Third olympiad, year 2007
Correspondence round. Solutions

1. (B.Frenkin) A triangle is cut into several (at least two) triangles. One of them is isosceles (non-regular) while the others are regular. Find the angles of the initial triangle.

   Answer. 30°, 60°, 90°.

   Solution. Among the vertices of the non-regular triangle at least one is not a vertex of the initial triangle. The sum of angles of the dissection triangles, adjacent to this vertex is equal to 180° or 360°. Hence the angle of the triangle is a multiple of 60°, and since the triangle is isosceles but non-regular this angle equals 120°. Then the two other angles of this triangle are equal to 30° and, as they are not multiples of 60°, the respective vertices are the vertices of the initial triangle. The angles of the triangle at these vertices can only be equal to either 30°, 90° or 150°. Moreover at least one of the two angles is not 30°, whereas their sum is less than 180°. The only possible alternatives are 30° and 90°. Such triangle can be split in the required manner, for instance, along the median drawn from the vertex of the right triangle (see the figure).

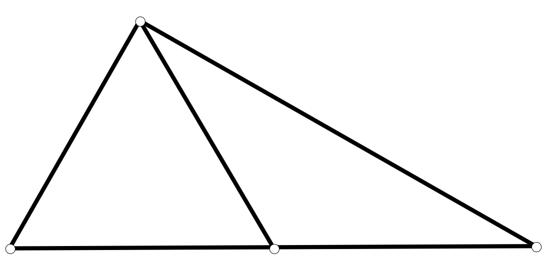

Fig. 1

2. (A.Blinkov) Each diagonal of a quadrilateral divides it into two isosceles triangles. Is it correct that the quadrilateral is a rhombus?

   Solution. No, this is incorrect. For example, let ABC be an isosceles triangle whereby the angle B is obtuse and not equal to 120°, and let the point D be the circumcenter of ABC. Then the quadrilateral $ABCD$ meets the problem conditions and is not a rhombus.

3. (B.Frenkin) The segments linking an inner point of a convex non-equilateral n-gon with its vertices split the n-gon into n congruent triangles. At what minimal n is it possible?

Answer. At $n = 5$.

Solution. Let us prove that for $n = 3,\ 4$ the above situation is impossible. If $n = 3$, the angles of the dissection triangles that meet at the inner point are equal, because the sum of any two distinct angles of these triangles is less than $180°$. But if so, then their opposite sides which are the sides of the polygon will also be equal.

The argument can follow a different line. Since the resulting triangles are congruent, the radii of their circumcircles and their areas are equal too. The first fact implies that the point determining the dissection is the orthocenter of the triangle. The second condition implies that this point is the mass center of the triangle. However, the orthocenter and the mass center can only coincide in a regular triangle.

Assume that the quadrilateral $ABCD$ is cut into congruent triangles by the segments drawn from the point O. Then $\angle OAB = \angle OCB$ as angles opposite to the same side of congruent triangles. Similarly $\angle OAD = \angle OCD$, $\angle OBC = \angle ODC$, $\angle OBA = \angle ODA$. Therefore, $\angle A = \angle C$, $\angle B = \angle D$, so $ABCD$ is a parallelogram. As line segments from O divide it into congruent triangles, O is the point of meet of its diagonals. Hence equality of the triangles implies that $ABCD$ is a rhombus.

If $n = 5$ the situation in question is possible (see the figure).

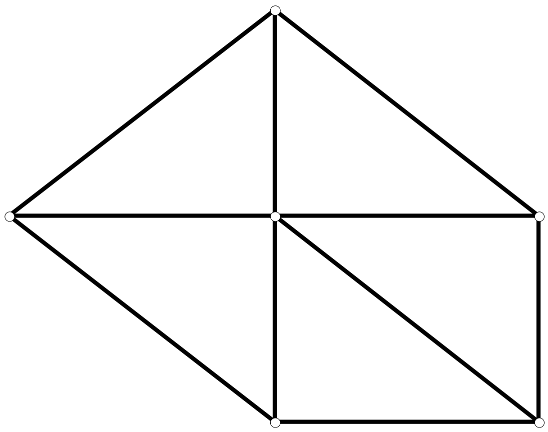

Fig. 2

4. (A.Blinkov) Does there exist a parallelogram such that all its pairwise intersections of bisectors lie outside it?

   Solution. No. Let side AD of the parallelogram $ABCD$ be not less than AB. Mark the segment $AE = AB$ on AD and draw a line parallel to AB through E. We will obtain a rhombus, wherein the intersection of bisectors is the incenter and as such lies within the rhombus. However it serves as an intersection of two bisectors of the initial parallelogram in which the rhombus is contained. Therefore this point belongs to the parallelogram.

5. (D.Shnoll)A non-convex n-gon has been split by a straight line into three pieces. Then two of them being put together, form a polygon equal to the third piece. Can n be equal to

   a) five?

   b) four?

   Answer. a) yes (see the figure).

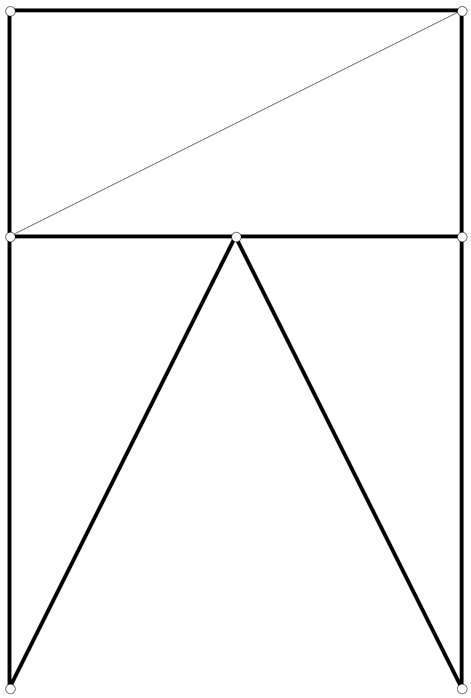

Fig. 3

b) yes (see the figure).

