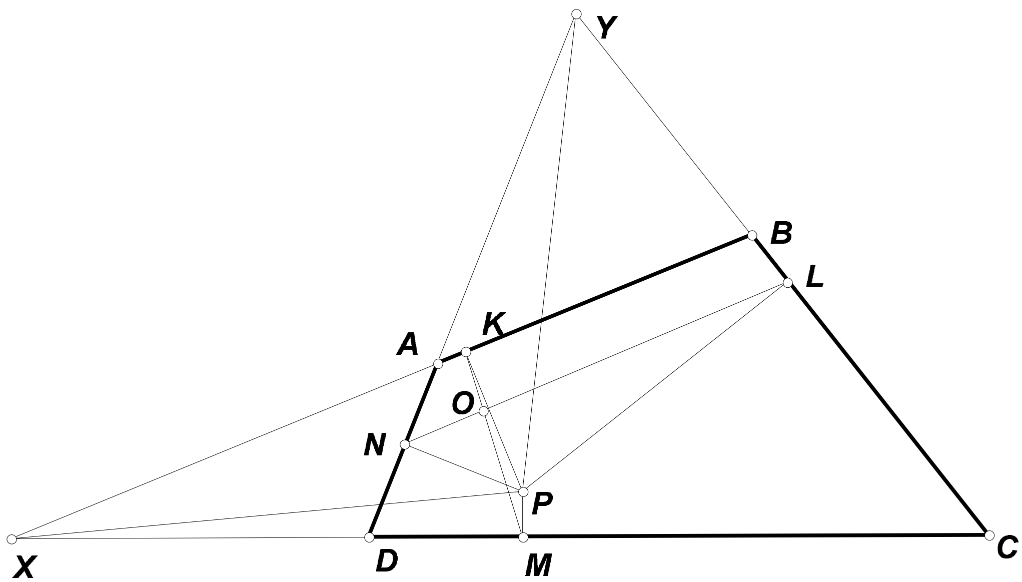

Fig.9.2.

3. (R.Pirkuliev) Prove the inequality

$$\frac{1}{\sqrt{2\sin A}}+\frac{1}{\sqrt{2\sin B}}+\frac{1}{\sqrt{2\sin C}}\le\sqrt{\frac{p}{r}},$$

where p and r are the semiperimeter and the inradius of triangle ABC.

Solution. Let R and S be the circumradius and the area of triangle ABC. Using the sinuses theorem and the formulaes $S = pr = abc/4R$, transform the right part of inequality:

$$\begin{aligned}\sqrt{\tfrac{p}{r}} &= \tfrac{p}{\sqrt{S}} = \tfrac{R(\sin A+\sin B+\sin C)}{\sqrt{2R^2\sin A\sin B\sin C}}\\ &= \sqrt{\tfrac{\sin A}{2\sin B\sin C}}+\sqrt{\tfrac{\sin B}{2\sin C\sin A}}+\sqrt{\tfrac{\sin C}{2\sin A\sin B}}.\end{aligned}$$

By Cauchi inequality:

$$\frac{2}{\sqrt{\sin A}}\le\sqrt{\frac{\sin B}{\sin C\sin A}}+\sqrt{\frac{\sin C}{\sin A\sin B}}.$$

Summing this inequality with two similar ones, we obtain the required assertion.

4. (F.Nilov, A.Zaslavsky) Let CC_0 be a median of triangle ABC; the medial perpendiculars to AC and BC intersect CC_0 in points A', B'; C_1 is the common point of AA' and BB'. Points A_1, B_1 are defined similarly. Prove that circle $A_1B_1C_1$ passes through the circumcenter of triangle ABC.

Solution. By the solution of problem 8.4, lines AA_1, BB_1, CC_1 concur in point L, and point C_1 lies on the circle passing through A, B and the circumcenter O of ABC. Hence $\angle OC_1L = \angle AC_1C-\angle AC_1O = \angle AC_1C-\angle ABO = (\pi-\angle C)-(\pi/2-\angle C) = \pi/2$, and C_1 lies on the circle with diameter OL. Similarly A_1 and B_1 lie on the same circle.

