Fourth Olympiad (year 2008)

Correspondence round. Solutions

1. (B.Frenkin, 8) Does a regular polygon exist such that just half of its diagonals are parallel to its sides?

   Answer. No, it doesn't.

   Solution. If the number of sides in the polygon is odd, then every its diagonal is parallel to certain its side. In turn, if the number of sides in the polygon equals $2k$, then there are $2k-3$ diagonals from every vertex, and $k-2$ of them are parallel to some sides. Therefore less than a half of diagonals are parallel to any sides.

2. (V.Protasov, 8) For a given pair of circles, construct two concentric circles such that both are tangent to the given two. What is the number of solutions, depending on location of the circles?

   Solution. Let the radii of the two circles with center at O be equal R and r (where $R > r$). Then there are two sets of circles tangent to them: with radii equal to $\frac{R+r}{2}$ and centers distant from O at $\frac{R-r}{2}$, as well as with radii equal to $\frac{R-r}{2}$ and centers distant from O at $\frac{R+r}{2}$. Furthermore any pair of circles from the same set will be symmetrical with respect to a certain line passing through O, while any pair of circles from different sets will either intersect each other or be tangent. This implies the following construction.

   If the radii of the circles are equal, then the center of the required concentrical circles belongs to the line about which these circles are symmetrical. Any point O of this line can serve as such center, except the points where the given circles intersect. In fact, draw a line through O and the center of one of the given circles, and determine the points A, B of its intersection with this circle (see the figure). The circles with radii OA, OB do fit.

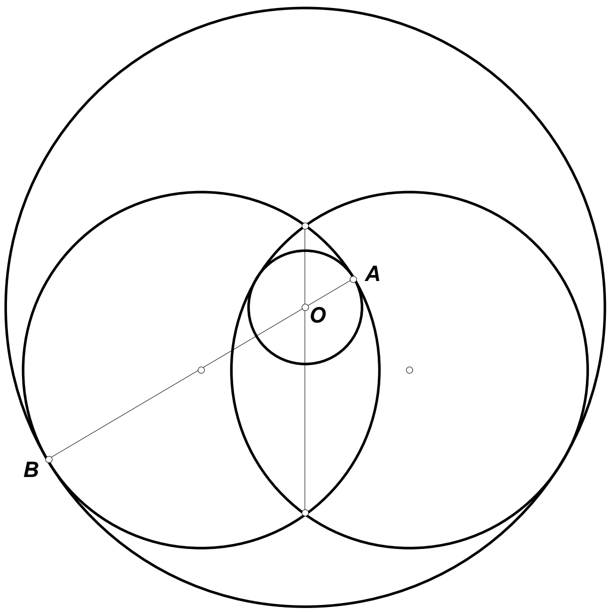

Fig. 2a

If the radii of the given circles are not equal, then the center of the required circles is at the distance from the center of each of the given circles, equal to the radius of another one. There are two such points if the given circles intersect, and there is only one such point if the circles are tangent. The required circles are constructed in the same way as in the previous case (see the figure).

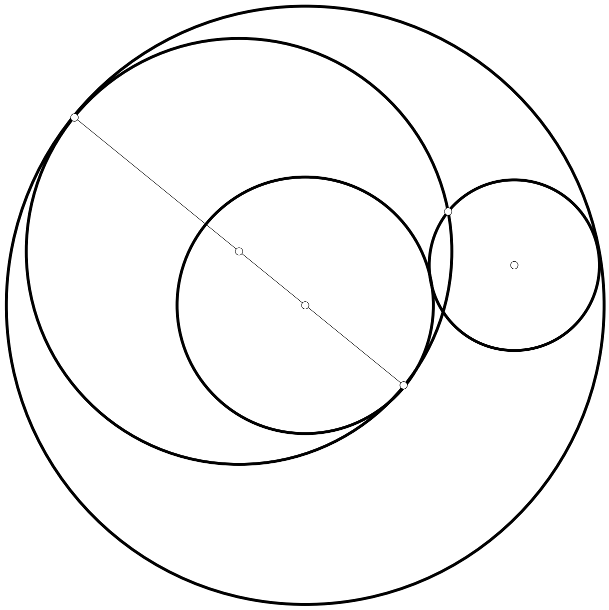

Fig. 2b

So, the problem has infinitely many solutions if the given circles are equal, two solutions if the radii are different and the circles intersect, only a single solution if the radii are different and the circles are tangent and, finally, none for the case of uneqal non-intersecting circles.

3. (A.Zaslavskiy, 8) A triangle can be dissected into three equal triangles. Prove that some its angle is equal to 60°.

   Solution. A triangle can be split into three triangles by either splitting along the lines linking some internal point with the vertices or by firstly splitting along the line passing through a certain vertex and secondly splitting in the same manner one of the resulting triangles. Let us consider both cases.

   1) Let the triangle ABC be split into three smaller ones by segments from the point M. Since the angle AMB is greater than any of the angles MAC, MBC, MCA, MCB, congruence of triangles is possible only if $\angle AMB = \angle BMC = \angle MCA = 120^\circ$. But then $MA = MB = MC$ and the initial triangle must be regular.

   2) Only an isosceles triangle can be cut into two equal triangles. Therefore one of the triangles obtained in the first split must be isosceles, while the other must be right and equal to "half" of the first one. A right triangle can be cut from the initial one only in one of the following three ways:

- by dropping an altitude in the initial triangle. But then the second triangle also is right and the first one cannot be equal to its half;

- by drawing a line CD perpendicular to BC through the vertex C of the obtuse triangle ABC. Then, since the area of triangle BCD equals half the area of triangle ACD, equations $AD = CD = 2BD$ must hold. That is impossible since BD is the hypotenuse of the triangle BCD;

- by constructing the line BD in a triangle with right angle C. Then, similarly to the previous case, we obtain that $AD = BD = 2CD$ and, therefore, $\angle B = 60°$ (see the figure).

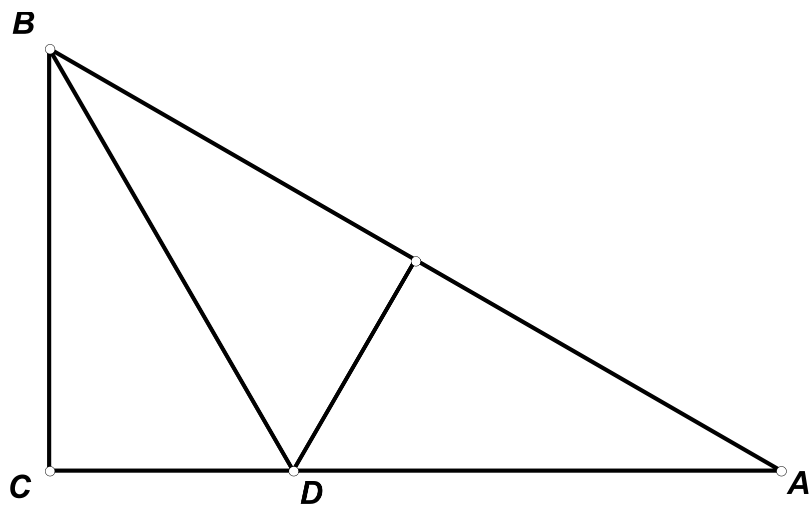

Fig. 3

4. (D.Schnoll, 8–9) The bisectors of two angles in an inscribed quadrangle are parallel. Prove that the sum of squares of some two sides in the quadrangle equals the sum of squares of two remaining sides.