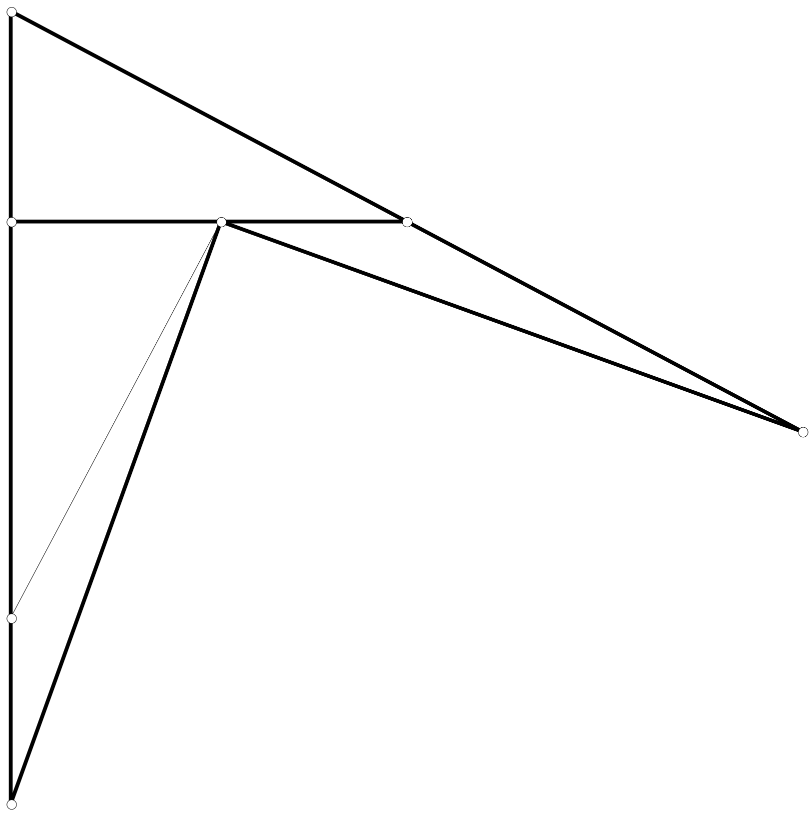

Fig. 4

6. (B.Frenkin) a) How many symmetry axes are possible for a checked polygon (i.e. a polygon with all its sides on the lines of a squared paper)? Indicate all possible values.) b) How many symmetry axes are possible for a checked polyhedron (i.e. a polyhedron built of equal cubes adjacent with their sides)?

   Answer. a) 0, 1, 2 or 4. b) 0, 1, 3, 5 or 9.

   Solution. a) Each symmetry axis of a checked polygon passes through its certain cell that will map into itself under the symmetry about this axis. Therefore the axis is parallel either to a side or to a diagonal of the cell. Hence, a checked polygon can possibly have only four symmetry axes.

   A restricted geometric shape cannot have two parallel symmetry axes. Indeed, a composition of symmetries with respect to a couple parallel lines is a parallel transfer. A restricted geometric shape cannot map onto itself under a transfer, therefore at least one of these lines is not its symmetry axis.

   Hence, a checked polygon has no more than 4 symmetry axes. If it has three axes of symmetry, then the composition of these symmetries is a symmetry about the fourth line. Examples of polygons having 0, 1, 2 or 4 axes are shown in the figure.

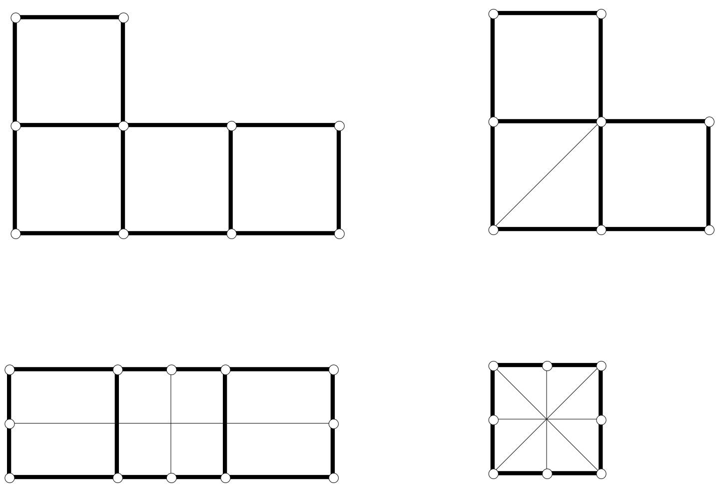

Fig. 5

b) Similarly to part a) we obtain that all the symmetry axes have different directions and are parallel either to edges of the cubes forming the polyhedron or to diagonals of its faces. Hence, the number of symmetry axes does not exceed 9. Let the lines l, l_1 be the symmetry axes. If the angle between them is not right, then the line l_2, symmetrical to l_1 with respect to l will also be a symmetry axis. If, alternatively, $l \perp l_1$, then the line perpendicular to both of them will also be a symmetry axis. So, all the symmetry axes except for l can be divided into pairs, i.e. their total number is odd. It is easily seen that all odd values are possible except for 7. If there are 7 axes then reflecting them with respect to each other we will obtain two additional axes.

7. (B.Frenkin) A convex polygon is circumscribed about a circle. Its points of tangency with the circle form a polygon with the same tuple of angles (the sequence of angles may be different). Is it correct that the polygon is regular?

Answer. Yes, it is.

