Second olympiad, year 2006
Final round. Solutions
Grade 8

8.1.(I.Yashchenko) Inscribe a regular triangle of the maximum perimeter into a given half-circle.

Solution. It is evident that there are two ways to inscribe a triangle into a half-circle: either two vertices of the triangle belong to the arch while the third belongs to the half-circle's diameter, or, vice-versa, two vertices belong to the diameter while the third belongs to the arch. Consider the first case. Suppose the vertices A, B lie on the arch. Then the perpendicular bisector to AB passes through the center of the half-circle. It follows that the third vertex coincides with the center and the side of the triangle equals the radius of the half-circle. (Fig. 8.1.1).

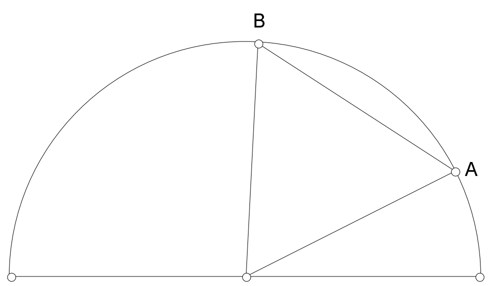

Fig. 8.1.1

In the second case, the altitude of the triangle does not exceed the radius of the half-circle. Specifically, in the case shown on Fig. 8.1.2 the equation holds. It follows that this triangle will be the one required.

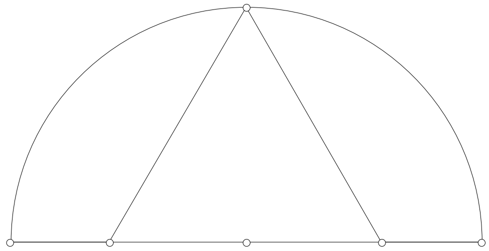

Fig. 8.1.2

8.2. (B.Frenkin) At what minimum n there exists an n-gon, that can be cut into a triangle, a quadrilateral, ,. . . , a 2006-gon?

Solution. Answer: $n = 3$. Fig. 8.2 suggests that at any $n \geq 3$ a triangle can be cut into an n-gon and a $(n + 1)$-gon. So, it is possible, first, to cut a triangle into 1002 triangles with the rays from the same vertex, and then to cut the first of those triangles into a triangle and quadrilateral, the second – into a pentagon and a hexagon, . . . , while the last one – into a 2005-gon and a 2006-gon.

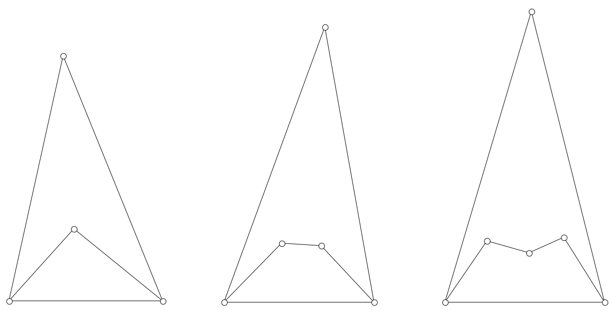

Fig. 8.2

8.3. (V.Protasov) Consider a parallelogram $ABCD$. Two circles with centers at vertices A and C pass through D. The line ℓ passes through D and has the secondary intersection with circles at points X, Y. Prove that $BX = BY$.

Solution. Let us consider for instance the case shown in the Fig. 8.3. We have $AX = AD = BC$ and $CY = CD = AB$. Furthermore $\angle BCY = \angle C - \angle DCY = \angle C - (\pi - 2\angle CDY) = 2\angle CDY - \angle D = \angle CDY - \angle ADX$, $\angle BAX = \angle DAX - \angle A = \pi - 2\angle ADX - \angle A = \angle D - 2\angle ADX = \angle CDY - \angle ADX$. Thus the triangles ABX and CYB are equal, which implies the desired equation. The other cases of location for X and Y are considered similarly.