   Solution. Firstly observe that the bisectors of adjacent angles cannot be parallel as the sum of these angles is less than 360°. If, in turn, the bisectors of angles A and C in the quadrilateral $ABCD$ are parallel, then $\frac{\angle A}{2} + \angle B + \frac{\angle C}{2} = 180°$ and $\angle B = \angle D$. Since the quadrilateral is inscribed, these angles are right and $AB^2 + BC^2 = CD^2 + DA^2$.

5. (From Kiev Olympiads, 8–9) Reconstruct the square $ABCD$, given its

vertex A and the distances of vertices B and D from a fixed point O in the plane.

Solution. Let O' be the point such that $AO = AO'$ and $\angle OAO' = 90°$. Then $\angle O'AB = \angle OAD$ and, since $AB = AD$, the triangles OAD and $O'AB$ are equal. Therefore $O'B = OD$ and knowing the lengths of segments OB, $O'B$, we can construct the point B, and then the whole square (see the figure). The problem has two solutions symmetrical about the line OA.

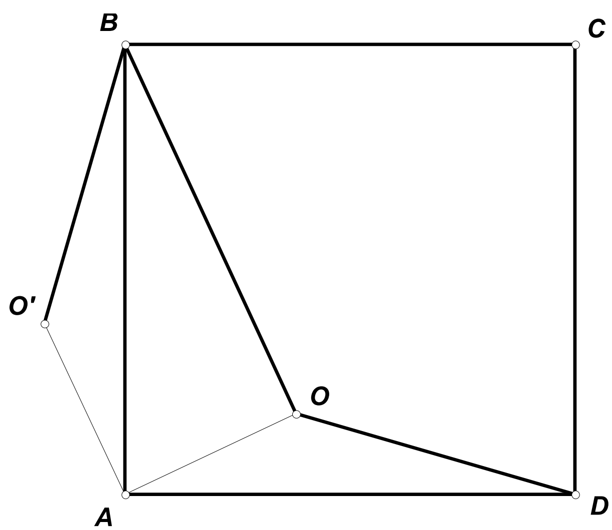

Fig. 5

6. (A. Myakishev, 8–9) In the plane, given two concentric circles with the center A. Let B be an arbitrary point on some of these circles, and C on the other one. For every triangle ABC, consider two equal circles mutually tangent at the point K, such that one of these circles is tangent to the line AB at point B and the other one is tangent to the line AC at point C. Determine the locus of points K.

   Solution. Let M, N be the centers of tangent circles. Then K is the midpoint of the segment MN, $\angle ABM = \angle ACN = 90°$ and $BM = MK = KN = NC$ (see the figure). Since AK is the median of the triangle AMN, $AK^2 = \frac{2AM^2+2AN^2-MN^2}{4} = \frac{AB^2+AC^2}{2}$ does not depend on the choice of points B, C. Therefore, K belongs to a fixed circle with center A. By rotating the triangle ABC around A one can obtain any other point of this circle.

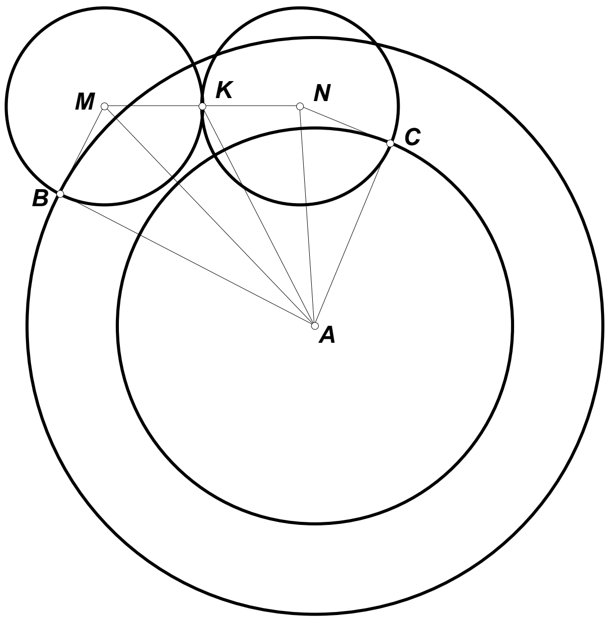

Fig. 6

7. (A.Zaslavskiy, 8–9) Given a circle and a point O on it. Another circle with center O meets the first one at points P and Q. The point C lies on the first circle, and the lines CP, CQ meet the second circle for the second time at points A and B. Prove that $AB = PQ$.

   Solution. If C coincides with O, then the problem statement is obvious. If the point C is diametrically opposite to O, then $\angle CPO = \angle CQO = 90^\circ$, i.e. the lines CP, CQ are tangent to the second circle, and the points A, B coincide with P, Q. In the remaining cases, since $OP = OQ$, CO is the bisector of the angle ACB. Under the symmetry about CO the lines CP and CQ map into each other, while the second circle maps into itself, hence the point P maps into either Q or B. But $CP \neq CQ$, so the first case is impossible. Therefore $CP = CB$. Similarly $CQ = CA$. This implies congruence of triangles CAB and CQP and, therefore, the problem statement itself (see the figure).

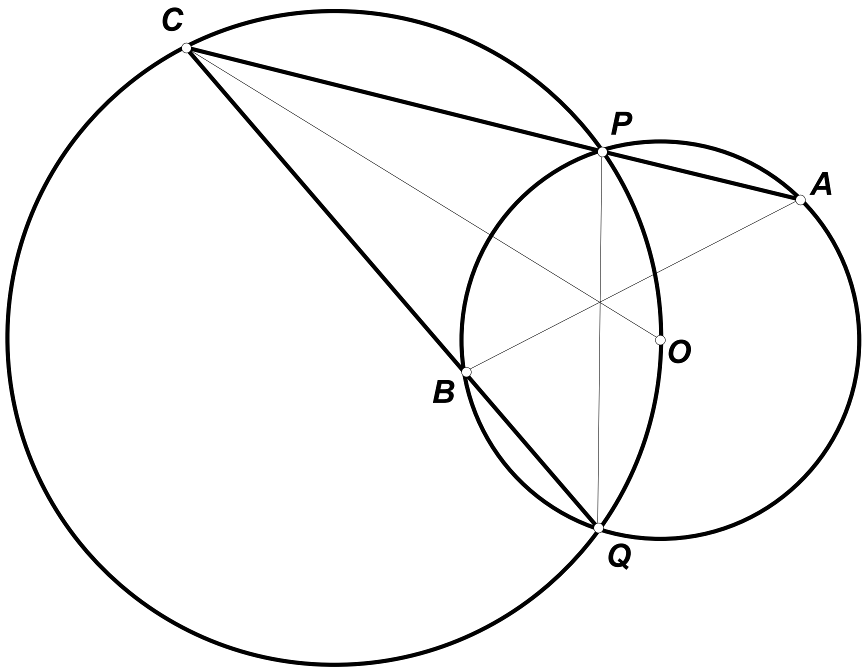

Fig. 7

8. (T.Golenishcheva-Kutuzova, B.Frenkin, 8–11) a) Prove that for $n > 4$, any convex n-gon can be dissected into n obtuse triangles.

   b) Prove that for any n, there exists a convex n-gon which cannot be dissected into less than n obtuse triangles.

   c) In a dissection of a rectangle into obtuse triangles, what is the least possible number of triangles?