5. (N.Avilov) Can the surface of a regular tetrahedron be glued over with equal regular hexagons?

   Solution. Yes. For example, sticking together a tetrahedron from the development in fig.9.5 and cutting its surface by bold lines, we obtain two equal regular hexagons (obscure and light).

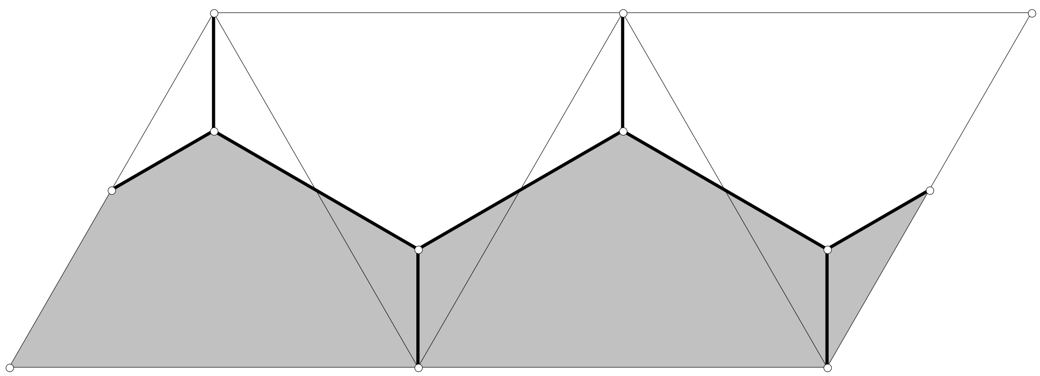

Fig.9.5.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions. 9 form. Second day

6. (B.Frenkin) Construct the triangle, given its centroid and the feet of an altitude and a bisector from the same vertex.

   Solution. Let C_1, C_2 be the feet of the bisector and of the altitude from vertex C of triangle ABC, and M be its centroid. It is evident that C lies on the perpendicular from C_2 to line C_1C_2. The projection of M to this perpendicular divides the altitude in relation $2:1$. This enables to construct point C and the midpoint C_0 of side AB.

   Let C' be the common point of CC_1 and perpendicular l to C_1C_2 from C_0. Then C' lies on the circumcircle of ABC (fig.9.6). So the medial perpendicular to CC' intersect l in the circumcenter O. Constructing the circumcircle, we find A, B as its common points with C_1C_2.

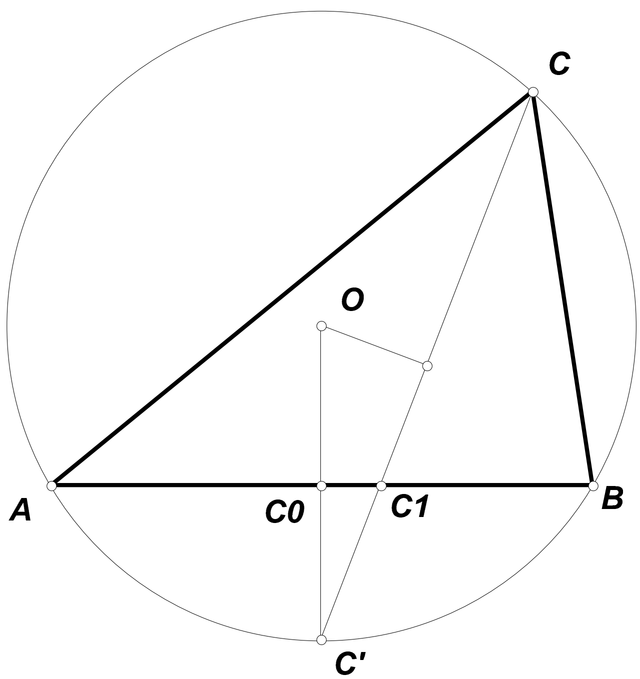

Fig.9.6.

