Second olympiad, year 2006
Correspondence round. Solutions

1. (V.Smirnov) Two lines in the plane, intersecting at an angle of $46°$, serve as symmetry axes for a geometric figure F. What is the minimal number of symmetry axes of this figure?

   Solution. Answer: 90.

   Let l_1, l_2 be the axes of symmetry for F. By applying consecutively the symmetry about l_1, then the symmetry about l_2 and then again the symmetry about l_1, we obtain the symmetry with respect to the line symmetrical to l_2 about the line l_1 . Therefore, the axes of symmetry for F will be all the lines which form angles of $46°, 2 \cdot 46°, \dots, n \cdot 46°, \dots$ with l_1. Since $46n$ is not divisible by 180 when $n < 90$, these lines will be distinct for $n = 1, \dots, 90$. I.e. the figure F has at least 90 axes of symmetry. On the other hand, a regular 90-gon satisfies the problem condition and has precisely 90 symmetry axes.

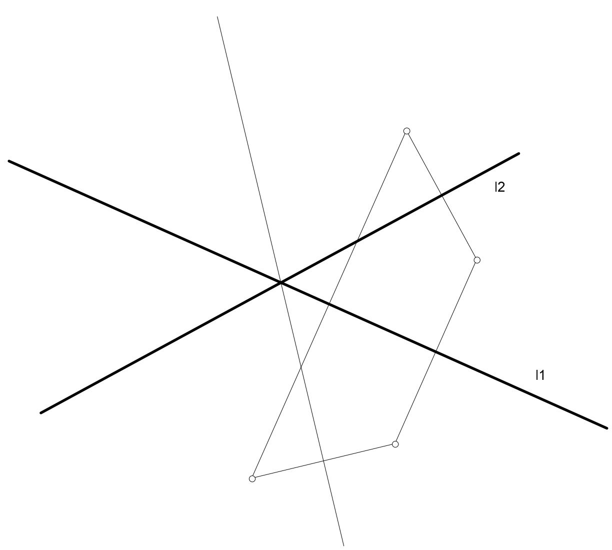

Fig. 1

2. (A.Akopyan) Points A and B move at equal speeds along equal circles. Prove that the perpendicular bisectors to AB concur at a fixed point.

   Solution. (Found by Nikita Bakanchev, 9-grader of Gymnasium 1543 in Moscow). Let l be the line of symmetry that maps the circles in question into each other. Let A' be the point symmetrical to A with respect to l. Then the points B and A' move along the same circle with opposite speeds, hence the perpendicular bisector to the segment $A'B$ does not change. The point of its intersection with l will be the center of the circumcircle of the triangle $AA'B$, therefore the perpendicular bisector to AB always passes through this point.

3. (Folklore) There is a map with segments of straight linear roads linking three villages. The villages themselves are beyond the boundaries of the map. Furthermore the fire station equidistant from the three villages is located inside the boundaries of the map but not indicated on the map. Can its location be determined using compass and ruler if the constructions are to be made within the map only?

   Solution. Choose an arbitrary point P on the map. A homothety centered in P with sufficiently small factor k maps the meet points of the roads into certain points A, B, C within the map. Therefore the circumcenter O of the triangle ABC can be found. Homothety with center P and factor $1/k$ maps the point O into the point in question.

4. (A.Gorskaya, I.Bogdanov) a) Two squares $ABCD$ and $DEFG$ are given, point E belongs to segment CD, while points F, G lie outside the square $ABCD$. Find the angle between the lines AE and BF.

   b) Two regular pentagons $OKLMN$ and $OPRST$ are given, where the point P belongs to segment ON, while the points R, S, T lie outside the pentagon $OKLMN$. Find the angle between the lines KP and MS.

   Solution. a) Let H be the second point of intersection between circumcircles of the squares (Fig. 4.1). Since $\angle AHD = 45°$, $\angle DHF = 90°$, $\angle EHF = 135°$, the points A, E, H are collinear. Similarly the points B, H, F are also collinear. Therefore, the angle in question equals $\angle BHA = 45°$. ==

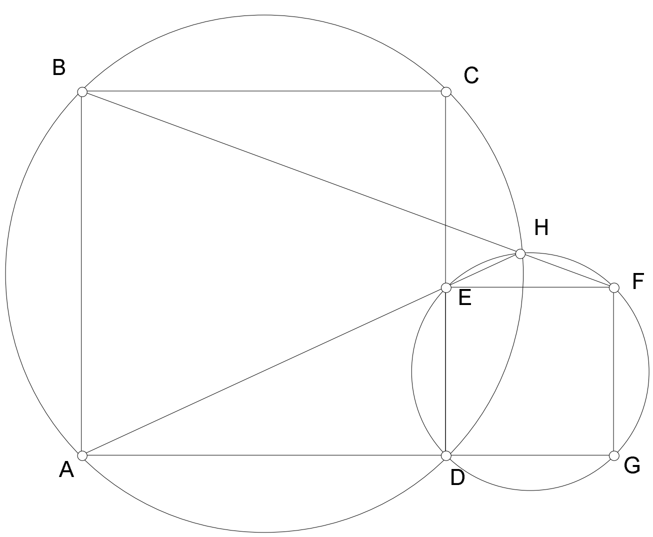

Fig. 4.1

   b) Answer: 72°. Solution is analogous to part a).