   Solution. a) If $n > 4$ then a convex n-gon must have an obtuse angle. A diagonal linking two vertices adjacent to the vertex of the obtuse angle splits the n-gon into an obtuse triangle and an $(n-1)$-gon. Therefore, if one proves the problem statement for $n = 5$, then for the rest of n it follows by induction.

   Observe that any triangle can be cut into three obtuse triangles. Indeed, link the vertices of the triangle with the center I of its incircle. Each of the resulting triangles will have an obtuse angle at the vertex I. This implies that a quadrilateral different from a rectangle can be split into four obtuse triangles.

   Now consider a pentagon $ABCDE$. Let its angle A be obtuse. If $BCDE$ is not a rectangle, then by constructing the diagonal BE and dissecting $BCDE$ in four obtuse triangles we obtain the cutting required. Otherwise, if $BCDE$ is a rectangle, then the angles B and E of the pentagon are obtuse, i.e. $ACDE$ cannot be a rectangle. Therefore, by constructing the diagonal AC and dissecting $ACDE$ into four triangles we will again obtain the required cutting pattern.

b) Let a convex n-gon be cut into $(n-1)$ obtuse triangles. The sum of their angles equals $(n-1)\pi$, while the sum of angles in the n-gon equals $(n-2)\pi$. Therefore, the sum of angles for the triangles that are non-adjacent to the vertices of the n-gon is equal to π. It means that there is no more than one obtuse triangle among them. Therefore, the vertices of the n-gon have at least $(n-2)$ adjacent obtuse angles of the triangles. Obviously, every single vertex of the convex n-gon cannot be adjacent to more than one obtuse angle. Therefore, the n-gon has no less than $(n-2)$ obtuse angles. This is true, however, not for every convex n-gon for any $n \geq 3$.

c) It is evident that by constructing a diagonal and dissecting the resulting triangles into three obtuse-angled ones we will obtain a cutting of a rectangle into six obtuse triangles.

Let is prove that it is impossible to cut a rectangle into fewer number of obtuse triangles. If a rectangle could be cut into less than 5 obtuse triangles then it could be cut into 5 as well: this follows from the fact that any obtuse triangle can be dissected into two obtuse triangles.

Assume that the rectangle is cut into five obtuse triangles. Then, following the argument presented in section b) above, we will obtain that the sum of angles in these triangles, that are not adjacent to the vertices of the rectangle equals 3π. If all of these angles are in the points belonging to the sides of the rectangle, then there are three such points and there is no more than one vertex of an obtuse angle at each one. Since there are no obtuse angles at the vertices of the rectangle, we have a contradiction. If certain vertices of triangles lie within the rectangle then we have one internal point adjacent to no more than three obtuse angles, and one point adjacent to at most one obtuse angle. I.e. the total number of obtuse angles does not exceed four, and we again get a contradiction.

9. (A.Zaslavskiy, 9-10) The lines symmetrical to diagonal BD of a rectangle $ABCD$ relative to bisectors of angles B and D pass through the midpoint of diagonal AC. Prove that the lines symmetrical to diagonal AC relative to bisectors of angles A and C pass through the midpoint of diagonal BD.

Solution. Let P be the midpoint of AC, and L be the point of intersection of the diagonals. Applying the sine theorem to triangles ABP, ABL, CBP, CBL we obtain $AL/CL = (AB/CB)^2$. Similarly $AL/CL = (AD/CD)^2$, i.e. $BC/CD = AB/AD = \frac{\sin\angle BDA}{\sin\angle DBA} = \frac{\sin\angle CDP}{\sin\angle CBP}$. Therefore, the lines BP and DP are symmetrical about AC. Let X be the second point of intersection of the line BP with the circumcircle of the triangle ABC. The point, symmetrical to X about the perpendicular bisector to AC, belongs both to PD and to BD and, therefore, coincides with D. Hence the quadrilateral $ABCD$ is inscribed and $AB \cdot CD = AD \cdot BC = (AC \cdot BD)/2$. Let the line symmetrical to AC about the bisector of the angle A, intersect BD at the point Q. Then the triangles ABQ and ACD are similar, hence $AB/AC = BQ/BD$ and $BQ = BD/2$ (see the figure).

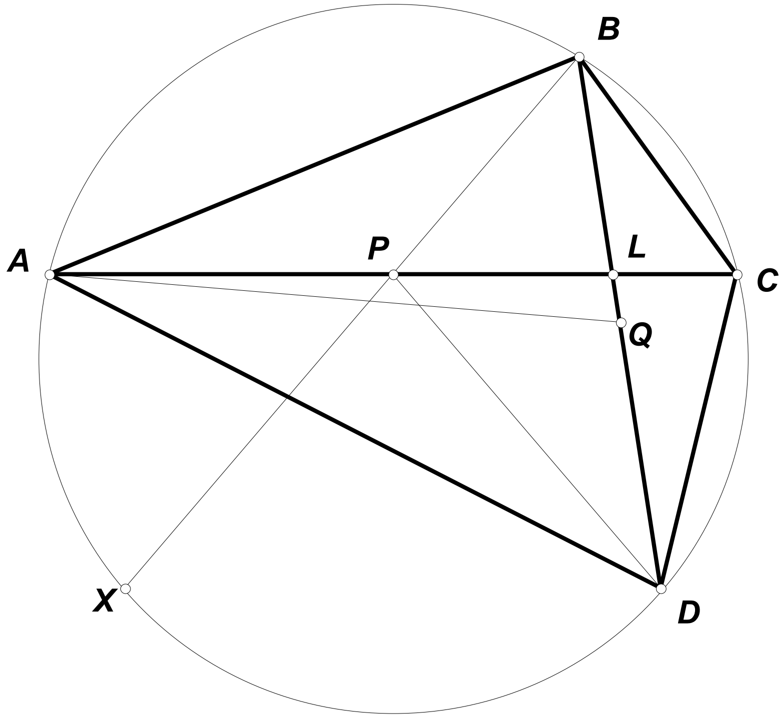

Fig. 9

10. (A.Zaslavskiy, 9–10) Quadrangle $ABCD$ is circumscribed around a circle with center I. Prove that the projections of points B and D to the lines IA and IC are concyclic.

Solution. It is obvious that the midpoint of BD is equidistant from projections of points B and D to any line. Let us prove that it is also equidistant from the projections X, Y of the point B to IA and IC.

Since $\angle BXI = \angle BYI = 90^\circ$, the points X, Y belong to the circle with diameter BI, i.e. the perpendicular bisector to the segment XY passes through the midpoint of BI. Therefore it is sufficient to prove that $XY \perp ID$. Clearly, in this case the perpendicular bisector to XY will coincide with the midline of the triangle BDI and will therefore pass through the midpoint of BD.

