Second olympiad, year 2006
Final round
Grade 8

8.1. Inscribe a regular triangle of the maximum perimeter into a given half-circle.

8.2. At what minimum n there exists an n-gon, that can be cut into a triangle, a quadrilateral, ,. . . , a 2006-gon?

8.3. Consider a parallelogram $ABCD$. Two circles with centers at vertices A and C pass through D. The line ℓ passes through D and has the secondary intersection with circles at points X, Y. Prove that $BX = BY$.

8.4. Two equal circles intersect at A and B. Point P that is distinct from A and B belongs to one of the circles, whereas points X, Y are secondary points of intersection of lines PA, PB with the other circle. Prove that the line containing P and perpendicular to AB splits one of the arcs XY into two equal arcs.

8.5. Does there exist a convex polygonin such that every its side is equal to some diagonal, whereas every diagonal is equal to some of the sides?

8.6. Consider a triangle ABC and a point P inside it. A', B', C' are the projections of P to the lines BC, CA, AB. Prove that the circumcenter of $A'B'C'$ lies inside the triangle ABC.

Grade 9

9.1. Consider a circle with radius R. Two other circles with the sum of radii also equal to R touch the first circle internally. Prove that the line linking the tangent points passes through one of the common points of these circles.

9.2. Given a circle, a point A on it and a point M inside it. Consider chords BC passing through M. Prove that the circles passing through midpoints of all triangles ABC, are tangent to some fixed circle.

9.3. Triangles ABC and $A_1B_1C_1$ are similar, and oriented differently. A point A' is chosen on the segment AA_1, such that $AA'/A_1A' = BC/B_1C_1$. Similarly we construct B' and C'. Prove that A', B' and C' are collinear.

9.4. In a non-convex hexagon each angle is equal either 90 or 270 degrees. Is it true that for certain lengths of its sides the given hexagon can be cut into a pair of non-equal hexagons similar to the given hexagon?

9.5. The line passing through the circumcenter and the orthocenter of a non-regular triangle ABC splits its perimeter and its area at the same ratio. Find it.

9.6. Consider a convex quadrilateral $ABCD$. Let A', B', C', D' be the orthocenters of triangles BCD, CDA, DAB, ABC. Prove that in quadrilaterals $ABCD$ and $A'B'C'D'$ the respective diagonals are split by the intersection points at one and the same ratio.

Grade 10

10.1. Five lines meet at the same point. Prove that there is a closed five-segment line for which the vertices and midpoints of edges lie on the these lines and there is strictly one vertex on each line.

10.2. Projections of the point X to the sides of the quadrilateral $ABCD$ belong to the same circle. Let Y be the point symmetrical to X about the center of this circle. Prove that the projections of the point B to the lines AX, XC, CY, YA also belong to the same circle.

10.3. Consider a circle and a point P inside it, distinct from its center. Consider also pairs of circles tangent to the given circle from the inside and tangent to each other at the point P. Find the locus of the meet points for the common external tangents of these circles.

10.4. The lines containing the medians of triangle ABC have secondary intersections with its circumcircle at points A_1, B_1, C_1. The lines passing through A, B, C and parallel to the opposite sides intersect the circumcircle at A_2, B_2, C_2. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

10.5. Can an unfolding of a tetrahedron to the plane occur to be a triangle with sides 3, 4 and 5 (The tetrahedron can be cut along the edges only)?

10.6. A quadrilateral was drawn on a board, both inscribed and circumscribed. The centers of the respective circles have been marked as well as the point of intersection of the lines linking the midpoints of the opposite sides. Then the quadrilateral itself was erased. Restore the quadrilateral in question by means of a compass and a ruler.