Solution. (M.Kayranbay, Kazakhstan) Let $A_1A_2\dots A_n$ be the given polygon, B_1, B_2,…,B_n be the points of tangency of the incircle with the sides A_1A_2, A_2A_3,...,A_nA_1. Let us denote values of the angles in the first polygon as $a_1, \dots, a_n$, in the second one as $b_1, \dots, b_n$. Then $b_i = (a_i + a_{i+1})/2$. Multiplying n such equations we obtain $2^n a_1 a_2 \cdots a_n = (a_1 + a_2) \cdots (a_{n-1} + a_n)(a_n + a_1)$. However, by AM-GM inequality we have $a_i + a_{i+1} \geq 2\sqrt{a_i a_{i+1}}$. The resulting equation implies that all angles of the polygons are equal, and since the polygon $B_1 \dots B_n$ is inscribed, the polygons are regular.

8. (A.Zaslavsky) Three circles intersect at point P, while their secondary points of intersection A, B, C are collinear; A_1, B_1, C_1 are secondary points of intersection for lines AP, BP, CP with respective circles; C_2 is the point of intersection for lines AB_1 and BA_1, and A_2, B_2 are defined similarly. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are congruent.

   Solution. Since quadrilaterals PAB_1C and $PBAC_1$ are inscribed, we have $\angle CAC_2 = \angle CAB_1 = \angle CPB_1 = \angle BAC_1$ (see the figure). Similarly, $\angle ABC_2 = \angle ABC_1$, i.e. the points C_1, C_2 are symmetrical with respect to line AB. Repeating the same argument for other two pairs of points, we obtain that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are symmetrical with respect to this line and therefore congruent.

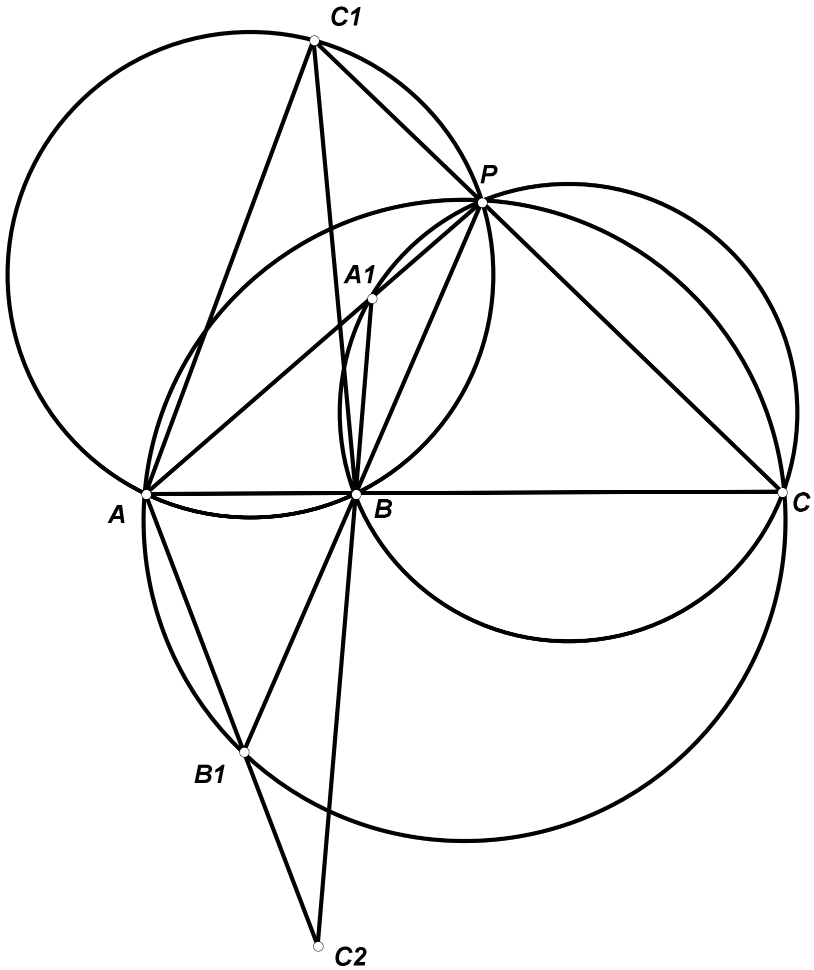

Fig. 6

9. (A.Zaslavsky) Two convex quadrilaterals are such that the sides of each one lie on perpendicular bisectors to the sides of the other. Find their angles.

Solution. Suppose the side $C'D'$ of the quadrilateral $A'B'C'D'$ belongs to the perpendicular bisector to side AB of the quadrilateral $ABCD$, while the side $D'A'$ belongs to the perpendicular bisector to BC. Then D' is the circumcenter of the triangle ABC. Similarly A', B', C' are the circumcenters of the triangles BCD, CDA, DAB. Therefore, $B'D'$ is the perpendicular bisector to AC. In turn, AC is a perpendicular bisector to one of the diagonals in $A'B'C'D'$, and since $AC \perp B'D'$ and $B'D' \nparallel A'C'$, AC is the perpendicular bisector to $B'D'$, i.e. $AB'CD'$ is a rhombus. Composition of symmetries with respect to lines $C'D'$, $D'A'$, $A'B'$ and $B'C'$ fixes the point A and therefore is a rotation centered at A'. On the other hand, it is the composition of the rotation centered at D', by doubled angle $C'D'A'$ and the rotation centered at B', by doubled angle $A'B'C'$. Therefore $\angle C'B'A' = \angle AB'D' = \angle B'D'A = \angle A'B'C'$. Similarly, $\angle B'C'D' = \angle D'A'B'$, i.e. $A'B'C'D'$ is a parallelogram. The sides of $ABCD$ are perpendicular to the sides of $A'B'C'D'$, hence $ABCD$ is a parallelogram with the same angles.

Since C is the circumcenter of the triangle $B'C'D'$, we have $\angle D'CB' = 2\angle C'D'A' = \angle B'D'C + \angle CB'D' = 90°$. Respectively, the acute angles of parallelograms $ABCD$ and $A'B'C'D'$ are equal to $45°$. It is easily seen that the two such parallelograms, mapped from one onto another by a rotation through $90°$ around a common center, meet the conditions of the problem (see the figure).