As the points B, I, X, Y belong to the same circle, the angle between XY and XA is equal to the angle between BY and BI, i.e. $\angle BIC - 90^\circ$. It follows that the angle between XY and ID is equal to $\angle AID + \angle BIC - 90^\circ = 90^\circ$ (see the figure).

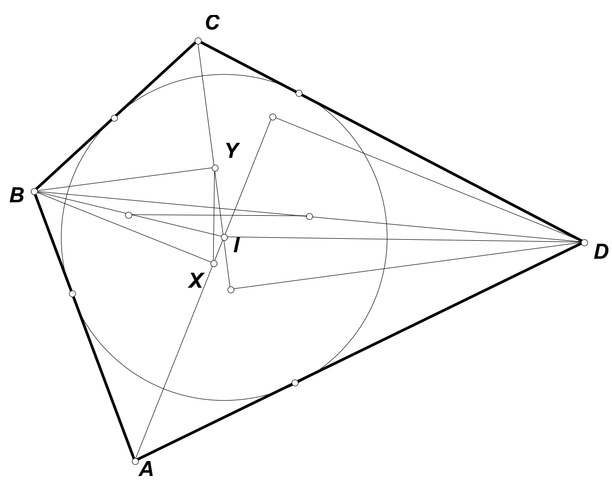

Fig. 10

11. (A.Zaslavskiy, 9–10) Given four points A, B, C, D. Any two circles such that one of them contains A and B, and the other one contains C and D, meet. Prove that common chords of all these pairs of circles pass through a common point.

Solution. Firstly let us consider the case when the points are not collinear. If, for example, C and D lie at the same side from the line AB then there is a circle ω passing through C and D and tangent to

AB. Then one can construct a circle with radius large enough, passing through A and B and not intersecting ω. Therefore, the segments AB and CD must intersect. Let O be the point of intersection of the perpendicular bisectors to these segments. Two circles with the center O and radii OA and OC either do not intersect at all or coincide. Therefore, the points A, B, C, D are concyclic. By the theorem about radical axes of three circles, the common chord of any two circles, passing through A, B and C, D respectively, passes through the meet point of AB and CD.

If all the points given are collinear, then it is obvious that the lines AB and CD intersect, while the common chord of the circles intersects with the line containing these points, at some point P that belongs to both segments and satisfies the equation $PA \cdot PB = PC \cdot PD$. These conditions define the point P unambiguously.

12. (A.Myakishev, 9–10) Given a triangle ABC. Point A_1 is chosen on the ray BA so that segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that lines A_1A_2, B_1B_2, C_1C_2 are parallel.

    Solution. Let O, I be the circumcenter and the incenter of the triangle. As BI is the bisector of the angle B in an isosceles triangle A_1BC, we have $A_1I = IC$. Similarly $A_2I = IB$. It follows that

$$A_1I^2 - A_2I^2 = IC^2 - IB^2 = (p-c)^2 - (p-b)^2 = a(b-c).$$

On the other hand, if B_0, C_0 are the midpoints of AC and AB, then

$$OA_1^2 - OA_2^2 = OC_0^2 - OB_0^2 + A_1C_0^2 - A_2B_0^2 =$$

$$= \left(\frac{b}{2}\right)^2 - \left(\frac{c}{2}\right)^2 + \left(a - \frac{c}{2}\right)^2 - \left(a - \frac{b}{2}\right)^2 = a(b-c).$$

Therefore, the lines A_1A_2 and OI are perpendicular (see the figure). Similarly we obtain that OI is perpendicular to the two other lines.

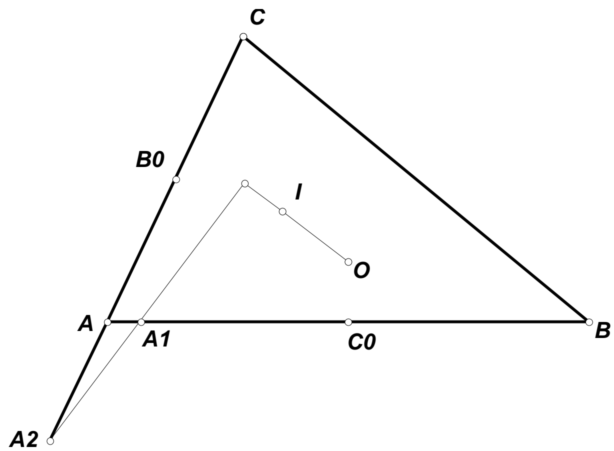

Fig. 12

13. (A.Myakishev, 9–10) Given triangle ABC. One of its excircles is tangent to the side BC at point A_1 and to the extensions of two other sides. Another excircle is tangent to side AC at point B_1. Segments AA_1 and BB_1 meet at point N. Point P is chosen on the ray AA_1 so that $AP = NA_1$. Prove that P lies on the incircle.

    Solution. The points of tangency of the triangle's sides with the excircles are symmetrical to its points of tangency with the incircle, relative to midpoints of sides. So $CA_1 = p-b$, $CB_1 = p-a$, $AB_1 = BA_1 = p-c$. Applying Menelaus theorem to the triangle ACA_1 and the line BB_1 we obtain that $A_1N/AA_1 = (p-a)/p$. The homothety with this ratio and the center at A maps the point A_1 into the point P. But the ratio of radii of the incircle and the excircle of the triangle is also equal to $(p-a)/p$, which means that the image of the point A_1 under this homothety belongs to the incircle (see the figure).

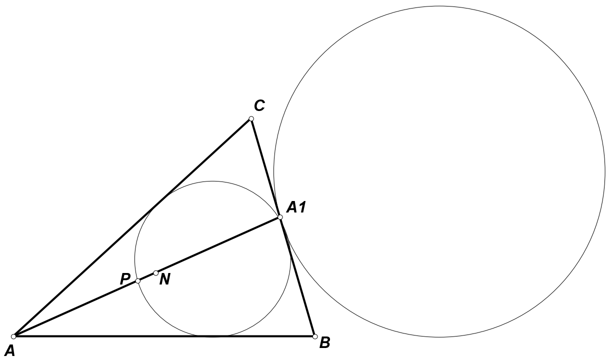

Fig. 13

14. (V.Protasov, 9–10) The line connecting the circumcenter and the orthocenter of a non-isosceles triangle is parallel to the bisector of one of its angles. Determine this angle.

   Answer. 120°.

   Solution. Let O be the circumcenter of the triangle ABC, let H be its orthocenter, let the line OH be parallel to the bisector of the angle C. As this bisector intersects the circumcircle at midpoint C' of the circle arc AB, we have $OC' \perp AB$, i.e. the quadrilateral $OC'CH$ is a parallelogram and $CH = OC' = R$. On the other hand $CH = 2R|\cos C|$, so the angle C equals either 60° or 120°. But in the first case the rays CO and CH are symmetrical about the bisector of the angle C, so the line OH cannot be parallel to this bisector. Therefore, $C = 120^\circ$ (see the figure).