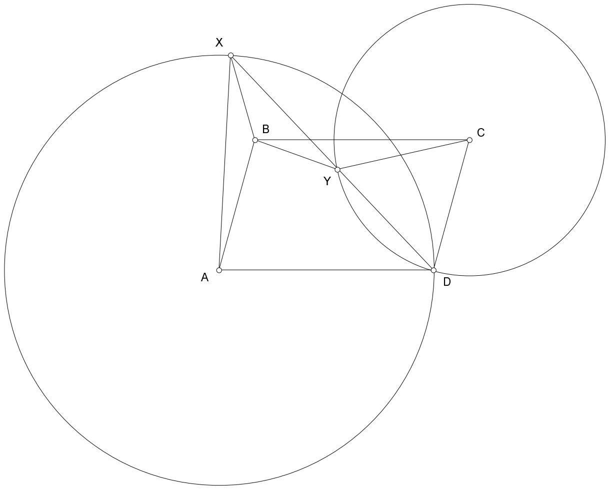

Fig. 8.3

8.4. (A.Zaslavsky) Two equal circles intersect at A and B. Point P that is distinct from A and B belongs to one of the circles, whereas points X, Y are secondary points of intersection of lines PA, PB with the other circle. Prove that the line containing P and perpendicular to AB splits one of the arcs XY into two equal arcs.

Solution. Consider the case when the point P lies within the second circle (Fig. 8.4). Suppose Q is its meet point with the line passing through P and perpendicular to AB, which lies outside the first circle. Then $\angle QPX = (\smile QX + \smile AP)/2$, $\angle QPY = (\smile QY + \smile BP)/2$. However $(\smile AP - \smile BP)/2 = \angle PBA - \angle PAB = \angle QPX - \angle QPY$, therefore the arcs QX and QY are equal. The other cases are considered similarly.

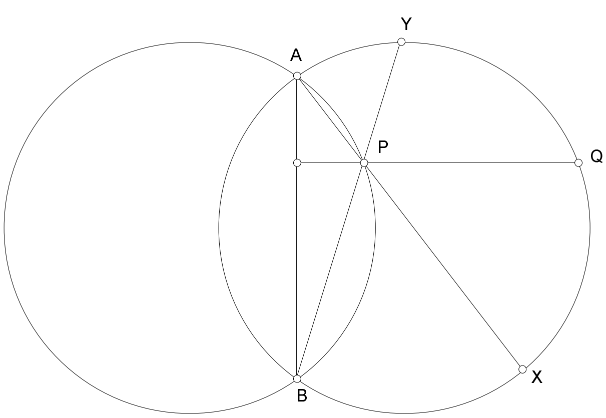

Fig. 8.4

8.5. (V.Gurovits, B.Frenkin) Does there exist a convex polygonin such that every its side is equal to some diagonal, whereas every diagonal is equal to some of the sides?

Solution. Answer: no. Let us assume the contrary and let AB be the longest side of the polygon, whereas CD be its shortest diagonal (AB and CD might share one common endpoint). Let E be the vertex lying on the other side of CD than A and B(Fig. 8.5). Then, since $AE \le AB$ and $BE \le AB$, we have $\angle AEB \ge 60^\circ$. On the other hand, since $CE \ge CD$ and $DE \ge CD$, we have $\angle CED \le 60^\circ$. But $\angle CED > \angle AEB$, a contradiction.

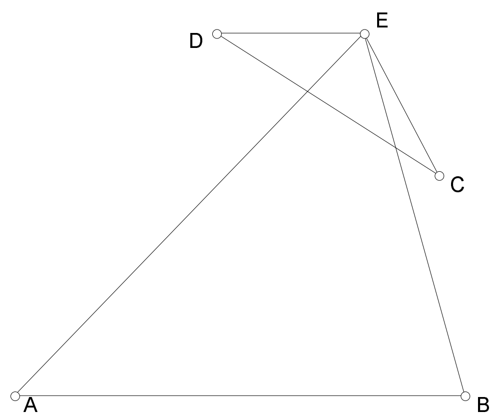

Fig. 8.5

8.6. (M.Volchkevich) Consider a triangle ABC and a point P inside it. A', B', C' are the projections of P to the lines BC, CA, AB. Prove that the circumcenter of $A'B'C'$ lies inside the triangle ABC.