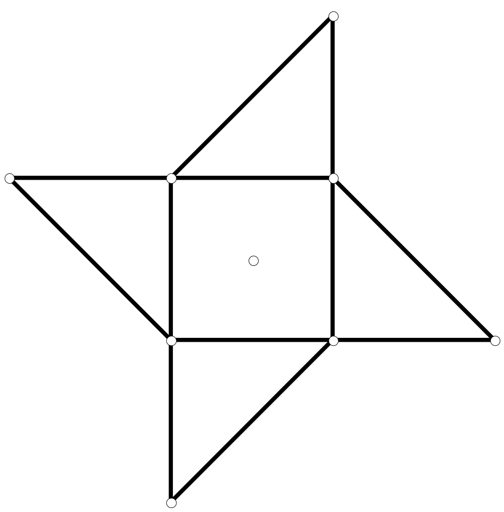

Fig. 7

10. (A.Zaslavsky) Find the locus of regular triangles' centers, the sides of which pass through the three given points A, B, C (i.e. there is exactly one of the given points on each side or its extension).

Solution. Let A, B, C be the given points. Construct circle arcs containing an angle of $60°$, on the sides and outside triangle ABC. Find the midpoints A', B', C' of the complementary circle arcs. The lines linking the center of the triangle with its vertices pass through A', B', C'. The vertices move along the constructed circles at constant angular speeds, hence the angles, at which the segments $A'B'$, $B'C'$, $C'A'$ are visible from the center, remain constant. So, the locus in question is the circle $A'B'C'$ (see the figure).

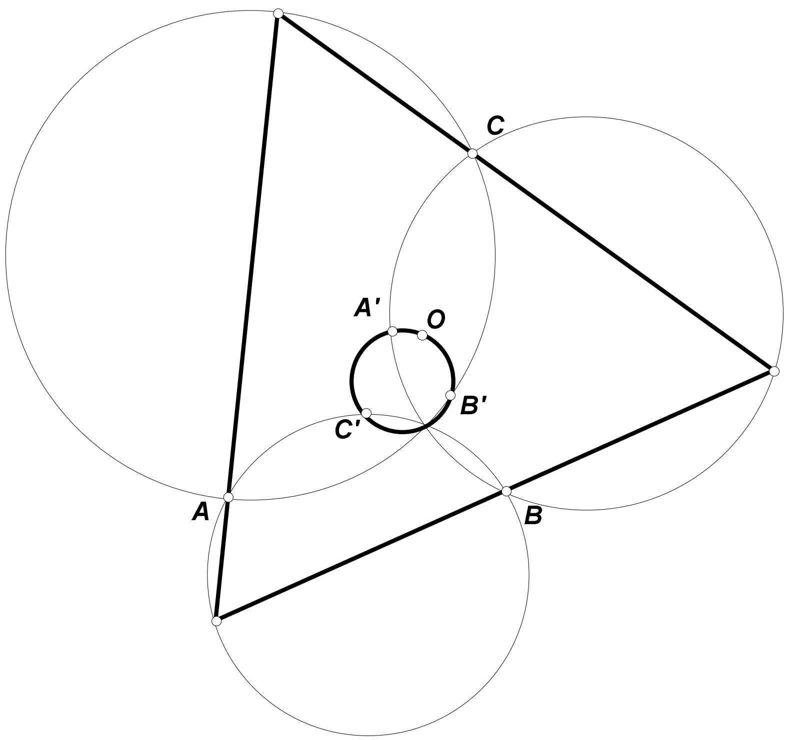

Fig. 8

11. (D.Schnoll) A boy and his father stand by the sea shore. If the boy stands on tiptoes his eyes are 1 meter high from the sea level. If he sits over father's shoulders his eyes are 2 meters high from the sea level. How many times farther does he see in the second case than in the first one? (Find the answer with precision 0.1, assuming the radius of the Earth equal to 6000 kilometers).

Solution. Visibility from height h above the sea level equals $d = \sqrt{(R+h)^2 - R^2} = \sqrt{2Rh - h^2}$, where R is the radius of the Earth (see the figure). If $h << R$, then with high enough accuracy $d = \sqrt{2Rh}$. Therefore the ratio in question can be accepted equal $\sqrt{2}$ or 1.4.

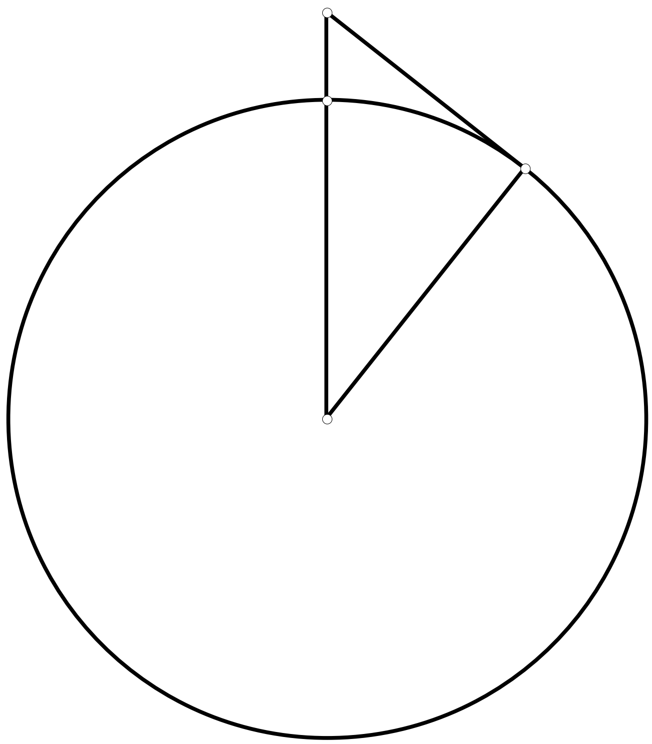

Fig. 9

12. (A.Zaslavsky) Consider a rectangle $ABCD$ and a point P. The lines passing through A and B and perpendicular to PC and PD respectively, intersect at the point Q. Prove that $PQ \perp AB$.