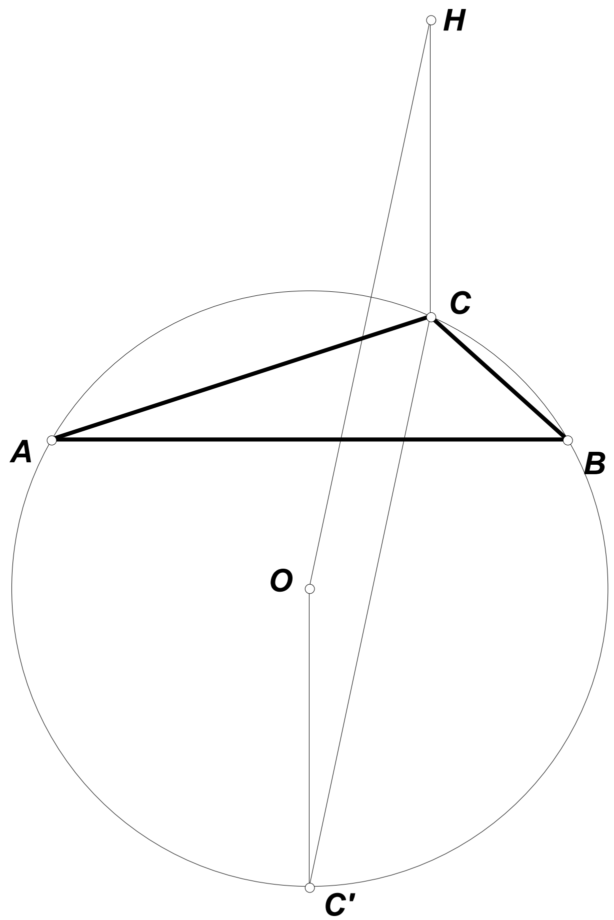

Fig. 14 -

15. (M.Volchkevich, 9–11) Given two circles and a point P not lying on them. Draw a line through P which cuts chords of equal length from these circles.

    Solution. Let A, B be the points of intersection of the required line with the first circle, and the points C, D be the midpoints of intersection of the required line with the second circle. Let M be the midpoint of segments AD and BC. Then the powers of point M with respect to the circles are equal, i.e. M belongs to their radical axis. Let L be the midpoint of the segment between the centers of the circles. Since the projections of these centers on the required lines are the midpoints of segments AB and CD, the point M is the projection of the point L on this line. Therefore, $\angle LMP = 90°$ and M belongs to the circle with diameter LP. So, in order to construct the line in question, one should find intersection points between this circle and the radical axis. The problem can have two, one or no solutions.

16. (A.Zaslavskiy, 9–11) Given two circles. Their common external tangent line touches them at points A and B. Points X, Y on the circles are such that some circle is tangent to the given two circles at these points, and in similar way (external or internal). Determine the locus of intersections of lines AX and BY.

Solution. The points X, Y are the centers for homotheties of the given circles with the tangent circle. Therefore, the line XY contains the center of homothety between these circles, i.e. the point of intersection between AB and the line of the centers. Let Y' be the point other than Y where this line intersects the second circle. Then $BY' \parallel AX$ и $\angle XYB = \angle Y'BA = \pi - \angle BAX$. Therefore the quadrilateral $AXYB$ is inscribed and the point P being the intersection between the lines AX and BY is the radical center for the given circles and the circumcircle of this quadrilateral. I.e. it belongs to the radical axis of these circles (see the figure). Obviously, any point of the radical axis belongs to the required locus.

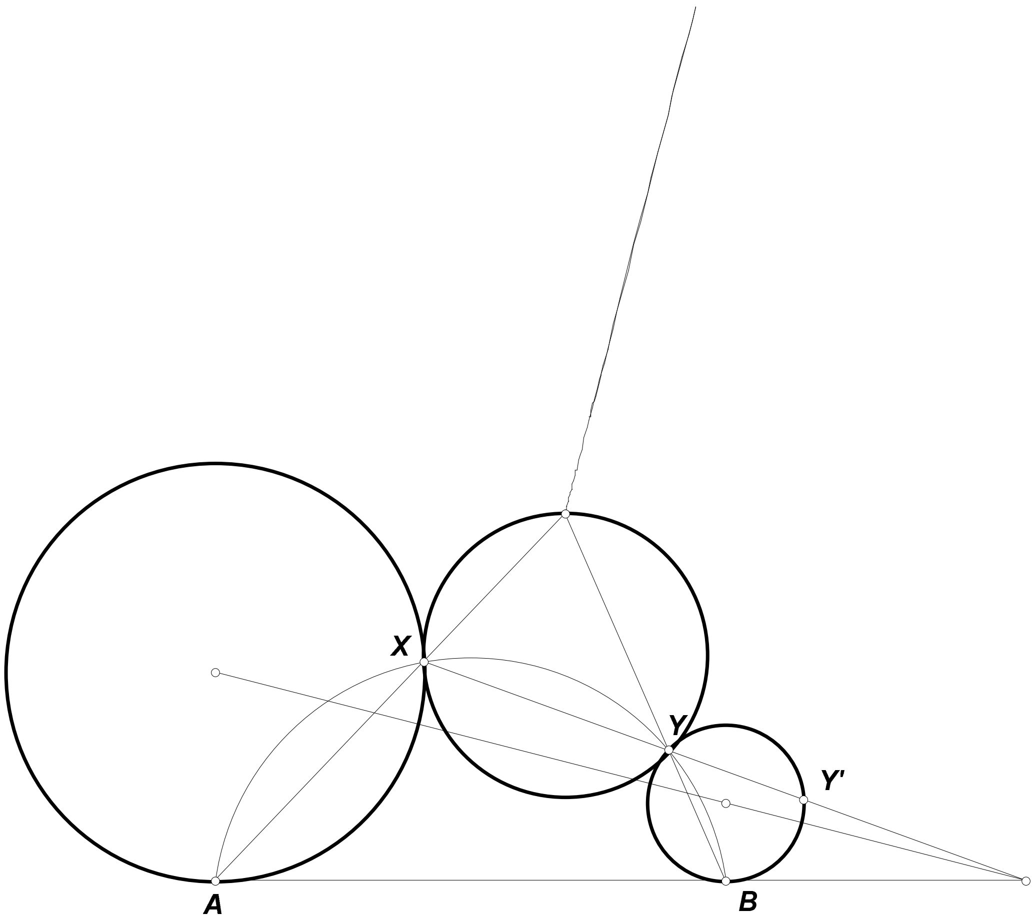

Fig. 16

17. (A.Myakishev, 9–11) Given triangle ABC and a ruler with two marked intervals equal to AC and BC. By this ruler only, find the incenter of the triangle formed by the midlines of triangle ABC.

Solution. Let us construct the segments $BC_1 = CB_1 = BC$ on the extension of the side AB beyond B and on the extension of the side AC beyond C respectively. Let A' be the point of intersection between BB_1 and CC_1. Then the line AA' passes through the point required (see the figure).