Solution. Let A_1, B_1, C_1 be the points symmetrical to P about BC, CA, AB. Since $CA_1 = CP = CB_1$, the perpendicular bisector to the segment A_1B_1 coincides with the bisector of angle A_1CB_1. Since $\angle A_1CB_1 = 2\angle ACB$, this bisector lies inside angle ACB (Fig. 8.6). Similarly, the perpendicular bisectors to segments A_1C_1 and B_1C_1 are inside the respective angles of triangle ABC. Therefore, the circumcenter Q of the triangle $A_1B_1C_1$ lies inside the triangle ABC. Since the triangle $A'B'C'$ is obtained from the triangle $A_1B_1C_1$ by homothety with center P and factor $\frac{1}{2}$, the circumcenter of $A'B'C'$ coincides with the midpoint of PQ and, therefore, lies inside ABC.

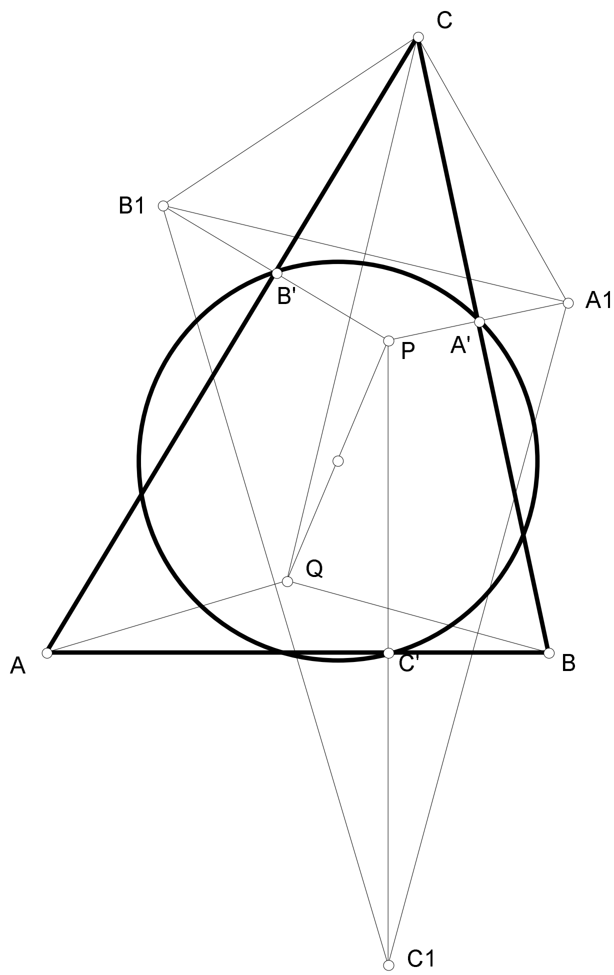

Fig. 8.6

Grade 9

9.1. (V.Protasov) Consider a circle with radius R. Two other circles with the sum of radii also equal to R touch the first circle internally. Prove that the line linking the tangent points passes through one of the common points of these circles.

Solution. Let O be the center of the outer circle, O_1, O_2 be the centers of the inner circles, A, B be the tangent points. Draw a line through O_1 parallel to OB, and a line through O_2 parallel to OA. By Thales theorem these lines meet at some point C of the segment AB. Furthermore $O_1C = O_1A$ and $O_2C = O_2B$, therefore point C belongs to both inner circles (Fig. 9.1).

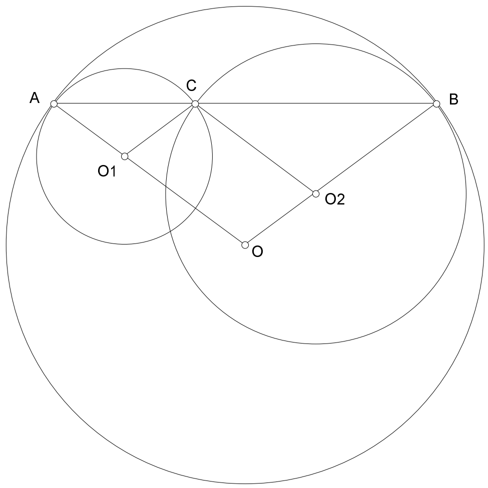

Fig. 9.1

9.2. (V.Protasov) Given a circle, a point A on it and a point M inside it. Consider chords BC passing through M. Prove that the circles passing through midpoints of all triangles ABC, are tangent to some fixed circle.