7. (A.Zaslavsky) The circumradius of triangle ABC is equal to R. Another circle with the same radius passes through the orthocenter H of this triangle and intersect its circumcirle in points X, Y. Point Z is the fourth vertex of parallelogram $CXZY$. Find the circumradius of triangle ABZ.

   Answer. R.

   Solution. We will prove that Z lies on circle ABH with radius equal to R. Let H' be the second common point of circles XYH and ABH, C' be the orthocenter of triangle ABH' (fig.9.7). Then C' lies on the circle which is the reflection of ABH in AB, i.e. on the circumcircle of ABC. So $CH = C'H' = 2R|\cos C|$ and $CHH'C'$ is a parallelogram. Since CC' and HH' are the chords of equal circles ABC and

XHY, they are symmetric wrt the midpont of segment XY. Thus, $XCYH'$ is the parallelogram and H' coincides with Z.

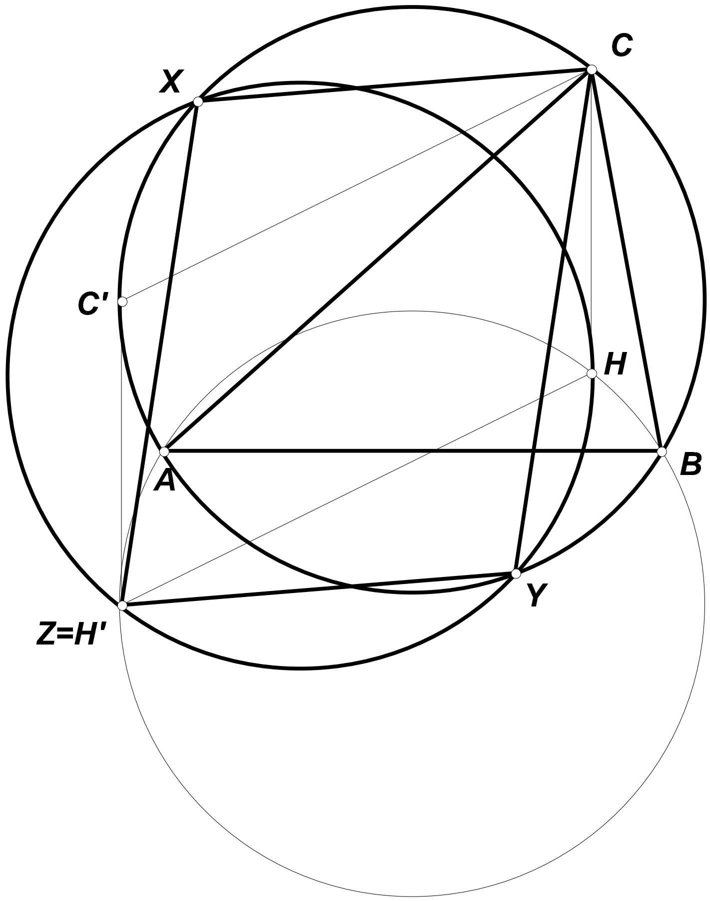

Fig.9.7.

8. (J.-L.Aime, France) Points P, Q lie on the circumcircle ω of triangle ABC. The medial perpendicular l to PQ intersects BC, CA, AB in points A', B', C'. Let A", B", C" be the second common points of l with the circles $A'PQ$, $B'PQ$, $C'PQ$. Prove that AA", BB", CC" concur.

   Solution. Let X, Y be the common points of ω and l. Consider the central projection from ω to l with center C. We obtain that $(AB; XY) = (B'A'; XY)$. Furthermore since $\angle A'PA - \angle B'PB - \angle XPY = \pi/2$, we have $(A'B'; XY) = (A"B"; YX)$. So $(AB; XY) = (A"B"; YX)$, and the common point of AA" and BB" lies on ω. Line CC" also passes through this point.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The final round. Solutions. 10 form. First day

1. (B.Frenkin) An inscribed and circumscribed n-gon is divided by some line into two inscribed and circumscribed polygons with different numbers of sides. Find n.