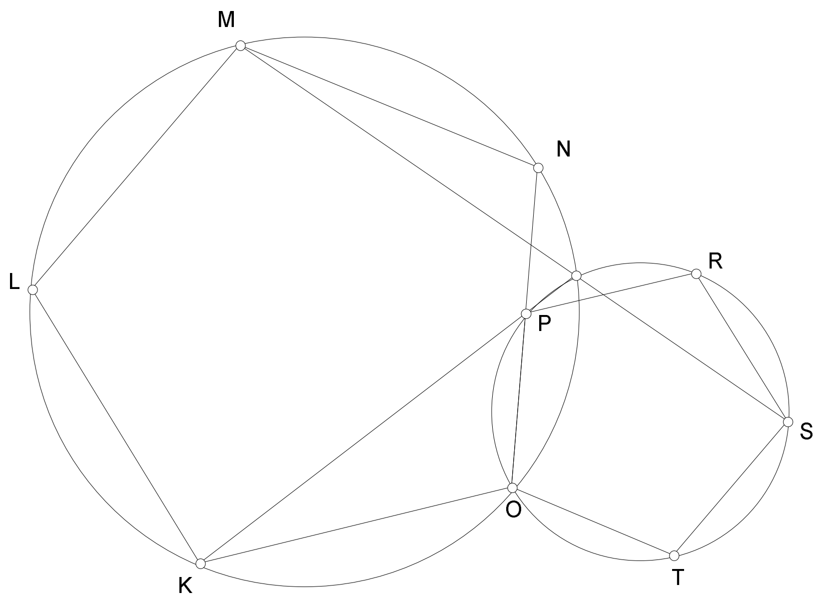

Fig. 4.2

5. (A.Tarasov) a) Fold a square 10×10 from a rectangular stripe 1×118.

   b) Fold a square 10×10 from a rectangular stripe $1 \times (100 + 9\sqrt{3})$ (approximately 1x115.58).

   In each case the stripe can be folded but cannot be ripped into parts.

   Solution. a) Let points A, B lie on the opposite sides of a stripe at the distance of 10 from the edge, while points C, D lie at the distance of 12. Folding the stripe along the bending lines as shown on Fig. 5.1, place its part lying to the right of CD next to its part lying to the left of AB. Repeating this folding procedure 9 times and then folding the resulting right triangles we will get the square in question.

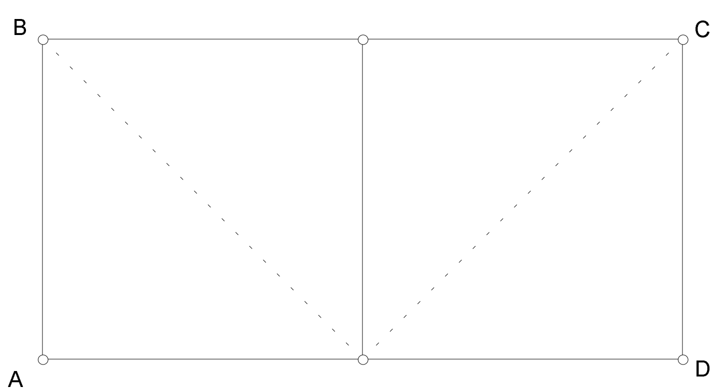

Fig. 5.1

b) Let points A, B lie on the opposite sides of the stripe at the distance of 10 from the edge, and let points C, D lie at the distance of $10+\sqrt{3}$. By folding the stripe along the bending lines as shown on Fig. 5.2, place its part lying to the right of CD next to its part lying to the left of AB. By repeating this folding procedure 9 times and then folding the resulting right triangles we will get the square in question.

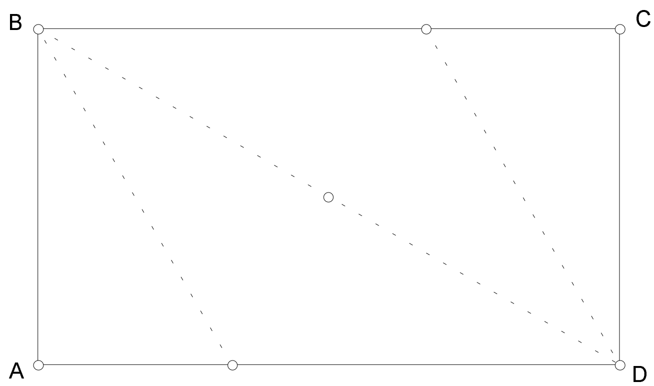

Fig. 5.2

6. (A.Afanasyev) a) (8-9) Given a segment AB with point C inside it. The segment is a chord of the circle with radius R. Inscribe a circle into the formed circle segment such that this circle contains point C and is tangent to the initial circle.

   b) (9-10) Given a segment AB with point C inside it, which is the tangency point of this segment with the circle of radius r. Draw a circle through points A and B such that it is tangent to the initial circle.

   Solution. Firstly let us prove the following fact.

   Lemma. Let the circle inscribed in the segment limited by the circle arc and chord AB, be tangent to the arc at point C and tangent to the chord at point D. Then CD is the bisector of angle ACB.

   Proof. Let O be the center of the major circle, whereas O' be the center of the minor one. Let L be the midpoint of the circle arc AB, to which the point C does not belong (Fig. 6). Since O' belongs to segment OC, while $O'D \parallel OL$, the isosceles triangles $O'DC$ and OLC are similar. Therefore D belongs to the segment CL while the line CD bisects the angle ACB.

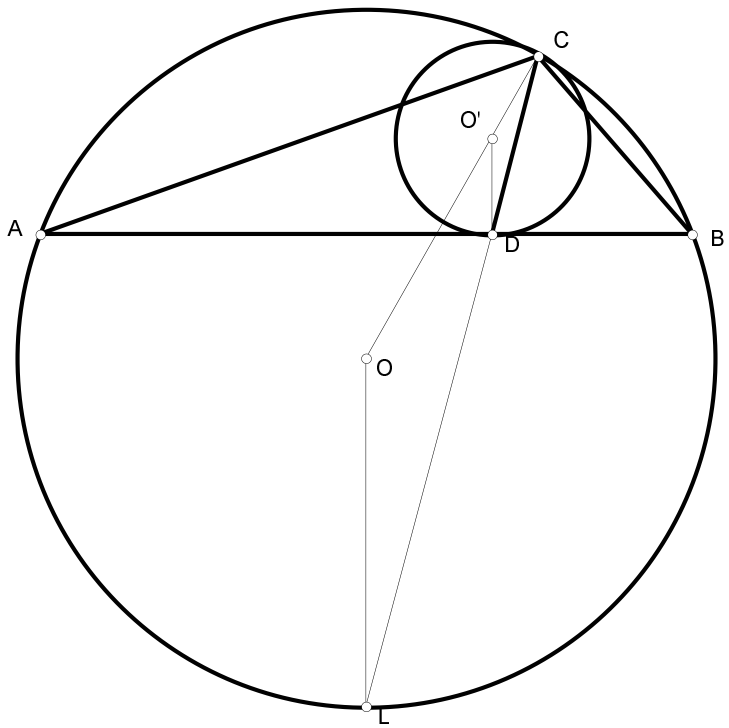

Fig. 6

We now turn to the solution of the problem.

