Second olympiad, year 2006

Correspondence round

1. (8) Two lines in the plane, intersecting at an angle of 46°, serve as symmetry axes for a geometric figure F. What is the minimal number of symmetry axes of this figure?

2. (8-9) Points A and B move at equal speeds along equal circles. Prove that the perpendicular bisectors to AB concur at a fixed point.

3. (8-9) There is a map with segments of straight linear roads linking three villages. The villages themselves are beyond the boundaries of the map. Furthermore the fire station equidistant from the three villages is located inside the boundaries of the map but not indicated on the map. Can its location be determined using compass and ruler if the constructions are to be made within the map only?

4. a) (8) Two squares $ABCD$ and $DEFG$ are given, point E belongs to segment CD, while points F, G lie outside the square $ABCD$. Find the angle between the lines AE and BF.

   b) (9-11) Two regular pentagons $OKLMN$ and $OPRST$ are given, where the point P belongs to segment ON, while the points R, S, T lie outside the pentagon $OKLMN$. Find the angle between the lines KP and MS.

5. a) (8) Fold a square 10×10 from a rectangular stripe 1×118.

   b) (9-11) Fold a square 10×10 from a rectangular stripe $1 \times (100 + 9\sqrt{3})$ (approximately 1x115.58).

   In each case the stripe can be folded but cannot be ripped into parts.

6. a) (8-9) Given a segment AB with point C inside it. The segment is a chord of the circle with radius R. Inscribe a circle into the formed circle segment such that this circle contains point C and is tangent to the initial circle.

   b) (9-10) Given a segment AB with point C inside it, which is the tangency point of this segment with the circle of radius r. Draw a circle through points A and B such that it is tangent to the initial circle.

7. (8-10) Given are a point E inside the square $ABCD$ and a point F outside it, so that triangles ABE and BCF are equal. Find the angles of the triangle ABE if it is known that the segment EF is equal to the side of the square, while the angle BFD is right.

8. (8-9) The segment AB splits the square into two equal parts. A circle can be inscribed into each of them. The radii of the circles equal r_1 and r_2 where $r_1 > r_2$. Find the length of AB.

9. (8-10) Let the line $L(\alpha)$ link the points of the unit circle, corresponding to the angles α and $\pi - 2\alpha$. Prove that if $\alpha + \beta + \gamma = 2\pi$, then the lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are concurrent.

10. (8-11) For what n a regular n-gon can be split by non-intersecting diagonals into isosceles (and, possibly, equilateral) triangles?

11. (9-10) Let point O be the center for the circumcircle of the triangle ABC; let A', B', C' be the points symmetrical to A, B, C about respective opposite sides; let A_1, B_1, C_1 be the points of intersection between the lines OA' and BC, OB' and AC, OC' and AB. Prove that the lines AA_1, BB_1, CC_1 intersect at the same point.

12. (9-10) In the triangle ABC the bisector of the angle A equals the half-sum of its median and altitude dropped from the vertex A. Prove that if $\angle A$ is obtuse then $AB = AC$.

13. (9-10) Consider two lines a and b, as well as two points A and B. The point X slides along a, whereas the point Y slides along b, so that $AX \parallel BY$. Find the locus of intersections between AY and XB.

14. (9-11) Consider a circle and a fixed point P not belonging to it. Find the locus for orthocenters of triangles ABP, where AB is the circle diameter.

15. (9-11) For a triangle ABC, consider the circumcircle and the incircle, the latter of which touches its sides BC, CA, AB at points A_1, B_1, C_1 respectively. The line B_1C_1 crosses the line BC at the point P, whereas the point M is the midpoint of the segment PA_1. Prove that the segments of the tangents from the point M to the incircle and to the circumcircle are equal.

16. (9-11) Sides of the triangle ABC are bases for regular triangles drawn outside it. Their outlying vertices form a regular triangle. Is that true that the initial triangle is regular?

17. (9-11) In two circles intersecting at points A and B, two parallel chords A_1B_1 and A_2B_2 are drawn. The lines AA_1 and BB_2 meet at the point X, while the lines AA_2 and BB_1 meet at the point Y. Prove that $XY \parallel A_1B_1$.

18. (9-11) Two perpendicular lines are drawn through the orthocenter H of the triangle ABC. One of them intersects BC at point X, another one intersects AC in point Y. The lines AZ, BZ are parallel to the lines HX and HY respectively. Prove that the points X, Y, Z are collinear.

19. (10-11) Through the midpoints of the sides of triangle T, the lines perpendicular to the bisectors of the opposite angles are drawn. These lines form the triangle T_1. Prove that its circumcenter is the midpoint of the segment linking the incenter and the orthocenter of T.

20. (10-11) In the plane, consider four points A, B, C, D. The points A_1, B_1, C_1, D_1 are the orthocenters of triangles BCD, CDA, DAB, ABC respectively. The points A_2, B_2, C_2, D_2 are the orthocenters of triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$, and so on. Prove that all the circles containing the midpoints of sides of these triangles, intersect in the same point.

21. (10-11) Consider points C', A', B' on the sides AB, BC, CA of the triangle ABC. Prove that the following inequality holds for areas of respective triangles:

$$S_{ABC}S^2_{A'B'C'} \geq 4S_{AB'C'}S_{BC'A'}S_{CA'B'};$$

moreover, the equality holds only if the lines AA', BB', CC' concur.

22. (10-11) Consider a circle and points A and B on it, as well as a point P in the plane. Let X be an arbitrary point of the circle, Y be the commom point of lines AX and BP. Find the locus of circumcenters of the triangles PXY.

23. Consider a convex quadrilateral $ABCD$, and let G be its center of mass as of a uniform plate (i.e., the point of intersection of two lines each of which links centroids of triangles sharing the same diagonal).

    a) (9-10) Suppose a circle can be circumscribed about $ABCD$, a point O being its center. Let us define point H similarly to point G by taking orthocenters instead of centroids. Prove that the points H, G, O are collinear, and $HG : GO = 2 : 1$.

    b) (10-11) Suppose a circle with center at point I is inscribed to $ABCD$. Let Nagel point N of a circumscribed quadrilateral denote the intersection of two lines, each of which links the points on the opposite sides of the quadrilateral, symmetrical to the tangent points of the incircle about midpoints of the sides. (These lines split perimeter of the quadrilateral in two equal parts). Prove that N, G, I are collinear, whereby $NG : GI = 2 : 1$.

24. a) (9-10) Consider the point P fixed within the circle and two perpendicular rays passing through it and crossing the circle at points A and B. Find the locus of projections of P on the lines AB.

    b) (10-11) Consider the point P fixed within the sphere and three pairwise-perpendicular rays passing through it and crossing the sphere at points A, B, C. Find the locus of projections of P on the plane ABC.

25. (11) In tetrahedron $ABCD$ the dihedral angles at edges BC, CD and DA are equal to α, whereas the dihedral angles at the remaining edges are equal to β. Find the ratio of AB/CD.

26. (11) Four cones with a common vertex and equal length of the generatrix are given. Radii of their bases are possibly not equal. Each of the cones is tangent with two others. Prove that the four tangent points of the circle bases of the cones are concyclic.