   Answer. 3.

   Solution. Suppose $n \neq 3$. If $n > 4$ then the boundary of at least one of obtained polygons contains the segments of three sides of the original polygon. So the incircle of this polygon coincides with the incircle of the original polygon. If $n = 4$ this also is correct because two obtained polygons have different number of sides and so the dividing line isn't a diagonal of original quadrilateral.

   Thus the dividing line is tangent to the incircle of the original n-gon, hence one of two parts obtained is a triangle. The vertices of the second part are $n-1$ vertices of the original polygon and two points lying on its sides. Since $n-1 \geq 3$, these vertices determine a unique circle which passes through the remain vertex of the n-gon and hence does not pass through the two points on the sides. Thus the cut polygon is not inscribed.

   Remark. It is possible to divide an arbitrary triangle by a line tangent to its incircle into a triangle and an inscribed-circumscribed quadrilateral.

2. (A.Myakishev) Let triangle $A_1B_1C_1$ be symmetric to ABC wrt the incenter of its medial triangle. Prove that the orthocenter of $A_1B_1C_1$ coincides with the circumcenter of the triangle formed by the excenters of ABC.

   Solution. Let H, I, O, M be the orthocenter, the incenter, the circumcenter, and the centroid of triangle ABC, I_0 be the incenter of its medial triangle. Obviously, the orthocenter H_1 of triangle $A_1B_1C_1$ is symmetric to H wrt I_0. On the other hand, in the triangle formed by the excenters of ABC, I is the orthocenter, ABC is the orthotriangle, and the circumcircle of ABC is the nine-points circle. So the circumcenter of excenter triangle is symmetric to I wrt O. Consider triangle IHH_1. Its median II_0 passes through M and is divided by this point in relation $2:1$. So M is the centroid of this triangle. But M also divides segment $HO0$ in relation $2:1$. Thus O is the midpoint of IH_1 (fig.10.2)

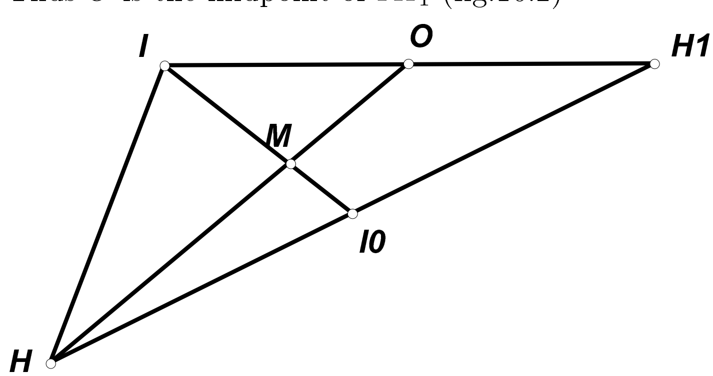

Fig.10.2.

3. (V.Yasinsky, Ukraine) Suppose X and Y are the common points of two circles ω_1 and ω_2. The third circle ω is internally tangent to ω_1 and ω_2 in P and Q respectively. Segment XY intersects ω in points M and N. Rays PM and PN intersect ω_1 in points A and D; rays QM and QN intersect ω_2 in points B and C respectively. Prove that $AB = CD$.

Solution. Point P is the homothety center of circles ω and ω_1. So $AD \parallel MN$ and segment AD is perpendicular to the center line of circles ω_1 and ω_2. Thus points A and D are symmetric wrt this line. Similarly B and C are symmetric wrt this line, and so $AB = CD$ (fig.10.3).