a) Let the circle in question be tangent to the given circle at point X. The lemma implies $AX/XB = AC/CB$. The set of the points satisfying this condition, is a circle with the center on the line AB (it is known as *Apollonius circle* for the points A and B). Let us take any of the intersection points of the given circle with Apollonius circle and link it with the center of the given circle. Then find the intersection point of this line with the perpendicular dropped from C to AB. We obtain the center of the circle in question. The problem has two solutions as the circle can be inscribed to either of the two segments in which chord AB splits the given disk.

b) Similarly to section a), let us draw the Apollonius circle and find its meet point with the given circle other than point C. The circle in question passes through this point as well as through points A, B. The problem has a single solution.

7. (D.Kalinin) Given are a point E inside the square $ABCD$ and a point F outside it, so that triangles ABE and BCF are equal. Find the angles of the triangle ABE if it is known that the segment EF is equal to the side of the square, while the angle BFD is right.

   Solution. Since the angle BFD is right, the point F lies on the circumcircle of the square, i.e. $\angle BFC = 135° = \angle AEB$ (because two other angles in the triangle AEB are evidently acute). As $\angle ABE = \angle CBF$[1], $\angle EBF = 90°$ and $BE/EF = \frac{1}{\sqrt{2}} = BE/AB$. Applying theorem of sines to the triangle ABE we get $\sin\angle EAB = BE\sin\angle AEB/AB = 1/2$. Therefore $\angle EAB = 30°$, $\angle EBA = 15°$ (Fig. 7).

[1]it is not hard to ensure that the case $\angle ABE = \angle BCF$ is impossible.

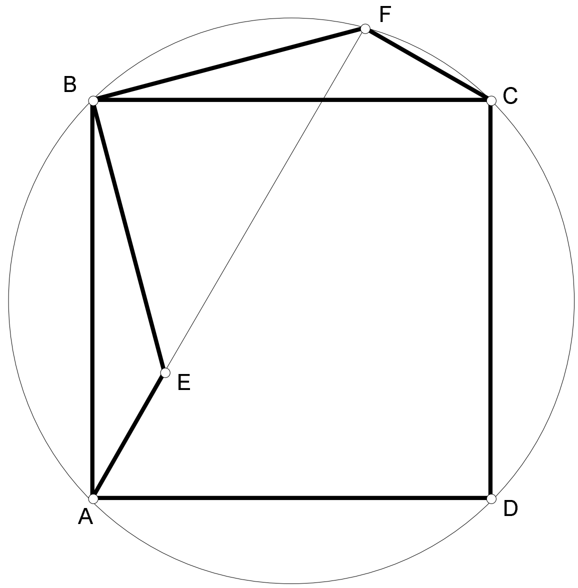

Fig. 7

8. (A.Blinkov) The segment AB splits the square into two equal parts. A circle can be inscribed into each of them. The radii of the circles equal r_1 and r_2 where $r_1 > r_2$. Find the length of AB.

Solution. If the segment AB is a diagonal of the square then it splits the square into two equal triangles, and $r_1 = r_2$, which contradicts the problem condition. If, on the other hand, one of the parts is a quadrilateral, then the sum of its side AB with the opposite side is greater than the sum of two other sides (Fig. 8.1), and therefore a circle cannot be inscribed into it. Hence AB splits the square into a triangle and a pentagon (Fig. 8.2).

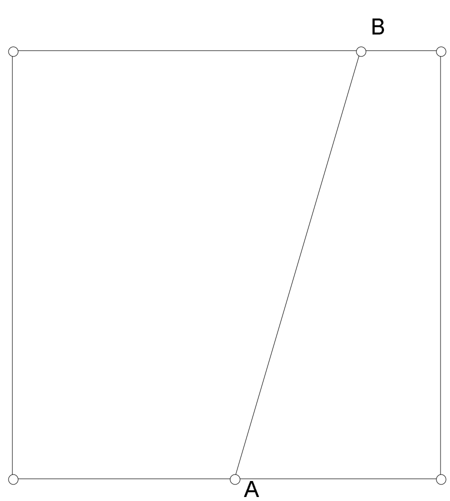

Fig. 8.1

The circles with radii r_1, r_2 are the excircle and the incircle of the right triangle ABC. So, $r_1 = (AB + BC + CA)/2$, $r_2 = (BC + CA - AB)/2$, and $AB = r_1 - r_2$.

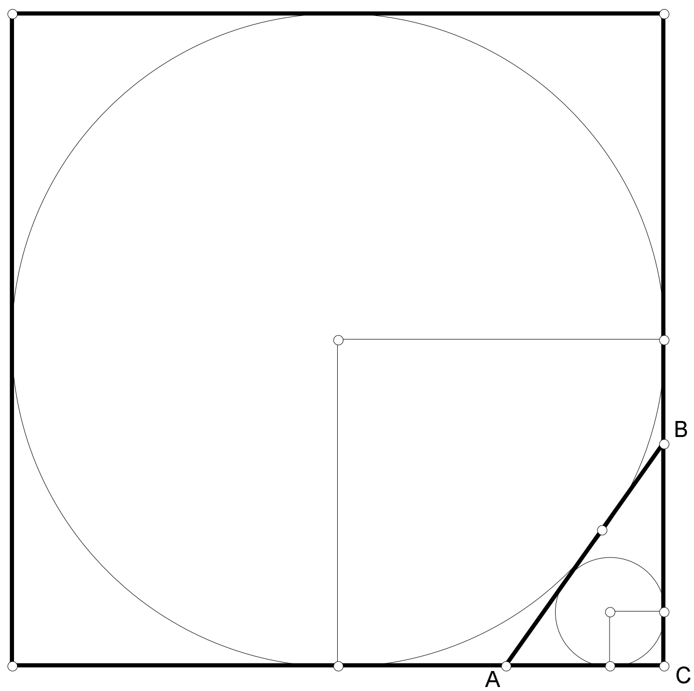

Fig. 8.2

9. (A.Kanel) Let the line $L(\alpha)$ link the points of the unit circle, corresponding to the angles α and $\pi - 2\alpha$. Prove that if $\alpha + \beta + \gamma = 2\pi$, then the lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are concurrent.