Solution. Let O be the center of the given circle, O' be the center of the circle passing through the midpoints of sides of ABC, and P be the center of mass of the triangle ABC. Since its vertices map into midpoints of its sides under homothety with center P and factor $-\frac{1}{2}$, point P lies on the segment OO' and splits it as $2:1$. Moreover, since the set of midpoints of the chords passing through M is a circle with diameter of OM, the set of mass centers of triangles ABC also is a circle obtained from it by homothety with center A and factor $\frac{2}{3}$. Therefore the set of points O' is also a circle (Fig. 9.2).

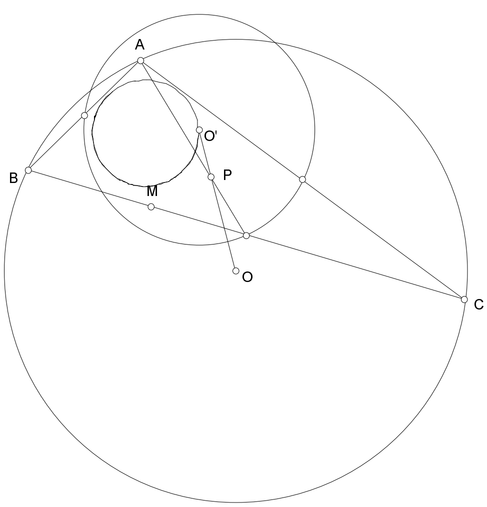

Fig. 9.2

Since the radii of all circles passing through midpoints of ABC sides are equal to half-radius of the given circle, all of these circles touch two circles concentric with the circle containig points O' (if point M coincides with O, then one of these circles degenerates to a point).

9.3. (A.Akopyan) Triangles ABC and $A_1B_1C_1$ are similar, and oriented differently. A point A' is chosen on the segment AA_1, such that $AA'/A_1A' = BC/B_1C_1$. Similarly we construct B' and C'. Prove that A', B' and C' are collinear.

Solution. The similarity that maps ABC into $A_1B_1C_1$ can be represented as a composition of symmetry about the line l and homothety with the center at certain point belonging to l and the factor k equal to the ratio of the respective sides of triangles. The segments AA_1, BB_1, CC_1 are obviously split in the ratio of k by l, i.e. the points A', B', C' belong to l.

9.4. (S.Markelov) In a non-convex hexagon each angle is equal either 90 or 270 degrees. Is it true that for certain lengths of its sides the given hexagon can be cut into a pair of non-equal hexagons similar to the given hexagon?

Solution. Assume t is the root of equation $t^4 + t^2 = 1$. Consider the hexagon $ABCDEF$ where $AB : BC = BC : CD = CD : AF = AF : FE = FE : ED = \frac{1}{t}$ and cut it as is shown in Fig. 9.4. Then the resulting hexagons will be similar to $ABCDEF$ with factors t and t^2.

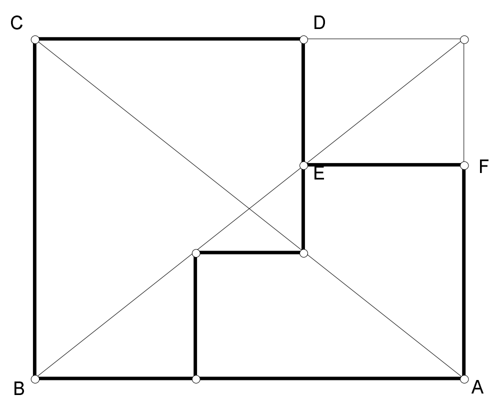

Fig. 9.4

9.5. (A.Zaslavsky) The line passing through the circumcenter and the orthocenter of a non-regular triangle ABC splits its perimeter and its area at the same ratio. Find it.

Solution. Answer: $1 : 1$.