    Solution one. Let U, V be projections of A and B on PC and PD respectively. Then U and V belong to the circumcircle of $ABCD$, and by applying Pascal theorem to the polygonal line $AUCBVD$ we obtain the statement of the problem (see the figure).

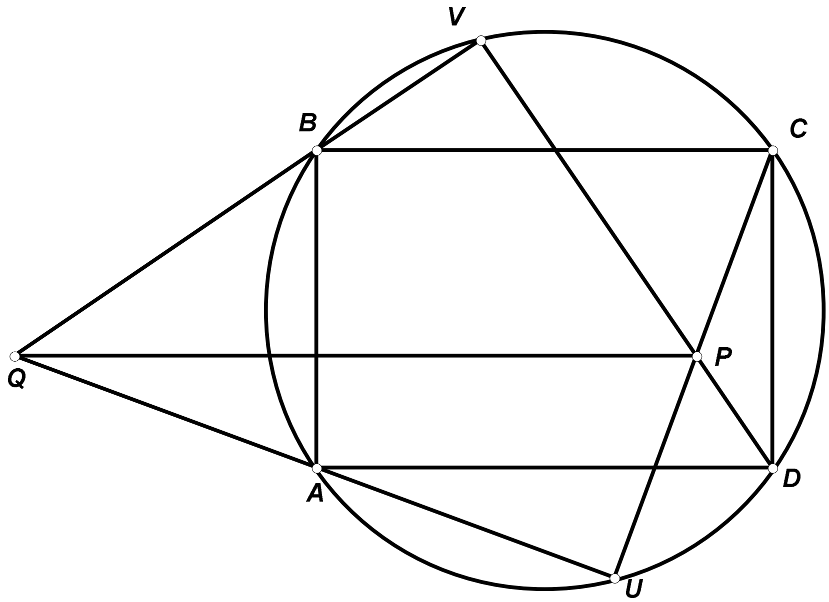

Fig. 10

Solution two. Since $ABCD$ is a rectangle, the scalar products of (PA, PC) and (PB, PD) are equal. However $(PA, PC) = (PC, PA) + (PC, AQ) = (PC, PQ)$. Similarly, $(PB, PD) = (PD, PQ)$. Therefore, $(PQ, CD) = 0$.

Solution three. Let Q' be the image of Q under the transfer by vector BC. Then $CQ' \parallel BQ \perp DP$, $DQ' \perp CP$. Therefore, P is the orthocenter in the triangle CDQ', and $PQ' \perp CD$.

13. (A.Zaslavsky) The points X,Y are chosen on the side AB of the triangle ABC so that $AX = BY$. The lines CX and CY have secondary intersections with the circumcircle of the triangle at the points U and V. Prove that all lines UV concur.

Solution one. Let Z be the point of intersection between AB and UV. Applying sine theorem to triangles ZAU and ZBV we obtain (see the figure)

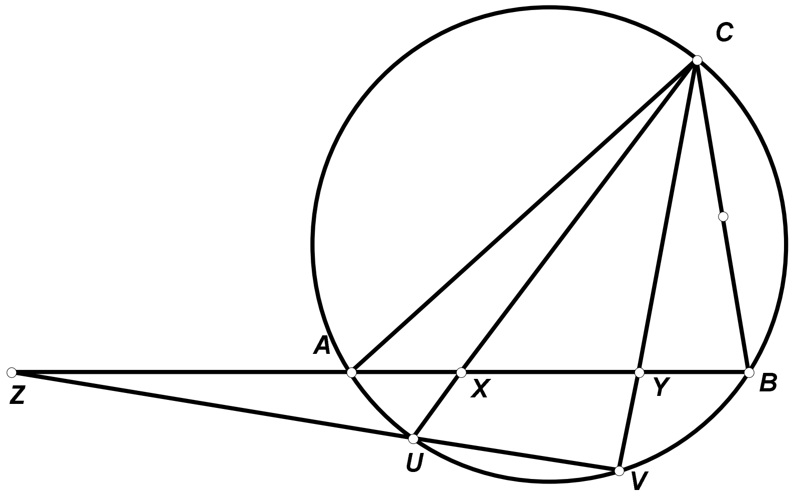

Fig. 11

$$\frac{ZA}{ZB} = \frac{AU \sin\angle AUZ}{BV \sin\angle BVZ} = \frac{AU \sin\angle ACY}{BV \sin\angle BCX} = \frac{\sin\angle ACX \sin\angle ACY}{\sin\angle BCX \sin\angle BCY}.$$

From triangle ACX we obtain $\sin\angle ACX = \frac{AX}{AC}\sin\angle AXC$. From the above and the three analogous relations we obtain $\frac{ZA}{ZB} = \frac{BC^2}{AC^2}$, i.e. this ratio does not depend on the choice of points X, Y.

Solution two. The point C', symmetrical to C about the perpendicular bisector to AB, belongs to the circumcircle of the triangle ABC. Let the line tangent at this point intersect AB at point Z. Draw an arbitrary secant to the circle through Z which will cross it at points U, V, and find the meet points X, Y of the lines CU and CV with AB. We have

$$\frac{AX}{BX} = \frac{S_{ACU}}{S_{BCU}} = \frac{AC \cdot AU}{BC \cdot BU} = \frac{AU \cdot BC'}{BU \cdot AC'}.$$

Similarly

$$\frac{AY}{BY} = \frac{AV \cdot BC'}{BV \cdot AC'}.$$

Multiplying these equations we obtain

$$\frac{AX \cdot AY}{BX \cdot BY} = \frac{AU}{BV}\frac{AV}{BU}\left(\frac{BC'}{AC'}\right)^2.$$

It follows from similarity of triangles ZAU and ZVB that $\frac{AU}{BV} = \frac{ZU}{ZB}$, while similarity of triangles ZAV and ZUB implies $\frac{AV}{BU} = \frac{ZV}{ZB}$. In addition, $ZU \cdot ZV = ZC'^2$. Therefore,

$$\frac{AX}{BX}\frac{AY}{BY} = \left(\frac{ZC' \cdot BC'}{ZB \cdot AC'}\right)^2.$$

From similarity of triangles ZAC' and $ZC'B$ we obtain that the right side of this ratio equals 1. It means that $AX = BY$ and the point Z is the common point of the lines from the problem statement.