   Solution. Let A, B, C be the concyclical points corresponding to angles α, β, γ. The perpendicular from the circle center to the line AB crosses the circle at the point corresponding to angle $(\alpha+\beta)/2 = \pi - \gamma/2$, whereas the perpendicular to the line $L(\gamma)$ crosses the circle at the point corresponding to the angle $(\gamma + \pi - 2\gamma)/2 = \pi/2 - \gamma/2$. Therefore $L(\gamma)$ is an altitude of the triangle ABC. Similarly $L(\alpha)$, $L(\beta)$ are altitudes of ABC, and therefore all the three lines intersect at its orthocenter.

10. (B.Frenkin) For what n a regular n-gon can be split by non-intersecting diagonals into isosceles (and, possibly, equilateral) triangles?

   Solution. Answer: n has to be the sum of two powers of two, possibly equal (in which case n itself is a power of two).

   Let us consider the triangle ABC from the decomposition, containing the center (Fig. 10.1). If the side AB is not a side of the initial polygon, then it splits a polygon, where it is the greatest distance between its vertices. Therefore AB has to be the base of a decomposition triangle whereas the number of sides that it separates must be even. For the lateral sides of the above triangle similar reasoning can be provided, therefore the number of sides being cut off by AB is a power of two. (If AB is a side of the initial polygon, then it cuts off 2^0 sides.) This is valid for sides BC and AC too. Since at least two sides in the triangle ABC are equal, we have $n = 2^k + 2^k + 2^l = 2^{k+1} + 2^l$.

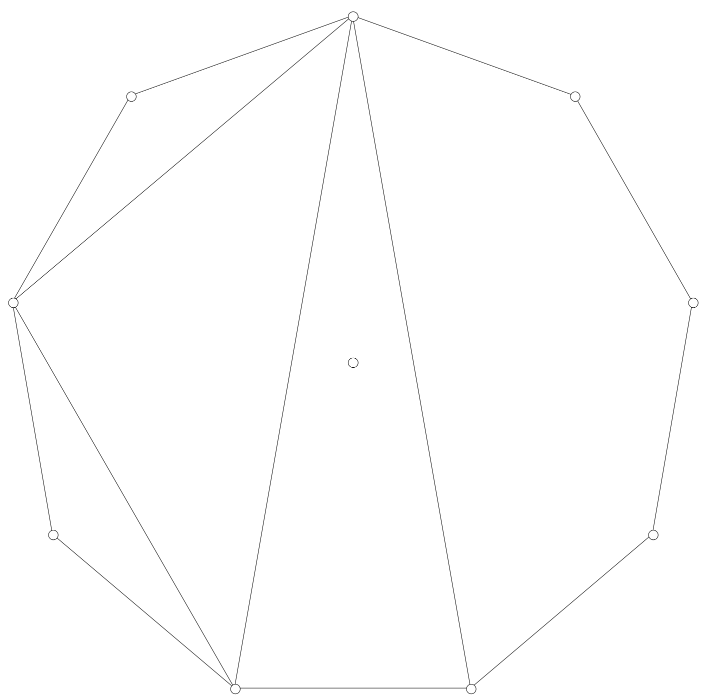

Fig. 10.1

Conversely, suppose $n = 2^k + 2^l$, where $k > 0$. Suppose A is one of the vertices of a regular n-gon, while vertices B and C lie in 2^{k-1} sides from it in two directions. Then $AB = AC$ and there exists a decomposition of the desired type containing $\triangle ABC$ (see Fig. 10.2).

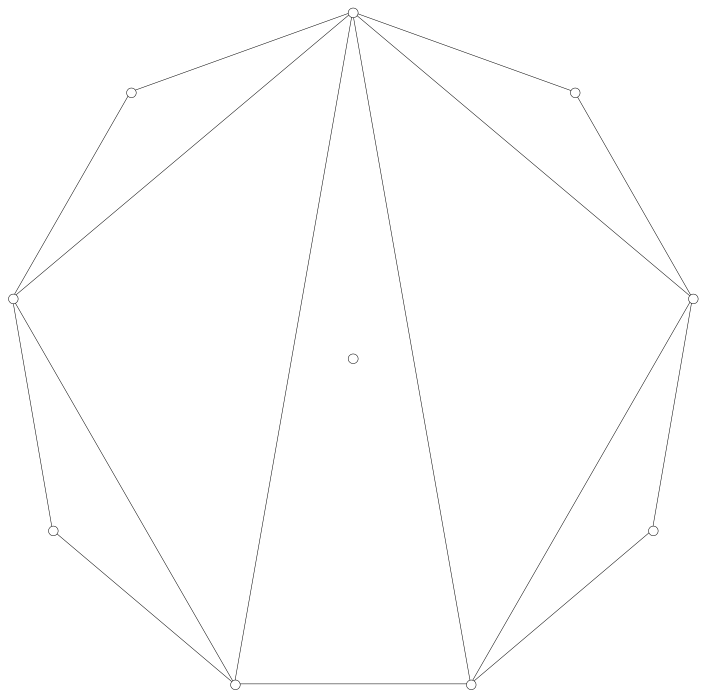

Fig. 10.2

11. (A.Zaslavsky) Let point O be the center for the circumcircle of the triangle ABC; let A', B', C' be the points symmetrical to A, B, C about respective opposite sides; let A_1, B_1, C_1 be the points of intersection between the lines OA' and BC, OB' and AC, OC' and AB. Prove that the lines AA_1, BB_1, CC_1 intersect at the same point.