First, let us prove that the line splits the perimeter and the area of the triangle in the same ratio if and only if this line passes through the incenter of the triangle. In fact, let the line cross the sides AC, BC at X and Y, and cross the bisector of angle C at J; let d_1 be the distance from J to the side AB, let d_2 be the distance from J to two other sides. Then $2S_{CXY} = (CX + CY)d_2$, $2S_{AXYB} = (AX + BY)d_2 + AB \cdot d_1$, and the ratios are equal if and only if $d_2 = d_1$, i.e. if J is the incenter.

Now let the circumcenter O, incenter I and orthocenter H be collinear. No more than one vertex of the triangle belongs to this line. Assume it does not contain vertices A and B. Since AI, BI are bisectors of angles HAO and HBO, we obtain that $AH/AO = HI/IO = BH/BO$. Since $AO = BO$, $AH = BH$, i.e. the triangle ABC is isosceles and the ratio in question equals $1:1$.

9.6. (Ya.Ganin, F.Rideau) Consider a convex quadrilateral $ABCD$. Let A', B', C', D' be the orthocenters of triangles BCD, CDA, DAB, ABC. Prove that in quadrilaterals $ABCD$ and $A'B'C'D'$ the respective diagonals are split by the intersection points at one and the same ratio.

Solution. Let us make use of the following statement.

Suppose $KLMN$ be a convex quadrilateral; points X, Y split the segments KL and NM at the ratio of α; points U, V split the segments LM and KN at the ratio of β. Then the intersection point of segments XY and UV splits the third one at the ratio of β, whereas the second one at the ratio of α (Fig. 9.6)

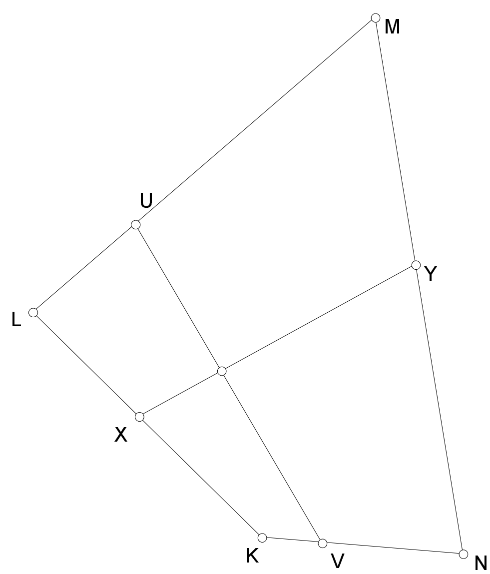

Fig. 9.6

It is not difficult to obtain the proof of this statement using method of masses.

Assume now that A_1, B_1, C_1, D_1 are the mass centers of triangles BCD, CDA, DAB, ABC; let A_2, B_2, C_2, D_2 be their circumcenters. The quadrilateral $A_1B_1C_1D_1$ is homothetic to the quadrilateral $ABCD$ about its mass center with ratio of $-\frac{1}{3}$. Therefore the respective diagonals of these quadrilaterals are split by the points of intersection in the same ratios. Let us prove that diagonals of the quadrilateral $A_2B_2C_2D_2$ are split by the point of intersection in the same ratios.

Let P be the point of intersection of diagonals in the quadrilateral $ABCD$. Then

$$\frac{AP}{CP} = \frac{AP}{BP}\frac{BP}{CP} = \frac{\sin\angle ABD \sin\angle ACB}{\sin\angle BAC \sin\angle CBD}.$$

Since the sides and the diagonals of the quadrilateral $A_2B_2C_2D_2$ are perpendicular to the sides and the diagonals of the quadrilateral $ABCD$ (for instance, the points A_2, B_2 belong to the perpendicular bisector to CD), diagonal A_2C_2 is split in the same ratio.

Now, let P_1, P_2 be the points of intersection for diagonals of quadrilaterals $A_1B_1C_1D_1$, $A_2B_2C_2D_2$; let P' be the point of segment $A'C'$, splitting it at the ratio of A_2P_2/P_2C_2. Since the points A_1, C_1 belong to the segments $A'A_2$, $C'C_2$ and split them at the ratio of $2:1$,it follows from the above statement that the point P_1 also splits the segment $P'P_2$ at the ratio of $2:1$. Considering the analogous point of the segment $B'D'$, we obtain the same result. This implies that P' is the point of intersection of diagonals in quadrilateral $A'B'C'D'$ and splits the diagonals at the same ratio as in the quadrilaterals $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ and $ABCD$.