14. (A.Zaslavsky) In a trapezoid with bases AD and BC, points P and Q are midpoints of diagonals AC and BD respectively. Prove that if $\angle DAQ = \angle CAB$ then $\angle PBA = \angle DBC$.

    Solution one. Let L be the point of intersection of diagonals in a trapezoid. Applying sine theorem to triangles AQD, AQB, ALD, ALB we obtain that $BL/DL = (AB/AD)^2$. Therefore $BC/AB = AB/AD$ and $CL/AL = (BC/AB)^2$ which is equivalent to the problem statement.

    Solution two. Let L and M be the midpoints of AB and AD respectively. Then, as $PL \parallel AD$, $QM \parallel AB$, we have $\angle AQM = \angle QAB = \angle CAD = \angle APL$ and therefore the triangles APL and AMQ are similar (see the figure). It follows that $AP/AQ = AL/AM = AB/AD$. Hence the triangles ABP and ADQ are similar, i.e. $\angle ABP = \angle ADQ = \angle CBQ$.

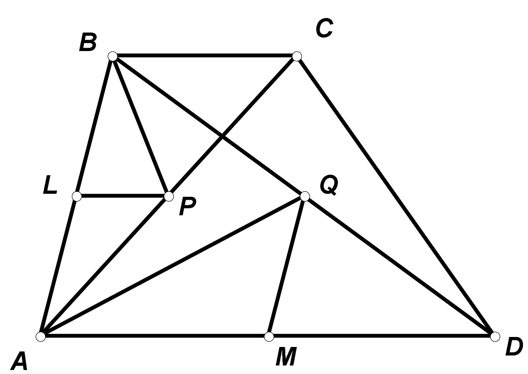

Fig. 12

15. (M.Volchkevich) Bisectors AA',BB' and CC' are drawn in the triangle ABC. Let $A'B' \cap CC' = P$ and $A'C' \cap BB' = Q$. Prove that $\angle PAC = \angle QAB$.

Solution. Applying sine theorem to triangles $AC'Q$ and $AA'Q$ we obtain (see the figure)

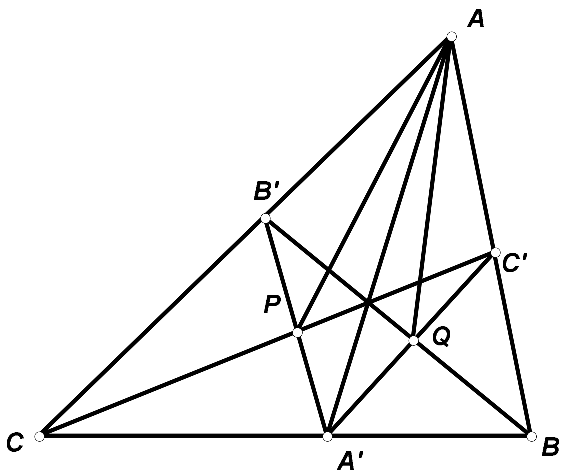

Fig. 13

$$\frac{\sin\angle C'AQ}{\sin\angle A'AQ} = \frac{C'Q}{A'Q}\frac{AA'}{AC'} = \frac{BA'}{BC'}\frac{AA'}{AC'}.$$

Similarly

$$\frac{\sin\angle B'AP}{\sin\angle A'AP} = \frac{CA'}{CB'}\frac{AA'}{AB'}.$$

According to Cheva theorem these ratios are equal, which is equivalent to the problem statement.

16. (V.Protasov) The points A, B are chosen on the sides of an angle. Two lines are drawn through midpoint M of the segment AB, one of which intersects the angle sides at points A_1, B_1, another one at points A_2, B_2. The lines A_1B_2 and A_2B_1 intersect AB at points P and Q. Prove that M is the midpoint of PQ.

Solution one. Let C be the vertex of the given angle. Considering central projections of the line AB on the line AC from points B_1, B_2 we obtain equality of the cross ratios (see the figure)

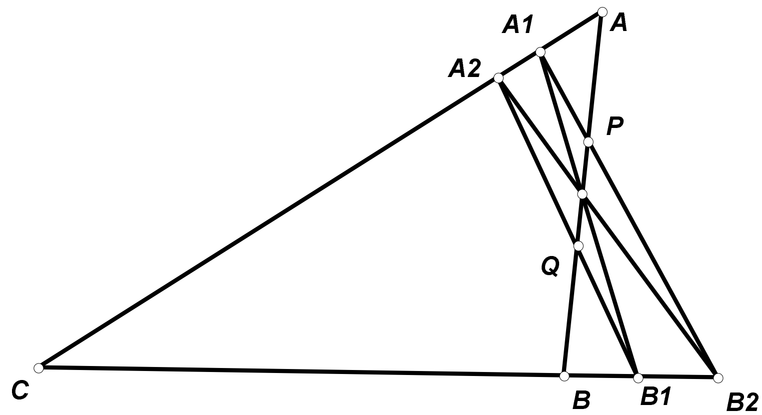

Fig. 14

$$(AP; MB) = (AA_1; A_2C) = (CA_2; A_1A) = (BQ; MA),$$

which is equivalent to the problem statement.