    Solution. Let O_a, O_b, O_c be the points symmetrical to O about BC, CA and AB respectively. Evidently, the lines CO_c, OC' and AB intersect at the same point, so in order to solve the problem it suffices to prove that the lines AO_a, BO_b and CO_c intersect at the same point.

    Since the triangle $O_aO_bO_c$ is homothetic to median triangle ABC with the center in O and the factor 2, it is centrally symmetrical to the triangle ABC. So the lines linking the respective vertices of these triangles contain the symmetry center. It is also easy to ensure that this point serves as the center of 9 points circle for each of the triangles.

12. (B.Frenkin) In the triangle ABC the bisector of the angle A equals the half-sum of its median and altitude dropped from the vertex A. Prove that if $\angle A$ is obtuse then $AB = AC$.

    Solution. Let us assume that the problem statement is wrong. Let H, L, M be the bases of the altitude, the bisector and the median, while P be the midpoint of the arc BC (not containing point A) of the circumcircle of the triangle (Fig. 12).

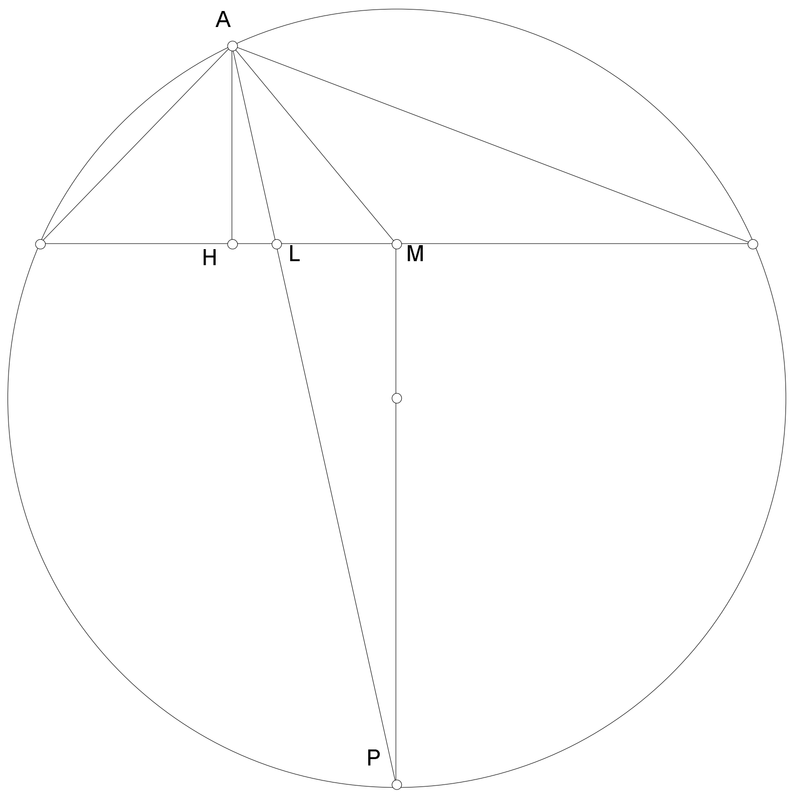

Fig. 12

If the angle A is obtuse, then $PM > AH$. As L belongs to the segment AP, it follows that $HL < LM$ and therefore the segment AL is less than the median of the triangle AHM, which is in turn less than the half-sum of the sides AH and AM. A contradiction.

13. (A.Akopyan) Consider two lines a and b, as well as two points A and B. The point X slides along a, whereas the point Y slides along b, so that $AX \parallel BY$. Find the locus of intersections between AY and XB.

    Solution. Let us draw a line through A parallel to b and crossing a at the point U. Similarly let us draw a line through B parallel to a and crossing b at the point V. For any points X, Y satisfying the condition, the respective sides of the triangles AUX and YUB are parallel. So these triangles are homothetic, i.e., the lines AY, BX and UV intersect at the homothety center. Evidently, one could obtain any point of the line UV in such a way.

14. (A.Zaslavsky) Consider a circle and a fixed point P not belonging to it. Find the locus for orthocenters of triangles ABP, where AB is the circle diameter.

    Solution. Let C be the orthocenter; let A', B', C' be the bases of altitudes in ABP dropped from A, B, P; let P' be the projection of C to the line OP (Fig. 14). Since $\angle CC'O = \angle CP'O = 90°$, the points O, C, C', P' belong to the circle and $CP \cdot PC' = OP \cdot PP'$. Similarly $CP \cdot PC' = BP \cdot PA'$. However, A' lies on the initial circle, therefore $BP \cdot A'P = |R^2 - OP^2|$. Thereby, the product $OP \cdot PP'$ And hence the point P' do not depend on the choice of diameter AB, i.e. the locus in question is the line perpendicular to OP and passing through the point P'.

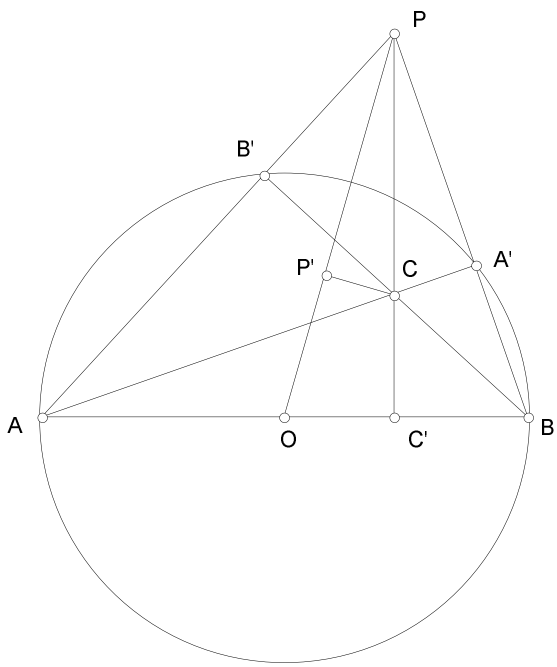

Fig. 14

15. (V.Protasov) For a triangle ABC, consider the circumcircle and the incircle, the latter of which touches its sides BC, CA, AB at points A_1, B_1, C_1 respectively. The line B_1C_1 crosses the line BC at the point P, whereas the point M is the midpoint of the segment PA_1. Prove that the segments of the tangents from the point M to the incircle and to the circumcircle are equal.