Grade 10

10.1. (Hiacinthos) Five lines meet at the same point. Prove that there is a closed five-segment line for which the vertices and midpoints of edges lie on the these lines and there is strictly one vertex on each line.

Solution. Let O be the point of intersection for the lines. Let us choose point A_1 on the line l_1 and find point A_2 on the line l_3 such that the midpoint B of segment A_1A_2 belongs to the line l_2 (Fig. 10.1). Applying sine theorem to triangles OA_1B and OA_2B, we obtain that $\frac{OA_2}{OA_1} = \frac{\sin\angle A_1OB}{\sin\angle A_2OB}$.

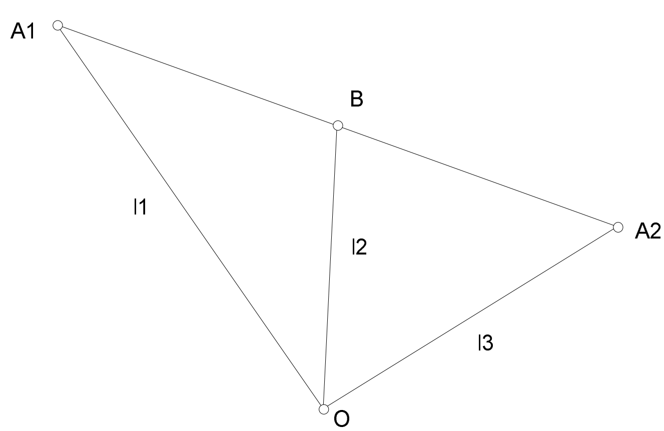

Fig. 10.1

Similarly, for the point A_2 let us choose point A_3 on the line l_5 such that the midpoint of A_2A_3 belongs to l_4 and so on. By multiplying the resulting ratios we obtain that A_6 coincides with A_1.

10.2. (A.Zaslavsky) Projections of the point X to the sides of the quadrilateral $ABCD$ belong to the same circle. Let Y be the point symmetrical to X about the center of this circle. Prove that the projections of the point B to the lines AX, XC, CY, YA also belong to the same circle.

Solution. Let us consider the case when X is within $ABCD$, the others are similar. Let K, L, M, N be the projections of X to AB, BC, CD, DA; let K', L', M', N' be the points symmetrical to X about these lines. Since K, L, M, N are concyclic, points K', L', M', N' are concyclic too. Since $BK' = BX = BL'$, the perpendicular bisector to the segment $K'L'$ passes through B and is the bisector of angle $K'BL'$, i.e. it is symmetrical to BX about the bisector of the angle B. So, the four lines symmetrical to the lines linking X with the vertices of $ABCD$, about the bisectors of respective angles, meet at the same point X', which is the circumcenter of the quadrilateral $K'L'M'N'$. Furthermore the center of the circle $KLMN$ will be at the midpoint of segment XX', and therefore X' coincides with Y. Further on, as the quadrilaterals $XKBL$, $XLCM$, $XMDN$, $XNAK$ are inscribed, we have $\angle AXB + \angle CXD = \angle KXA + \angle KXB + \angle CXM + \angle DXM = \angle KNA + \angle BLK + \angle CLM + \angle MND = (\pi - \angle KLM) + (\pi - \angle MNK) = \pi$. This implies that the lines XB and DX are symmetrical about the bisector of angle AXC. Similarly the lines YB and DY are symmetrical about the bisector of angle AYC. In addition, as demonstrated before, the bisectors of angles BAD and XAY, BCD and XCY coincide. Hence the lines symmetrical to BA, BX, BC, BY about the bisectors of respective angles of $AXCY$ intersect at point D. Therefore, following the argument from the beginning of this solution we obtain the problem statement. (Fig. 10.2).