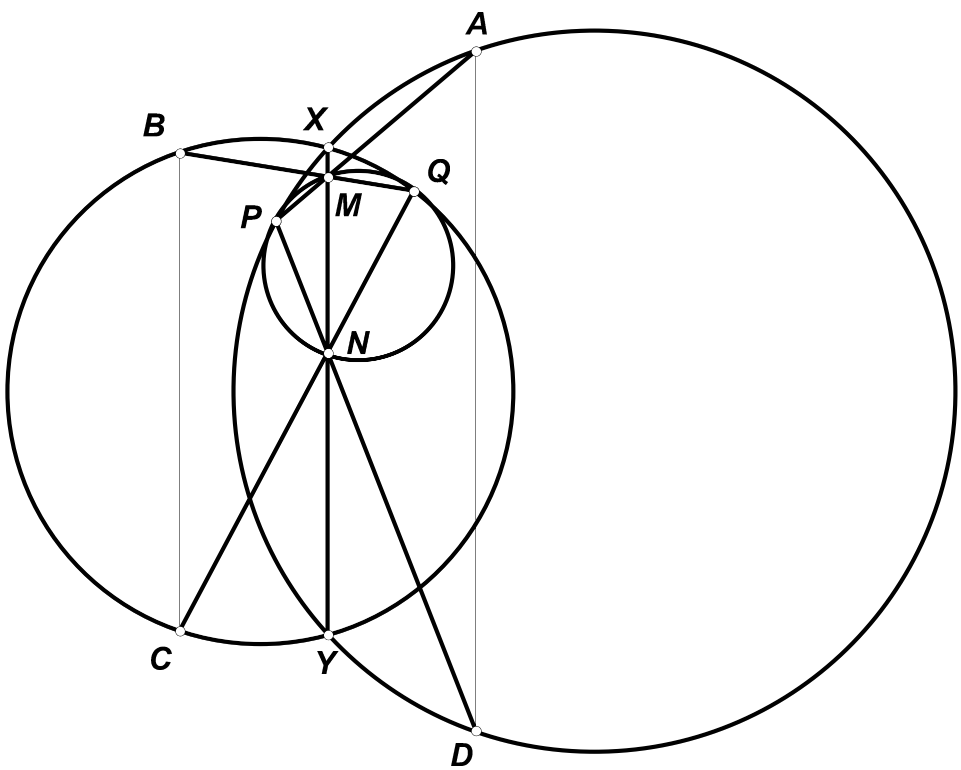

Fig.10.3.

4. (A.Zaslavsky) Given three points C_0, C_1, C_2 on the line l. Find the locus of incenters of triangles ABC such that points A, B lie on l and the feet of the median, the bisector and the altitude from C coincide with C_0, C_1, C_2.

Answer. The perpendicular to l passing through point C' on segment C_0C_2, such that $C_0C'^2 = C_0C_1 \cdot C_0C_2$.

Solution. Let C_3, C_4 be the points of contact between side AB and the incircle and the excircle of triangle ABC. Then C_0 is the midpoint of segment C_3C_4. On the other hand, points C_3, C_4 are the projections of incenter I and excenter I_c to AB (fig.10.4). Since these centers lie on CC_1, we have

$$\frac{C_2C_3}{C_2C_4} = \frac{CI}{CI_c} = \frac{r}{r_c} = \frac{C_1I}{C_1I_c} = \frac{C_1C_3}{C_1C_4}.$$