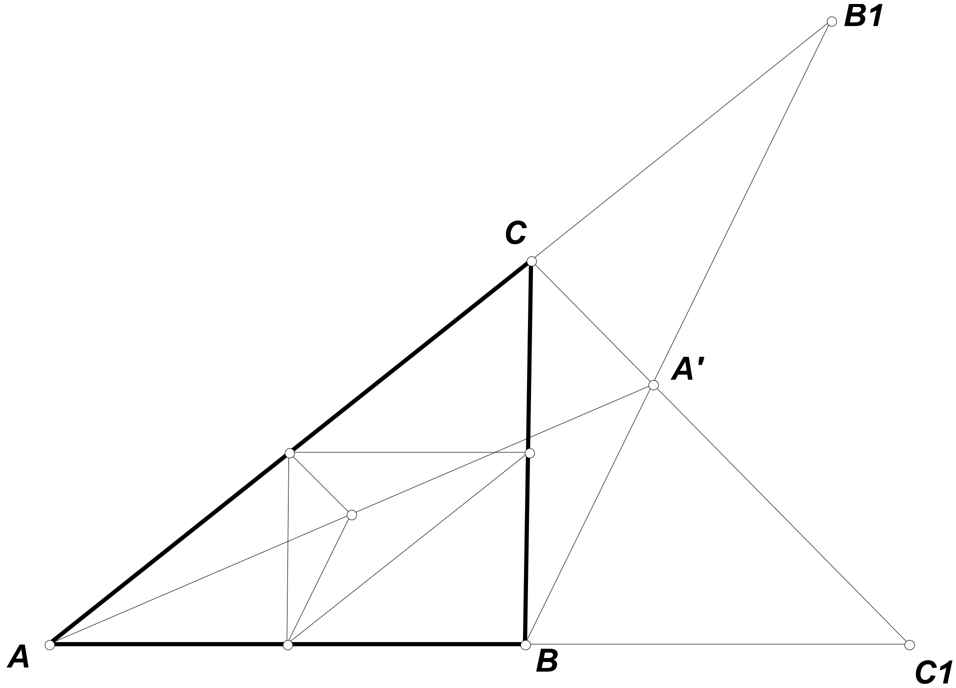

Fig. 17

Clearly, since the triangles BCB_1 and CBC_1 are isosceles, the lines BB_1 and CC_1 are parallel to bisectors of the angles C and B. Therefore under the homothety with center A and ratio of $1/2$ these lines will map into the bisectors of angles of the middle triangle, whereas the point A' will map into the center in question. Similarly, making use of the second segment marked on the ruler we can construct the line passing through B and the point in question.

18. (A.Abdullayev, Azerbaijan, 9–11) Prove that the triangle having sides a, b, c and area S satisfies the inequality

$$a^2+b^2+c^2-\frac{1}{2}(|a-b|+|b-c|+|c-a|)^2 \geq 4\sqrt{3}S.$$

Solution one. Let C be the middle angle of the triangle. Then $|b-c|+|c-a|=|a-b|$ and the left part of the ineqity equals

$$a^2+b^2+c^2-2(a-b)^2=4ab-(a^2+b^2-c^2)=2ab(2-\cos C).$$

As the right part of the inequation is equal to $2\sqrt{3}ab\sin C$, the given inequity is equivalent to

$$2-\cos C \geq \sqrt{3}\sin C.$$

However $\cos C + \sqrt{3}\sin C = 2\cos(C - \frac{\pi}{3})$, so this inequity always holds and turns into equation only if $C = 60^\circ$.

Solution two. Assuming again that c is the middle side of the triangle, let us denote $x = p-a$, $y = p-b$, $z = p-c$, where p is the half-perimeter, and let us write the left part as

$$a^2+b^2-(a-b)^2+c^2-(a-b)^2 = 2ab+4xy = 2(x+z)(y+z)+4xy = 2pz+6xy.$$

Since the right part is equal to $4\sqrt{3pxyz}$, the inequality takes the form

$$pz + 3xy - 2\sqrt{3pxyz} = (\sqrt{px} - \sqrt{3xy})^2 \geq 0.$$

19. (V.Protasov, 10-11) Given parallelogram $ABCD$ such that $AB = a$, $AD = b$. The first circle has its center at vertex A and passes through D, and the second circle has its center at C and passes through D. A circle with center B meets the first circle at points M_1, N_1, and the second circle at points M_2, N_2. Determine the ratio M_1N_1/M_2N_2.

Solution. The points M_1, N_1 are symmetrical about the line AB, so M_1N_1 equals double distance from M_1 to AB. Similarly M_2N_2 equals double distance from M_2 to BC. Additionally, $CM_2 = CD = AB$, $AM_1 = AD = BC$, $BM_1 = BM_2$, which means that triangles ABM_1 and CM_2B are equal. Therefore the required ratio which equals the ratio of the altitudes of these triangles, is inverse to the ratio of respective sides, i.e. it is equal to b/a (see the figure).

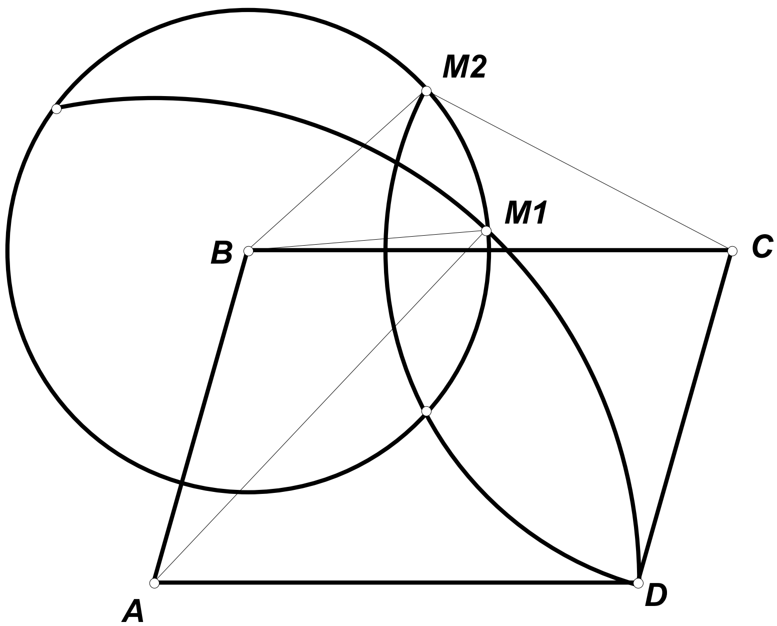

Fig. 19

20. (A.Zaslavskiy, 10–11) a) Some polygon has the following property: if a line passes through any two points which bisect its perimeter then this line bisects the area of the polygon. Is it true that the polygon is central symmetrical?

b) Is it true that any figure with the property from part a) is central symmetrical?

Solution. a) Yes, it is. Let X, Y be two points splitting the perimeter of the polygon in halves, which are not its vertices; let X', Y' be the points on the same sides that satisfy condition $XX' = YY'$; let P be the point of intersection for XY and $X'Y'$. Since each of these two lines splits the polygon into two pol with equal areaygons, the areas of triangles PXX' and PYY' are equal. As $XX' = YY'$, the altitudes dropped on these sides are also equal. Furthermore $\angle XPX' = \angle YPY'$ as vertical. Therefore these triangles are congruent. If the lines XX' and YY' are not parallel, then the congruence of angles $PX'X$ and PYY' follows. However if the pair of X, Y is fixed then this equation cannot hold for arbitrary X', Y'. So, when one of the opposite points moves along a side of the polygon, the other moves along the parallel side, while the lengths of these sides are equal. It means that the polygon has a center of symmetry.