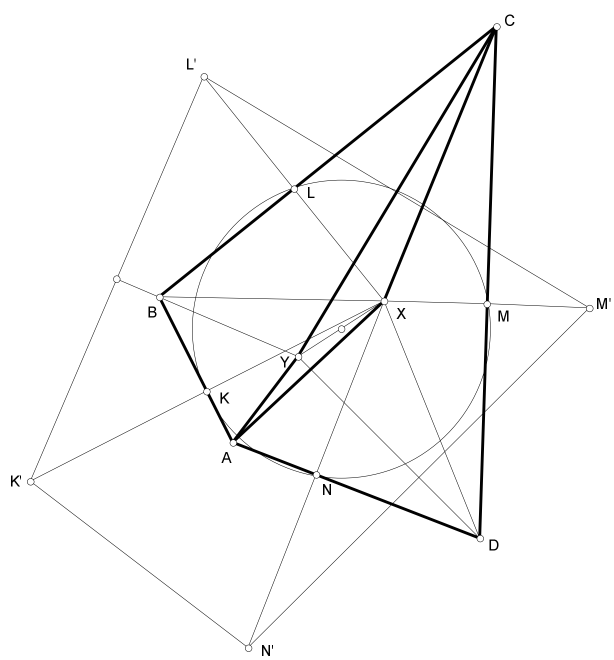

Fig. 10.2

10.3. (P.Kozhevnikov) Consider a circle and a point P inside it, distinct from its center. Consider also pairs of circles tangent to the given circle from the inside and tangent to each other at the point P. Find the locus of the meet points for the common external tangents of these circles.

Solution. Let X be the point of intersection for the tangent lines. Let us draw a circle with center X and radius XP and consider the inversion about it. The circles tangent at point P will map into one another, as they are tangent to the circle of inversion and two lines mapping into themselves. Therefore the initial circle will map into itself. It means that the circle of inversion is orthogonal to the given one, i.e. the tangent from X to the given circle is equal to XP, and X belongs to the radical axis of the point P and the given circle. Clearly, any point of the radical line can be obtained in the same manner, i.e. the locus in question coincides with the radical axis of point P and the given circle.

10.4. (A.Zaslavsky) The lines containing the medians of triangle ABC have secondary intersections with its circumcircle at points A_1, B_1, C_1. The lines passing through A, B, C and parallel to the opposite sides intersect the circumcircle at A_2, B_2, C_2. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

Solution (M.Ilyukhina). Let A' be the point of intersection for the lines tangent to the circumcircle ω at points B and C (similarly let us denote points B' and C'). As is known, the line AA' is a symmedian of the triangle ABC (i.e. the line symmetrical to AA_1 about the bisector of angle A). Let the line AA' have a secondary intersection with ω at point A_0. Then $\angle A_1AB = \angle A_0AC$, and therefore the arcs BA_1 and CA_0 are equal.

Since the triangle $A'BC$ is isosceles and ω is its excircle, then the arcs BA_1 and CA_0 are symmetrical about the bisector ℓ of the angle $BA'C$. The equality of the arcs implies that under this symmetry the points A_1 and A_0 map into one another. Note that ℓ is the perpendicular bisector to BC, therefore A under this symmetry is mapped to A_2 (Fig. 10.4), hence the line A_1A_2 maps into the line AA'. So, since the lines AA', BB', CC' intersect at the same point L as symmedians of triangle ABC, the lines A_1A_2, B_1B_2, C_1C_2 also intersect at the same point that is isogonally adjoint L about the triangle $A'B'C'$.

The case when either of the points A', B', C' does not exist is considered similarly.

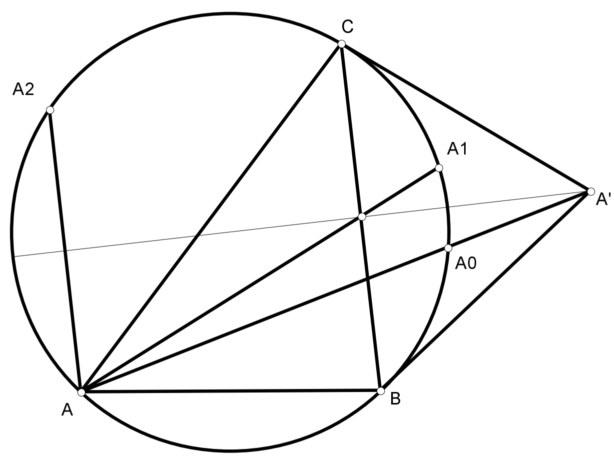

Fig. 10.4

10.5. (S.Markelov) Can an unfolding of a tetrahedron to the plane occur to be a triangle with sides 3, 4 and 5 (The tetrahedron can be cut along the edges only)?