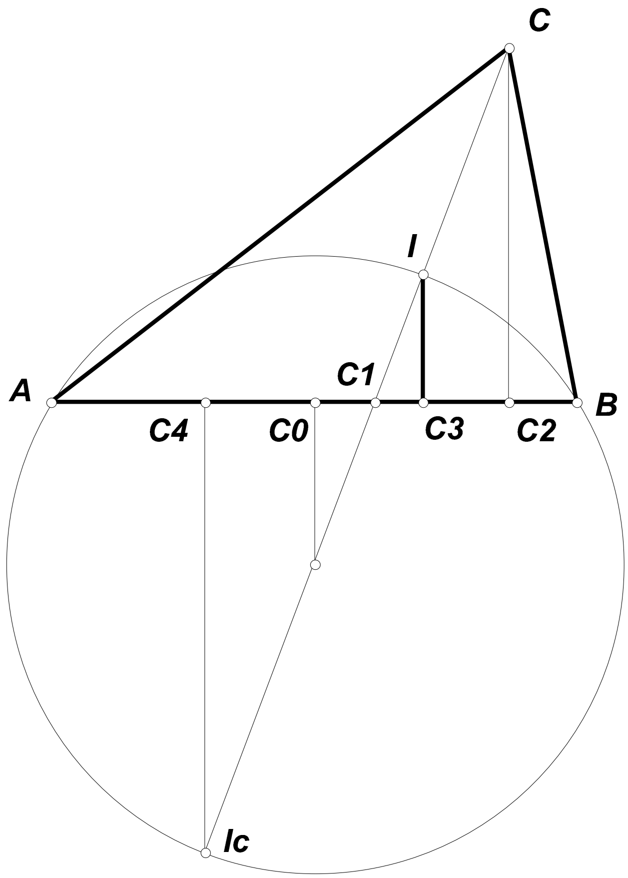

Fig.10.4.

This implies that C_3 coincides with C'. Now take an arbitrary point I, the projection of which to l coincides with C_3. The perpendiculars to l from C_2 and from C_0 intersect line C_1I in point C and in the circumcenter of triangle IAB. Finding the common points of the corresponding circle with l, we obtain the required triangle.

5. (I.Bogdanov) A section of a regular tetragonal pyramid is a regular pentagon. Find the ratio of its side to the side of the base of the pyramid.

   Answer. $\frac{3-\sqrt{5}}{\sqrt{2}}$.

   Solution. Let the plane of the section meet sides CD and DA of pyramid's base $ABCD$ in points X, Y. Then pentagon $ABCXY$ is the central projection of a regular pentagon. So the double relation of A, Y, D and the point at infinity of line AD is equal to double relation of four points in which one sideline of the regular pentagon intersects its remaining sidelines (fig.10.5). Thus:

$$\frac{DY}{AD} = \frac{3-\sqrt{5}}{2}.$$

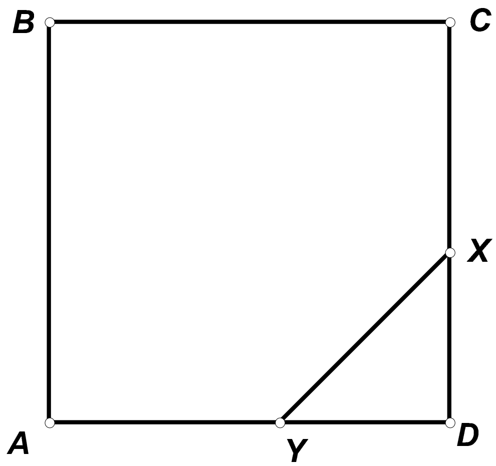

fig.10.5.

Point X divides segment CD in the same ratio. So the required ratio is

$$\frac{XY}{AB} = \frac{3-\sqrt{5}}{\sqrt{2}}.$$

Remark. Since the ratio of a side of the pentagon to a side of the base is determined unambiguously, the ratio of a lateral edge to a side of the base also is determined unambiguously. On the other hand, the planes of 8 faces of an icosahedron bound a regular octahedron. So the pyramid satisfying the conditions of the problem is a half of an octahedron, and its lateral edge is equal to the side of the base.

IV RUSSIAN GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

The final round. Solutions. 10 form. Second day

6. (B.Frenkin) The product of two sides in a triangle is equal to $8Rr$, where R and r are the circumradius and the inradius of the triangle. Prove that the angle between these sides is less than 60°.