Solution two. Let us perform the central symmetry with respect to the point M. Let the points A_1 and B_2 map into the points A_1' and B_2' respectively. We need to prove that the lines AB, A_2B_1 and $B_2'A_1'$ concur. This follows from Desargues theorem applied to the triangles AA_2B_2' and BB_1A_1'. Since the intersection points of the lines AA_2 and BB_1, A_2B_2' and B_1A_1', AB_2' and BA_1' belong to the same line, the lines AB, A_2B_1 and $B_2'A_1'$ intersect in the same point.

Solution three. Let us perform the central symmetry with respect to the point M. Let the points A_1 and B_2 map into the points A_1' and B_2' respectively. Draw a line parallel to BC through M. It follows from similarity of triangles that $\frac{B_2'A_2}{A_2M} = \frac{2B_2'A}{BC}$, $\frac{MA_1'}{A_1'B_1} = \frac{BC}{2BB_1}$ and $\frac{B_1X}{XB_2'} = \frac{B_1B}{B_2'A}$, where X is a point of intersection of the lines AB and B_1B_2'. Having multiplied the equalities, we will obtain $\frac{B_2'A_2}{A_2M} \cdot \frac{MA_1'}{A_1'B_1} \cdot \frac{B_1X}{XB_2'} = 1$. Cheva theorem applied to the triangle MB_1B_2' implies that the lines MX, B_1A_2 and $B_2'A_1'$ concur as has been required.

17. (L.Yemelyanov) What triangles can be cut into three triangles with equal radii of circumcircles?

Answer. All, except for isosceles not acute-angled triangles.

Solution. If a triangle ABC is acute-angled then the radii of circumcircles for the triangles ABH, BCH and CAH, where H is the orthocenter, are equal.

Suppose $\angle C \geq 90^\circ$ and $AC > BC$. Let us choose a point D on the side AC such that $AD = BD$. Let us also choose a point E on the side AB such that $\angle AED = \angle C$ (this is possible because $\angle DBA = \angle A < \angle C$). By the sine theorem the radii of the circumcircles about the triangles ADE, BDE and BDC are equal (see the figure).

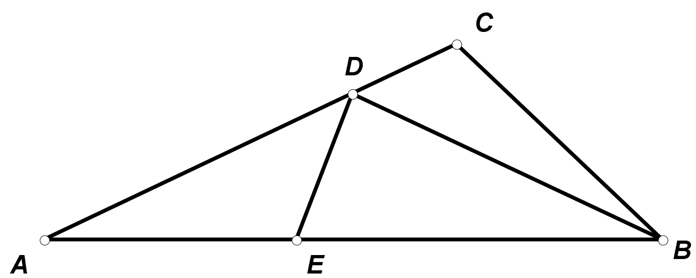

Fig. 15

Suppose $\angle C \geq 90^\circ$ and $AC = BC$. Let us show that the triangle ABC cannot be cut in the required manner. If the cutting is done from an internal point then the circumradii of the resulting triangles can only be equal if this point is the orthocenter, which is impossible. If, alternatively, the triangle is cut by a chevian into two triangles, then one of those which is further cut by a second chevian has to be isosceles. Therefore, the first cut needs to be done by segment CD, where $AD = AC$. But then for any cutting of the triangle ACD from the vertex A, the circumradii of the resulting triangles will be less then the circumradius of the triangle BCD (see the figure).

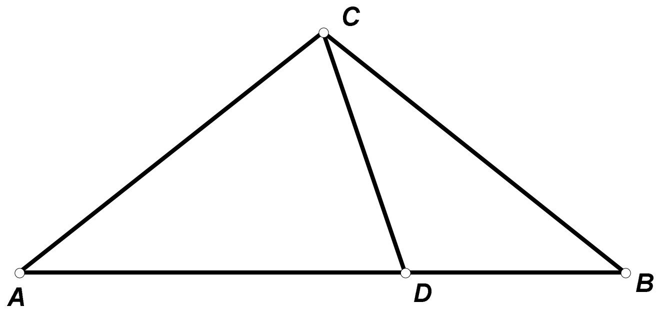

Fig. 16

18. (B.Frenkin) Find the locus of the vertices of the triangles with a given orthocenter and circumcenter.

    Solution. Let O be the circumcenter of the triangle ABC, H be the orthocenter, C_0 be the midpoint of the side AB. Then $\vec{C}H = 2\vec{O}C_0$, and since C_0 lies within the circumcircle, $CH < 2OC$. The points satisfying this condition lie outside the circle, diametrically opposite points of which are the point M (dividing the segment OH in ratio of $1 : 2$ and being the center of mass of the triangle) and the point M' symmetrical to H with respect to O. For such points C the required triangle is constructed in the following way: for the point C_0, take the image of C under homothety with the center M and the ratio of $-1/2$. Then draw a line through C_0 that is perpendicular to CH. Now find the points A, B which are the meet points of this line and the circle with center O and radius OC. However, this construction can lead to a degenerated triangle, the vertices A, B, C of which are collinear. This happens when $\angle OC_0C = \angle MCH = 90°$, i.e. the point C belongs to the circle with diameter MH. However point H is an exception. For it, the required triangle exists. It can be any right triangle with hypotenuse equal to the diameter of the circle with the center O and radius OH. Thereby the required locus is the outer area of the circle with diameter MM' excluding the circle with diameter MH but including the point H. So, the locus in question is the outer area of the circle with diameter of MM' excluding the circle with diameter MH but including the point H (see the figure).