Solution. Answer: yes, it can. For instance, a tetrahedron can be constructed out of the unfolding shown in Fig. 10.5 (the lesser cathetus is split into three equal parts, while the hypotenuse is split at the ratio of $4:1$). It is easily seen that each of the three angles, into which the lesser angle of the triangle is divided, is less than the sum of the remaining two; therefore such unfolding is indeed suitable to form a tetrahedron.

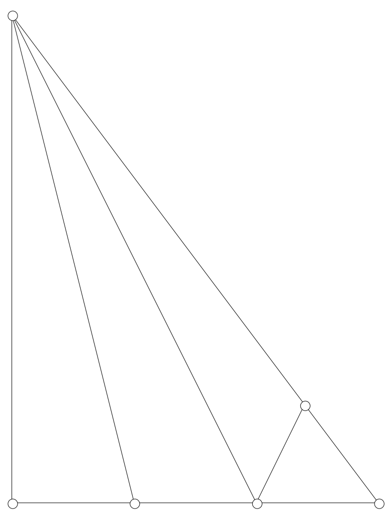

Fig. 10.5

10.6. (A.Zaslavsky) A quadrilateral was drawn on a board, both inscribed and circumscribed. The centers of the respective circles have been marked as well as the point of intersection of the lines linking the midpoints of the opposite sides. Then the quadrilateral itself was erased. Restore the quadrilateral in question by means of a compass and a ruler.

Solution. The construction is based on two lemmas.

1. The diagonals of all quadrilaterals inscribed in the given circle with center O and circumscribed about the given circle with center I intersect at the same point L belonging to the extension of the segment OI beyond the point I.

2. The center of incircle of the quadrilateral lies on the line linking the midpoints of its diagonals (Monge theorem).

Let us also note that in any quadrilateral the point of intersection of the lines linking the midpoints of opposite sides is also the midpoint of the segment linking the midpoints of its diagonals (point M).

It follows from lemma 1 that the midpoints of diagonals of the required quadrilateral belong to the circle with diameter OL. From this and from lemma 2 we obtain that the point M lies on the circle in which I and the midpoint of OL are diametrically opposite points. Therefore, drawing the line through M that is perpendicular to IM and finding its intersection with OI, we obtain the midpoint of OL and hence the point L itself. Furthermore, drawing the circle with diameter of OL and finding its points of intersection with the line MI, we obtain the midpoints of quadrilateral's diagonals. Moreover by considering the quadrilateral, two vertices of which lie on the line OI, it is easily seen that for the third vertex X the line XI is the bisector of angle OXL (Fig. 10.6). This enables us to restore the circumcircle of the quadrilateral and find its vertices as the points of intersection of this circle with the diagonals.

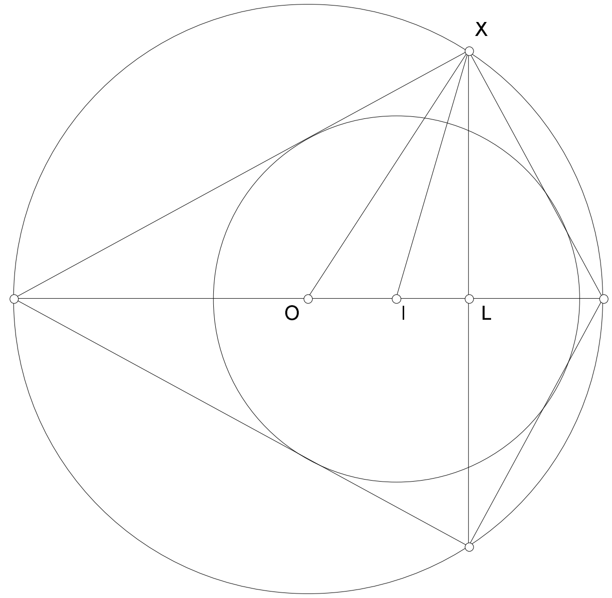

Fig. 10.6