Remark. The above argument implicates that no two sides of the polygon lie on the same line. If this condition does not hold then the polygon can possess the required property while not having a center of symmetry (see the figure).

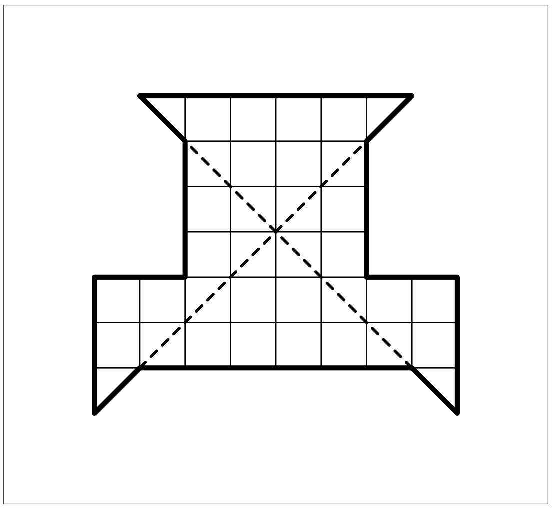

Fig. 20a

b) No, it isn't. Let, for instance, ABC be a regular triangle, whereas A', B', C' be the midpoints of its sides. Let us draw six circle arcs $60°$ each with centers at A', B', C' and endpoints A, B, C, A', B', C'. Let X, Y be a pair of points that bisect the perimeter of the shape formed by these circle arcs, and assume that the point X belongs, for instance, to the circle arc AB'. Then the point Y belongs to the circle arc $A'B$, and the circle arcs AX and $A'Y$ are equal. Since the circle arcs AB' and $A'B$ are parts of the same circle with center at C' it means that $\angle AC'X = \angle A'CY$. Therefore the area of $AXYB$ equals the sum of areas of sectors $C'AX$ and $C'BY$ which are not dependent on location of points X, Y, of the area of triangle $C'XY$, that also does not depend on location of these points, and of the areas of two constant segments. Therefore this area is constant and evidently equal to half of the area of the total shape (see the figure).

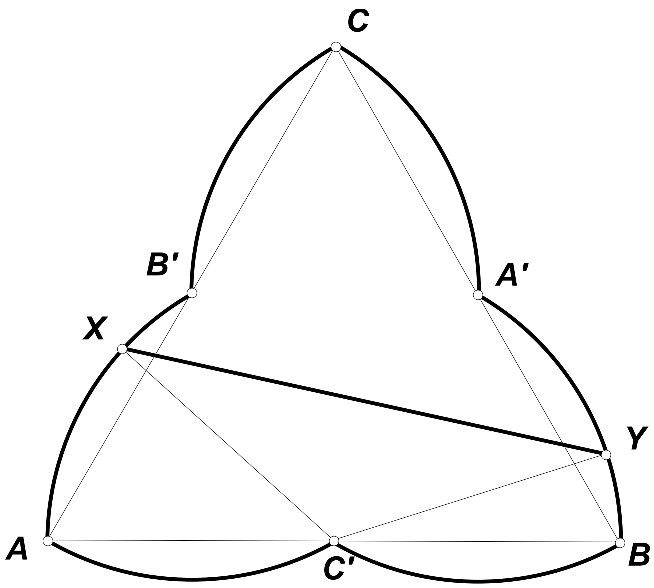

Fig. 20b

Remark. There are even some convex figures with the above property, for instance, the figure formed by the line

$$x = 12\cos\phi + \cos 2\phi + \frac{1}{2}\cos 4\phi,\ y = 12\sin\phi - \sin 2\phi + \frac{1}{2}\sin 4\phi, \quad 0 \le \phi \le 2\pi.$$

However the proof in this case is considerably more complicated.

21. (A.Zaslavskiy, B.Frenkin, 10–11) In a triangle, one has drawn middle perpendiculars to its sides and has measured their segments lying inside the triangle.

a) All three segments are equal. Is it true that the triangle is equilateral?

b) Two segments are equal. Is it true that the triangle is isosceles?

c) Can the segments have length 4, 4 and 3?

Solution. a) Yes, it is true. Observe that in the triangle ABC we have $\angle A < \angle B \le \angle C$. Then the perpendicular bisectors to sides AC and BC intersect with the side AB. The segments of these perpendiculars that lie within the triangle ABC have equal projections on the lines perpendicular to AB, but they form unequal angles with these lines. Hence they are not equal.

Now assume $\angle A \le \angle B < \angle C$. Then the perpendicular bisectors to AB and AC cross AC and AB respectively and therefore they cut similar yet unequal triangles from the triangle ABC. The segments of perpendicular bisectors lying within the triangle are the respective sides of these triangles and therefore they are not equal.

Observe that the above reasoning implies that the segment of a perpendicular bisector with minimal length is the one dropped to the middle side of the triangle.

b) No, it isn't. For instance, consider the triangle with angles $A = \pi/8$, $B = \pi/4$, $C = 5\pi/8$ and the circumcircle of unit radius. Then the perpendicular bisector to AB intersects the side AC, and its segment within the triangle is equal to $AB \operatorname{tg} \angle A/2 = \sin(5\pi/8) \operatorname{tg}(\pi/8) = \cos(\pi/8) \operatorname{tg}(\pi/8) = \sin(\pi/8)$. The perpendicular bisector to BC intersects AB, and the length of the respective segment is equal to $BC \operatorname{tg} \angle B/2 = \sin(\pi/8)$. Therefore these segments are equal. Any triangle with $\angle A < \angle B < \angle C$ and $\cos A \operatorname{tg} B = \sin C$ also meets the required condition.

c) No, it isn't. If the triangle is isosceles then as follows from part a) the segments of perpendicular bisectors to the lateral sides are shorter then the altitude to the base. If $\angle A < \angle B < \angle C$ and $\cos A \operatorname{tg} B = \sin C$, then the ratio of perpendicular bisectors' segments to the longest and the middle sides of the triangle is equal to the ratio of these sides themselves. I.e.

$$\frac{\sin C}{\sin B} = \frac{\cos A}{\cos B} = \frac{\cos B}{\sqrt{1 - 2\cos B + 2\cos^3 B}}.$$

Examining the right side of this equation we can conclude that its maximum is less then $4/3$.

22. (A.Khachaturyan, 10–11) a) All vertices of a pyramid lie on the facets of a cube but not on its edges, and each facet contains at least one vertex. What is the maximal possible number of the vertices of the pyramid?

b) All vertices of a pyramid lie in the facet planes of a cube but not on the lines including its edges, and each facet plane contains at least one vertex. What is the maximal possible number of the vertices of the pyramid?

Answer. a)13. b) Arbitrary large number.