    Solution. Assume $AB < AC$. As soon as the lines AA_1, BB_1 and CC_1 concur, by Ceva and Menelaus theorems we obtain that $PB/PC = A_1B/AC$. Moreover, $MB = (PB - A_1B)/2$, $MC = (PC + A_1C)/2$, $MA_1 = (PB + A_1B)/2 = (PC - A_1C)/2$. So, $MB/MA_1 = MA_1/MC = A_1B/A_1C$, which is equivalent to the problem's statement.

16. (P.Pushkar') Sides of the triangle ABC are bases for regular triangles drawn outside it. Their outlying vertices form a regular triangle. Is that true that the initial triangle is regular?

    Solution. Answer: yes, it is true. Let us assume the contrary. Then one of the angles in ABC, for instance the angle A, exceeds $60°$. Then the ray $B'C'$ lies outside of the angle $AB'C$, and since $\angle A'B'C' = \angle AB'C = 60°$, the ray $B'A'$ lies within this angle. Therefore the ray $A'B'$ lies within the angle $B'AC'$ (Fig. 16). Similarly the ray $A'C'$ lies within this angle, which contradicts the equation $\angle B'A'C' = \angle BA'C = 60°$.

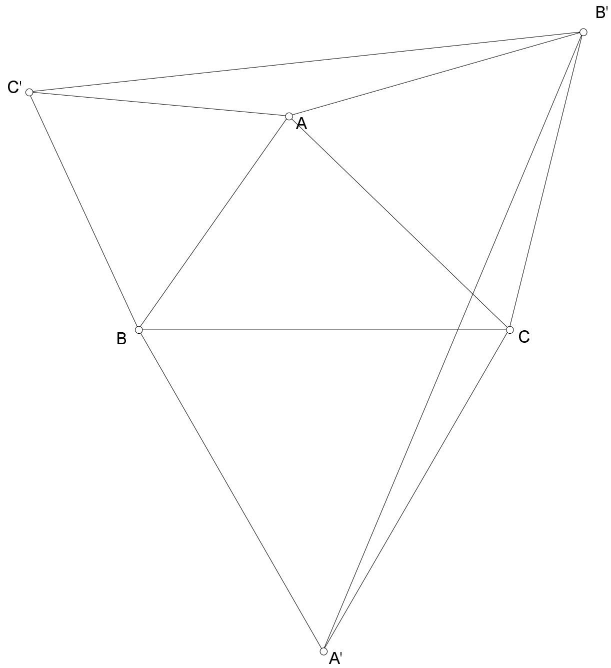

Fig. 16

17. (A.Zaslavsky) In two circles intersecting at points A and B, two parallel chords A_1B_1 and A_2B_2 are drawn. The lines AA_1 and BB_2 meet at the point X, while the lines AA_2 and BB_1 meet at the point Y. Prove that $XY \parallel A_1B_1$.

    Solution. The problem statement is equivalent to the fact that the points A, B, X, Y belong to the same circle, i.e., $\angle XAY = \angle XBY$. However $\angle XAY = \angle BAA_2 - \angle BAX = \angle BAA_2 - \angle BB_1A_1$, $\angle XBY = \angle B_2BA - \angle AA_1B_1$. At the same time, the fact that A_1B_1 and A_2B_2 are parallel implies $\angle ABB_1 + \angle A_1B_1B = \angle BAA_2 + \angle B_2A_2A$, and this evidently implies the required statement.

18. (A.Akopyan) Two perpendicular lines are drawn through the orthocenter H of the triangle ABC. One of them intersects BC at point X, another one intersects AC in point Y. The lines AZ, BZ are parallel to the lines HX and HY respectively. Prove that the points X, Y, Z are collinear.

    Solution. For definiteness, consider the case shown on Fig. 18. Let U be the point of intersection between HX and BZ, and V be the point of intersection between HY and AZ. Then the problem statement is equivalent to $HU/UX = YV/HV$ or $HU/YV = HV/XU$. In the right triangles AYV and BUH the angles AYV and BUH are equal because their sides are perpendicular. Therefore the triangles are similar and $HU/YV = BU/AV$. Similarly $HV/XU = BU/AV$. The other cases are considered in the same way.

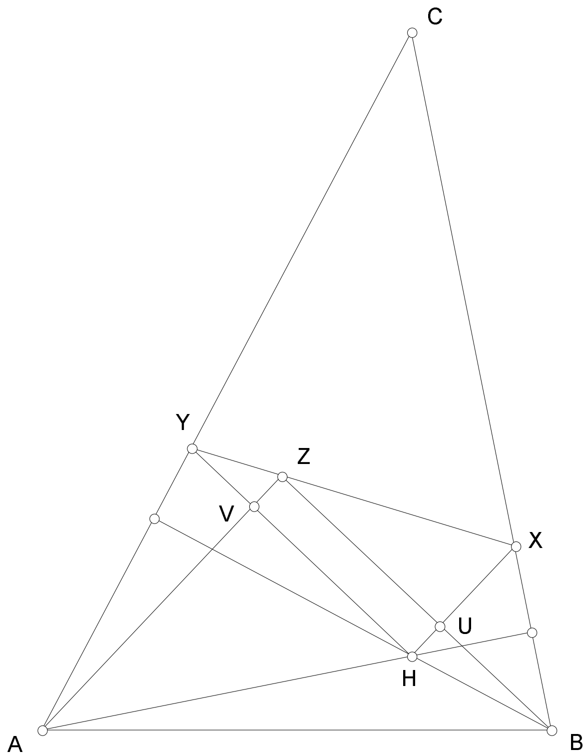

Fig. 18

19. (L.Yemelyanov) Through the midpoints of the sides of triangle T, the lines perpendicular to the bisectors of the opposite angles are drawn. These lines form the triangle T_1. Prove that its circumcenter is the midpoint of the segment linking the incenter and the orthocenter of T.