   Solution. Let the product of sides $AC = b$ and $BC = a$ of triangle ABC be equal to $8Rr$. Since the area of ABC is $S = pr = abc/4R$ where p is the semiperimeter, we have $4prR = abc = 8Rrc$. So $p = 2c$ or $a + b = 3c$. As $b < a + c$, this implies that $2a > 2c$ and $c < a$. Similarly $c < b$. Thus C as the strictly smallest angle of the triangle is less than 60°.

7. (F.Nilov) Two arcs with equal angular measure are constructed on the medians AA' and BB' of triangle ABC towards vertex C. Prove that the common chord of the respective circles passes through C.

   Solution. Let the circle constructed on AA' intersect AC in point X, and the circle constructed on BB' intersect BC in point Y (fig.10.7). Since $\angle AXA' = \angle BYB'$, triangles CXA' and CYB' are similar. So $CX/CA' = CY/CB'$ and

$$CX \cdot CA = 2CX \cdot CB' = 2CY \cdot CA' = CY \cdot CB.$$

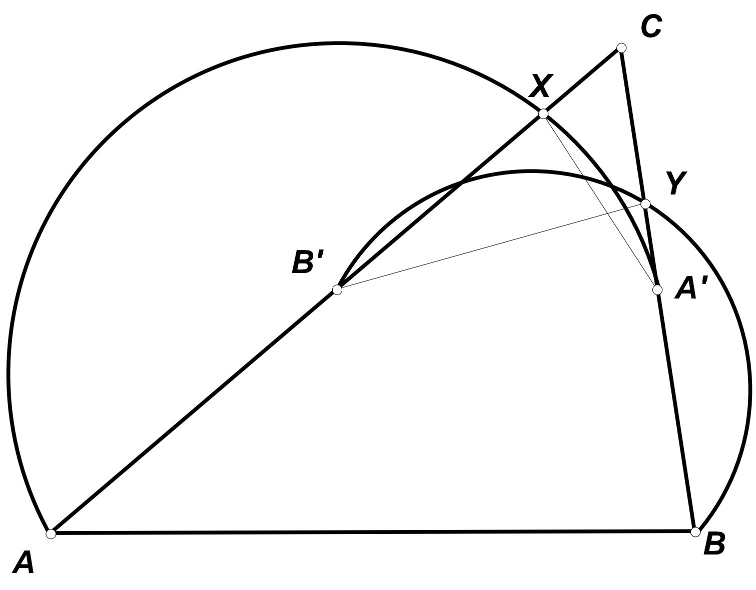

Рис.10.7.

   Thus the degrees of point C wrt both circles are equal and C lies on their radical axis.

8. (A.Akopyan, V.Dolnikov) Given a set of points inn the plane. It is known that among any three of its points there are two such that the distance between them doesn't exceed 1. Prove that this set can be divided into three parts such that the diameter of each part does not exceed 1.

Solution. Call two points 1-close if the distance between them does not exceed 1.

If the diameter of the given set V doesn't exceed $\sqrt{3}$ then V can be covered by a circle with radius 1. We can choose this circle so that it contains some points of V on its boundary. Denote the center of the circle and some point of V on its boundary by X and Y respectively.

Note that any two points in set $V \setminus B(Y,1)$ are 1-close. So the diameter of this set doesn't exceed 1. Furthermore segment $[X,Y]$ divides $V \bigcap B(Y,1)$ into two parts with diameters less or equal to 1. Thus we obtain the required dissection.

Let now two points $X, Y \in V$ exist such that $d(X,Y) > \sqrt{3}d$. Then the join of sets $V \setminus B(X,1)$, $V \setminus B(Y,1)$ and $V \bigcap B(X,1) \bigcap B(Y,1)$ contains V and each of these sets has the diameter less or equal to 1. In fact, any two points of $V \setminus B(X,1)$ or $V \setminus B(Y,1)$ are 1-close. Furthermore set $V \bigcap B(X,1) \bigcap B(Y,1)$ lies inside the set $B(X,1) \bigcap B(Y,1)$ with diameter less or equal to 1 (this diameter is the segment between common points of circles $S(X,1)$ and $S(Y,1)$).