Solution. a) The section of a cube by the plane of the pyramid's base intersects all its facets and is therefore a convex hexagon. The vertices of the base lie on the sides of this hexagon but not in its vertices. It is easily seen that if any side contains more than two vertices of the base then it is impossible to link them by a non-self-intersecting polygonal line lying within the hexagon. Therefore the base has no more than 12 vertices, and the pyramid has no more than 13 vertices. A pyramid with 13 vertices evidently exists.

b) In the considered case the vertices of the base can belong to the lines containing the sides of the hexagon, and their number can be arbitrarily large (see the figure).

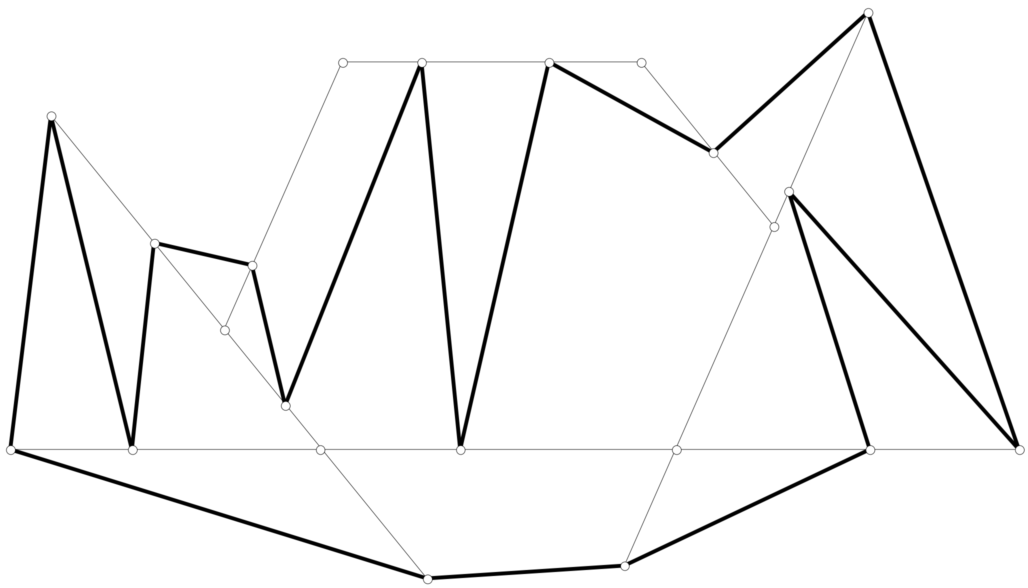

Fig. 22

23. (V.Protasov, 10–11) In the space, given two intersecting spheres of different radii and a point A belonging to both spheres. Prove that there is a point B in the space with the following property: if an arbitrary circle passes through points A and B then the second points of its meet with the given spheres are equidistant from B.

Solution. Draw a line through A that is parallel to the line containing centers of the given spheres, and find secondary points C, D of its intersection with the spheres. Let us show that the midpoint B of the segment CD is the required point. Choose an arbitrary circle passing through A and B, and consider the cuts of the spheres by the plane of this circle. These cuts are two circles one of which passes through A and C, another passes through A and D. The centers of these circles are at the projections O_1, O_2 of the centers of spheres on the plane of the cut. Therefore the lines O_1O_2 and CD are parallel. Thus it suffices to prove the plane version of the problem statement.

Let X_1, X_2 be the secondary points of intersection of the circle passing through A and B with the given circles; let A' be the secondary point of intersection of the given circles. Then O_1O_2 is the midline of the triangle $A'CD$, i.e. $CB = BD = O_1O_2$. It follows that $O_1B = O_2D = O_2X_2$, $O_2B = O_1C = O_1C_1$. In addition, the center O of the circle ABX_1X_2 is equidistant from O_1 and O_2, therefore $\angle BO_1X_1 = \angle BO_1O + \angle OO_1X_1 = \angle BO_1O + \angle AO_1O = \angle AO_2O + \angle BO_2O = \angle BO_2O + \angle OO_2X_2 = \angle BO_2X_2$. So the triangles O_1X_1B and O_2BX_2 are equal and hence $BX_1 = BX_2$ (see the figure).

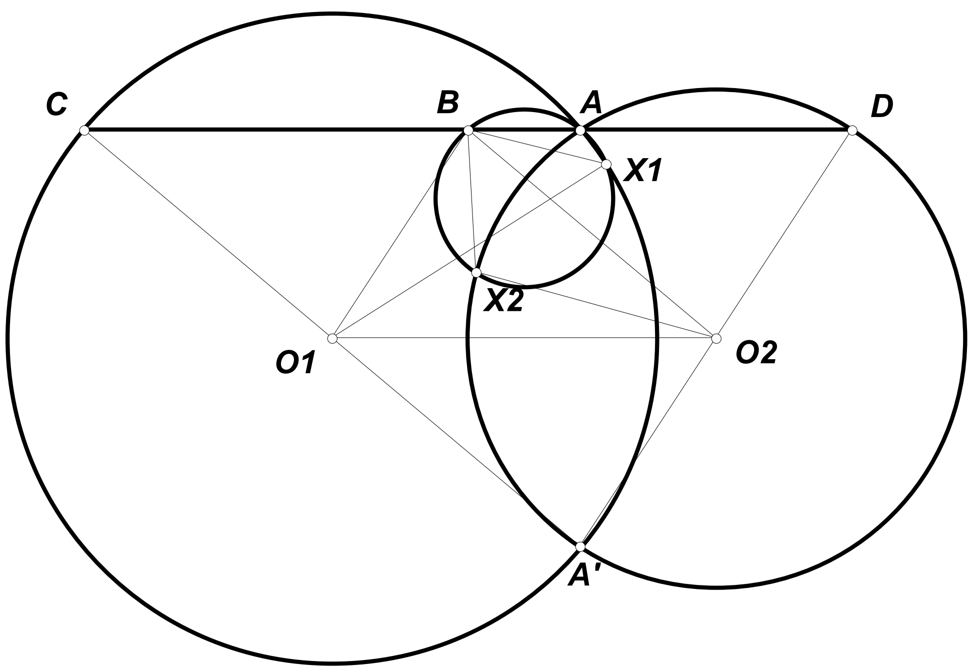

Fig. 23

24. (I.Bogdanov, 11) Let h be the least altitude of a tetrahedron, and d the least distance between its opposite edges. For what values of t the inequality $d > th$ is possible?

Answer. For $t < 3/2$.

Solution. Let ABC be the facet of the tetrahedron $ABCD$ with the largest area. Then its volume is equal to $V = S_{ABC}h/3$. On the other hand it is equal to half the product of lengths of the opposite edges and the distance between them multiplied by the sine of the angle between them. Let $A'B'C'$ be the triangle whose midlines are the sides of ABC. Then, for instance, $S_{A'B'D} = AB \cdot CD \sin\phi$, where ϕ is the angle between AB and CD. Since the sum of areas of lateral facets of the tetrahedron is greater than the area of its base, the area of triangle $A'B'C'$ does not exceed the triple maximal area of triangles $A'B'D$, $B'C'D$, $C'A'D$, i.e. $d < 3h/2$. This inequality cannot be strengthened, because if one takes a regular pyramid and its altitude tends to zero, then the ratio d/h tends to $3/2$.