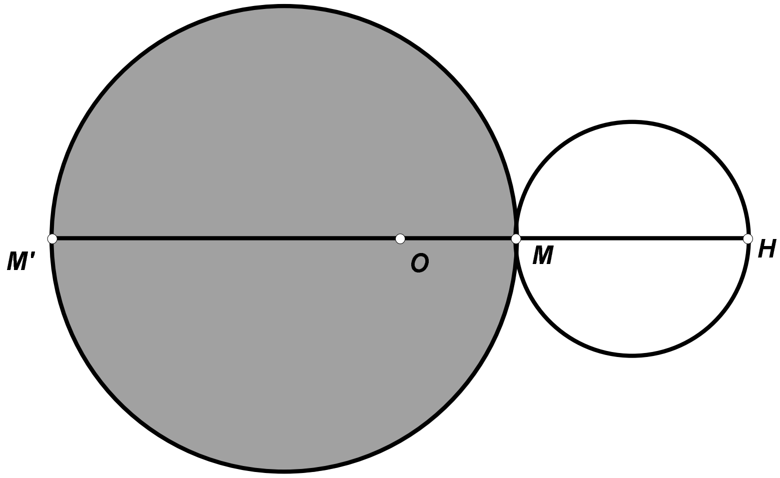

Fig. 17

19. (V.Protasov) Into angle A equal to α, a circle is inscribed that is tangent to its sides at points B and C. The line tangent to the circle at some point M intersects the segments AB and AC at points P and Q respectively. At what minimal α the inequality $S_{PAQ} < S_{BMC}$ is possible?

Solution. Denote by I the incenter of the triangle PAQ. We have $\angle MCB = \angle MBP = \frac{1}{2}\angle MPA = \angle IPQ$. Similarly, $\angle MBC = \angle IQP$. Therefore, $\triangle IPQ \sim \triangle MCB$, hence $\frac{S_{IPQ}}{S_{MCB}} = \left(\frac{x}{a}\right)^2$, where $x = PQ$, $a = CB$. On the other hand, the ratio $\frac{S_{PAQ}}{S_{IPQ}}$ equals the ratio of the perimeter of the triangle PAQ to the side PQ, i.e., equals $\frac{2b}{x}$, where $b = AB = AC$ (because the perimeter of the triangle PAQ equals $2b$). By multiplying the two ratios we obtain $\frac{S_{PAQ}}{S_{MCB}} = \frac{2bx}{a^2}$. The ratio $\frac{S_{PAQ}}{S_{MCB}}$ is minimal when the length of segment $x = PQ$ is minimal. Let us show that the minimal length of PQ is attained when $\triangle APQ$ is isosceles. Let O be the center of the circle inscribed into $\angle A$; R be its radius; M be the midpoint of the arc BC; M' be another point on this arc; PQ and $P'Q'$ be the respective tangent segments between the sides of the angle. Assume $\beta = \angle BOP (= \angle POM = \angle MOQ = \angle COQ), \gamma = \angle BOP' (= \angle P'OM'), \delta = \angle COQ' (= \angle Q'OM')$. Then $PQ = PM + MQ = 2R\mathrm{tg}\beta$, $P'Q' = P'M' + M'Q' = R(\mathrm{tg}\gamma + \mathrm{tg}\delta)$. As $2\beta = \gamma + \delta$ and the tangent function is convex downwards at the interval $(0;\pi/2)$, we have $PQ < P'Q'$ as required.

Thus, it suffices to consider the case when $\triangle APQ$ is isosceles. Then $a = 2b\sin\frac{\alpha}{2}$; $AM = AO - OM = b\left(\frac{1}{\cos\frac{\alpha}{2}} - \mathrm{tg}\frac{\alpha}{2}\right)$; $PQ = 2AM\mathrm{tg}\frac{\alpha}{2} =$

$2b\frac{\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}$. Substituting the resulting expressions in the ineqality $\frac{2bx}{a^2} < 1$, we obtain:

$$\alpha > 2\arcsin\frac{\sqrt{5}-1}{2}.$$

20. (D.Schnoll) The base of the pyramid is a regular triangle with side 1. Two of three angles at the apex of the pyramid are right. Find the maximal volume of the pyramid.

    Solution. Let ABC be the base of the pyramid, while the sides AC, BC are visible from the apex at right angles. Then S belongs to the intersection of spheres with diameters AC and BC, i.e. to the circle lying in the plane perpendicular to the base, with diameter CD, where D is the midpoint of AB. The maximal volume is attained when S is the point of this circle, the most remote from the plane ABC. Then the altitude of the pyramid equals $CD/2$, whereas its volume is $1/16$.

21. (N.Dolbilin) There are three pipes on the plane (equal circular cylinders, 4 meters in circumference). Two of them are lying parallel, tangent to each other at common generatrix, and thus creating a tunnel above the plane. The third one, perpendicular to the first two, cuts out a chamber in the tunnel. Find the area of the border surface of this chamber.

    Answer. $8/\pi$.

    Solution one. The horizontal cross-sections of the chamber are rectangles with perimeters equal to doubled diameter of the pipes. For each of these rectangles the angle between its plane and the tangent to the chamber surface is the same for all points. The midpoints of sides of these rectangles form quarters of pipes' circles when the cross-section moves. Hence the surface area of the chamber is equal to the surface area of a tetrahedron, the sides of which are isosceles triangles with the base equal to the diameter of the pipe and the altitude equal to the quarter of the circumference of the pipe.

    Solution two. Let us call the tangent cylinders lengthwise, and the one perpendicular to them — lateral. It is obvious that the plane, tangent to both cylinders along their common generatrix and the vertical plane passing through the axis of the lateral cylinder, are the symmetry planes of the chamber that split it into four equal parts. Let us consider one of these quarters. Its surface contains two parts: the part of the surface of the lengthwise cylinder lying within the half of the lateral cylinder, and the part of the surface of the lateral cylinder lying between the lengthwise one and the vertical plane tangent to it. The line of cylinders' intersection is an ellipse which lies in a vertical plane. The cylinders map one into another under the symmetry about this plane. Under this symmetry the image of the part of chamber's surface lying on the lengthwise cylinder supplements the part of chamberвЂ™s surface lying on the lateral cylinder, up to a curvilinear rectangle. Its sides are equal to half of the diameter and the quarter of the circumference of the cylinder. Accordingly, the area of the chamber's surface equals the quadrupled area of such rectangle.