Solution. The sides of the triangle T_1 are external bisectors of angles in the triangle T_0 formed by the midlines of T. Hence they cross at its excenters. Furthermore the bisectors of internal angles of T_0 serve as altitudes in T_1, i.e. its incenter I_0 coincides with the orthocenter of T_1, whereas the circumcenter O_0 is the center of the circle passing through the midpoints of sides in T_1, and therefore it is the midpoint of segment I_0O_1, where O_1 is the circumcenter of T_1. Moreover, O_0 is the midpoint of the segment OH, where O, H are the circumcenter and the orthocenter of T, while the center of mass M of T splits the segment HO in the ratio of $2:1$ (Fig. 19). The homothety with the center I_0 and the factor $\frac{1}{3}$ maps the incenter I of T into M, while the homothety with center O_0 and ratio of -3 maps M to H. Since the composition of these homotheties is the central symmetry with center O_1, point O_1 is the midpoint of IH.

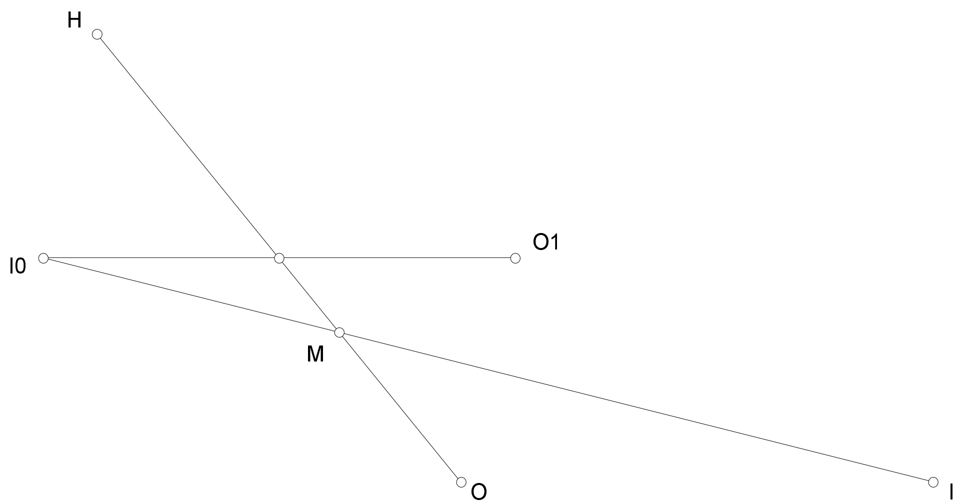

Fig. 19

20. (A.Zaslavsky) In the plane, consider four points A, B, C, D. The points A_1, B_1, C_1, D_1 are the orthocenters of triangles BCD, CDA, DAB, ABC respectively. The points A_2, B_2, C_2, D_2 are the orthocenters of triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$, and so on. Prove that all the circles containing the midpoints of sides of these triangles, intersect in the same point.

    Solution. First, let us prove that the circles containing the midpoints of the sides of triangles ABC, BCD, CDA and DAB intersect at the same point. Let X be the point of intersection for the nine point circles of triangles ACD and BCD, that is different from the midpoint of AB. Let Y, Z, U be the midpoints of AC, BC, CD. Then $\angle YXZ = \angle YXU + \angle XUZ = \angle DCA + \angle BDC = \angle BCD$, i.e. X belongs to the nine point circle of the triangle ABC. Similarly X also belongs to the nine point circle of the triangle ABD. Now, as the nine point circles of the triangles CDA and ACB_1 coincide, the point X also belongs to the nine point circles of the triangles ABB_1 and CBB_1. Similarly it belongs to the nine point circles of the triangles ABA_1 and BCC_1. Hence it also belongs to the nine point circles of the triangles A_1B_1B, BB_1C_1 and $A_1B_1C_1$. This implies the problem statement.

    A shorter solution is based on the following fact.

    Let the points U, V, W belong to an equilateral hyperbola. Then the orthocenter of the triangle UVW also belongs to this hyperbola, and its nine point circle passes through the center of the hyperbola.

    In fact, drawing an equilateral hyperbola through the points A, B, C, D we obtain that all the circles pass through its center.

21. (A.Zaslavsky) Consider points C', A', B' on the sides AB, BC, CA of the triangle ABC. Prove that the following inequality holds for areas of respective triangles:

$$S_{ABC}S^2_{A'B'C'} \geq 4S_{AB'C'}S_{BC'A'}S_{CA'B'};$$

    moreover, the equality holds only if the lines AA', BB', CC' concur.

    Solution. Denote $P_1 = AB' \cdot BC' \cdot CA'$, $P_2 = BA' \cdot AC' \cdot CB'$. It is easily seen that $S_{A'B'C'} = (P_1 + P_2)/4R$, where R is the circumradius of ABC, and therefore

$$\frac{S_{AB'C'}S_{BC'A'}S_{CA'B'}}{S_{ABC}S^2_{A'B'C'}} = \frac{P_1P_2}{(P_1+P_2)^2} \leq \frac{1}{4}.$$

Moreover, the equality holds only if $P_1 = P_2$, which is equivalent to the fact that the lines AA', BB' and CC' are concurrent.

22. (A.Zaslavsky) Consider a circle and points A and B on it, as well as a point P in the plane. Let X be an arbitrary point of the circle, Y be the commom point of lines AX and BP. Find the locus of circumcenters of the triangles PXY.

    Solution. Let Q be the point of intersection of the circles ABX and PXY distinct from X. Then $\angle ABQ = \angle AXQ = \angle YXQ = \angle YPQ = \angle BPQ$. So $\angle BQP = \pi - (\angle BPQ + \angle QBP) = \pi - \angle ABP$ does not depend on the choice of the point X. Therefore, all the circles PXY pass through Q and their centers belong to the perpendicular bisector of PQ.

23. (A.Myakishev) Consider a convex quadrilateral $ABCD$, and let G be its center of mass as of a uniform plate (i.e., the point of intersection of two lines each of which links centroids of triangles sharing the same diagonal).

    a) (9-10) Suppose a circle can be circumscribed about $ABCD$, a point O being its center. Let us define point H similarly to point G by taking orthocenters instead of centroids. Prove that the points H, G, O are collinear, and $HG : GO = 2 : 1$.

    b) (10-11) Suppose a circle with center at point I is inscribed to $ABCD$. Let Nagel point N of a circumscribed quadrilateral denote the intersection of two lines, each of which links the points on the opposite sides of the quadrilateral, symmetrical to the tangent points of the incircle about midpoints of the sides. (These lines split perimeter of the quadrilateral in two equal parts). Prove that N, G, I are collinear, whereby $NG : GI = 2 : 1$.

    Solution. a) Let M_a and H_a be the centroid and the orthocenter of triangle BCD respectively. Similarly denote the centroids and orthocenters of the remaining three triangles. All of the triangles have a common circumcircle with center at O. Consideration of Euler's lines of these triangles whows that the quadrilateral $M_aM_bM_cM_d$ maps into quadrilateral $H_aH_bH_cH_d$ under the homothety with center O and factor 3. Therefore, the points of diagonals' intersection of these quadrilaterals map into each other.

    b) Let us denote by M_1 the center of mass of the contour of the quadrilateral. The point G belongs to the segment IM_1 and splits it in ratio of $2 : 1$. In fact, M_1 is the center of mass of four points, placed at the midpoints of quadrilateral sides with masses proportional to their lengths, while G is the center of mass of four points, placed at the centers of mass of the triangles IAB, IBC,ICD, IDA with masses proportional to areas of these triangles. Obviously, these two systems of points are homothetic with center I and factor $\frac{2}{3}$.

    Let a, b, c, d be the lengths of tangent segments to the incircle from the vertices A, B, C, D. It is evident that if the masses a, b, c, d are placed to A, B, C, D, then the mass center of the resulting system is at the point N. If alternatively we place masses $2a + b + d$, $2b + a + c$, $2c + b + d$, $2d + c + a$ to the vertices then the mass center is the point M_1. It remains to show that I is the mass center for masses $b+d$, $a + c$, $b + d$, $a + c$.

    The point I satisfies the equation $S_{IAB} - S_{IBC} + S_{ICD} - S_{IDA} = 0$. The same equation holds for the midpoints U and V of diagonals of the quadrilateral. Therefore, these three points are collinear (these statement is known as *Monge theorem*). Now let X, Y be the tangency points of the incircle and the sides BC and AD. Then the line XY forms equal angles with these sides and by Brianchon theorem it passes through the intersection point L of diagonals. By applying sine theorem to triangles LXB and LYD we obtain that $BL/DL = b/d$. Similarly, $AL/CL = a/c$. Together with the equations $S_{UBC}/S_{UAD} = BL/DL$, $S_{VBC}/S_{VAD} = CL/AL$, $S_{IBC}/S_{IAD} = (b+c)/(a+d)$ this implies that I divides the segment AC in the ratio of $(a+c)/(b+d)$, as has been required.

24. (Folklore) a) Consider the point P fixed within the circle and two perpendicular rays passing through it and crossing the circle at points A and B. Find the locus of projections of P on the lines AB.

    b) Consider the point P fixed within the sphere and three pairwise-perpendicular rays passing through it and crossing the sphere at points A, B, C. Find the locus of projections of P on the plane ABC.

    Solution. a) Let P_1 be the point symmetrical to P about the line AB, and P_2 be the point symmetrical to P about the midpoint of segment AB. Then the triangles ABP_1 and ABP_2 are symmetrical about the perpendicular bisector to AB, therefore $OP_1 = OP_2$. Since $APBP_2$ is a rectangle, $OA^2+OB^2 = OP^2+OP_2^2$, i.e. the distance OP_2 does not depend on choice of rays PA, PB. Therefore points P_1 and P_2 belong to the circle with center O, whereas the projection of P to AB lies on the circle with the radius twice less and with the center in the midpoint of OP.

    b) Let us complete the pyramid $PABC$ to a rectangular parallelepiped $PAC'BCB'P'A'$. Similarly to part a) we obtain that $OP'^2 = 3R^2 - 2OP^2$, i.e. the point P' lies on the sphere with center O. Since the mass center M of the triangle ABC lies on the segment PP' and splits it in ratio of $1:2$, M belongs to the sphere with the center at the point of segment OP dividing it in ratio of $2:1$. Furthermore, the projection of O into the plane ABC is the center O' of the circumcircle of the triangle ABC, whereas the projection of P is its orthocenter H. Since M belongs to the segment $O'H$ and $MH = 2MO'$, we have $MK = KH$, i.e. the locus in question is the sphere with center K and radius of $\sqrt{3R^2 - 2OP^2}/3$.

25. (A.Zaslavsky) In tetrahedron $ABCD$ the dihedral angles at edges BC, CD and DA are equal to α, whereas the dihedral angles at the remaining edges are equal to β. Find the ratio of AB/CD.

    Solution. It follows from the problem statement that the trilateral angles at the vertices A and B as well as at C and D are respectively equal. So, $\angle CBD = \angle CBA = \angle DAC = \angle DAB$, $\angle ADB = \angle CDB = \angle DCA = \angle BCA$, and all the faces of the tetrahedron are similar. Furthermore $\frac{AB}{BC} = \frac{BC}{BD} = \frac{BD}{CD} = \frac{\sin\angle BAC}{\sin\angle BAD} = \frac{\sin\alpha}{\sin\beta}$. So, $\frac{AB}{CD} = \left(\frac{\sin\alpha}{\sin\beta}\right)^3$.

26. (D.Tereshin) Four cones with a common vertex and equal length of the generatrix are given. Radii of their bases are possibly not equal. Each of the cones is tangent with two others. Prove that the four tangent points of the circle bases of the cones are concyclic.

    Solution. The circle bases of the cones lie on the sphere with center in the vertex of the cones and the radius equal to their generatrix. Inversion with the center at any point of this sphere maps it to a plane, whereas the circles are mapped to the circles on this plane, each of which is tangent to two others. Now the theorem about the angle between a tangent and a chord directly implies that the four tangent points belong to the same circle, which is the image of a circle on the sphere.