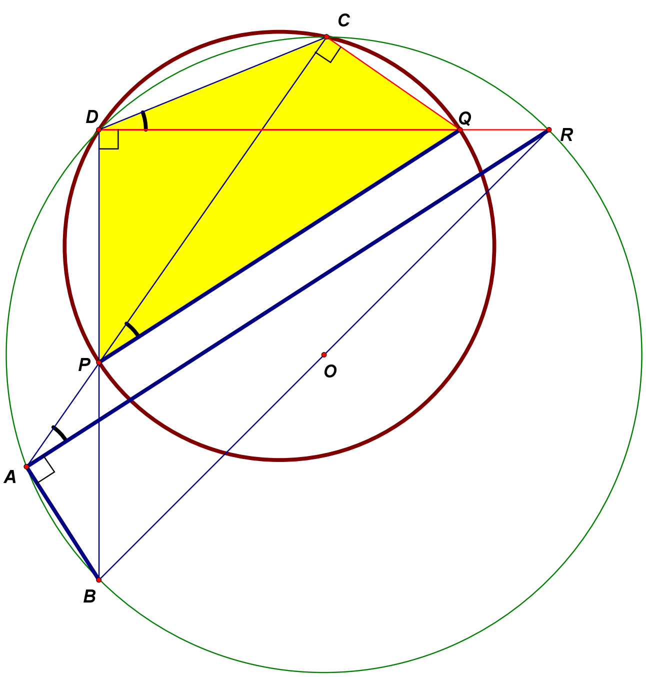
The first Olympiad, 2005

Correspondence round. Solutions

1. (A.Zaslavsky) The circle chords AC and BD intersect at point P. Perpendiculars to AC and BD at points C and D respectively intersect at point Q. Prove that the lines AB and PQ are perpendicular.

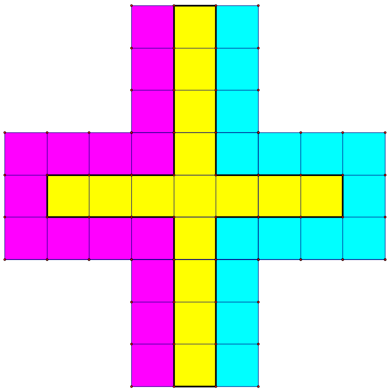
Solution. Let the perpendiculars intersect inside the circle (the case when the meet point is outside the circle is considered similarly). Denote by R the second intersection of line DQ with the circle.



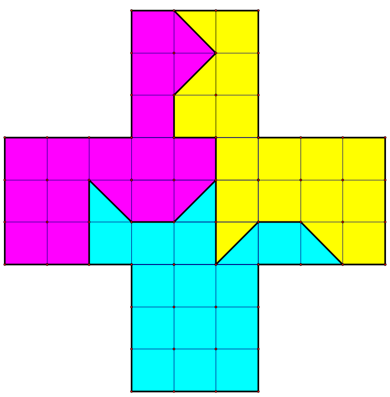
The quadrilateral $PDCQ$ is inscribed in the circle (it is formed by two right-angled triangles sharing the common hypotenuse PQ). Therefore $\angle CDQ = \angle CPQ$ as they are subtended by the same circle arc. For the same reason $\angle CDQ = \angle CDR = \angle CAR$, so lines PQ and AR are parallel (the corresponding angles are equal). Since BR is a diameter as follows from the problem statement, $\angle BAR = 90°$.

2. (L.Yemelyanov) Cut the cross composed of five equal squares into three polygons of equal area and perimeter.

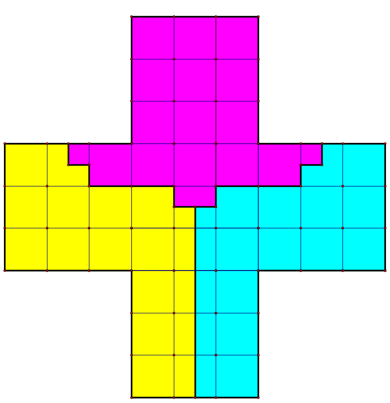
Solution. Here are some of the possible ways to cut as required:



(Arthur Darbinyan, city of Yerevan, physic-mathematical school of Yerevan)



(Igor Borodulin, city of Yekaterinburg, gymnasium no. 9)



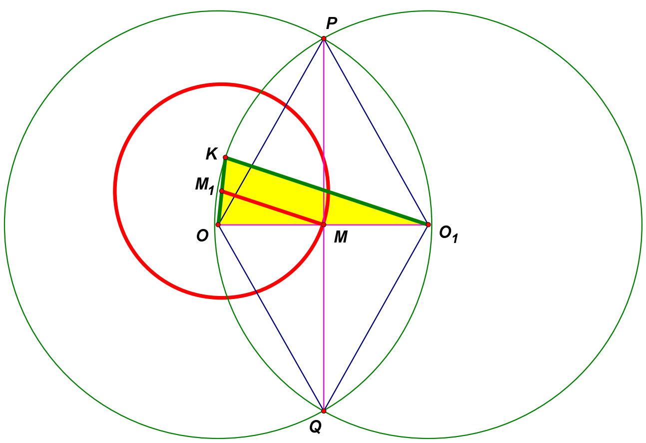
(Aleksander Makarets, Kharkov, physic-mathematical school no. 27)

3. (V.Protasov) A circle and a point K inside it are given. An arbitrary circle of the same size

passes through point K and shares a chord with the initial circle. Find the locus of midpoints of such chords.

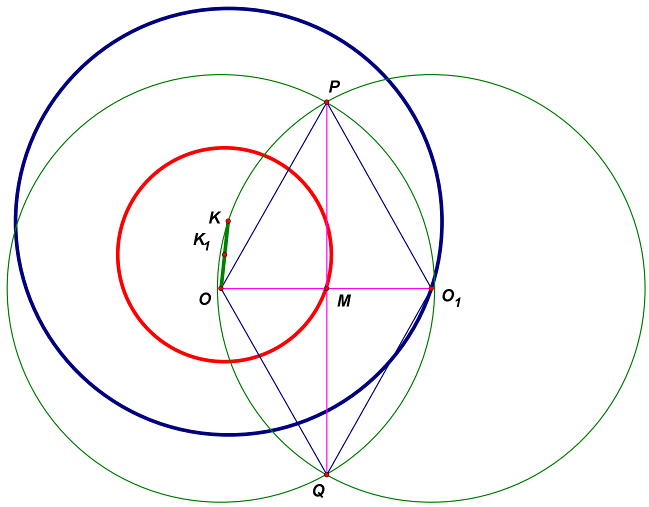
Solution. The locus of interest will be the circle with the midpoint of OK as its center (where O is the center of the initial circle) and $\frac{R}{2}$ as its radius (where R is the radius of the initial circle).

Method one. Indeed, let PQ be the common chord, M its midpoint, whereas O_1 be the center of an arbitrarily chosen circle. As it follows from the problem statement, OPO_1Q is a rhombus, so M will also be the midpoint of OO_1. Midline MM_1 of triangle KO_1 equals half of KO_1, i.e. half of the radius. Therefore, all midpoints of the chords lie on the circle with the midpoint of OK as its center and $\frac{R}{2}$ as its radius (see the figure).



Testing that any point of the resulting circle is the midpoint of one of the chords is straightforward.

Method two. Centers of the circles equal to the initial circle and passing through point K lie on the circle having K as its center and R as its radius. If O_1 is the center of some circle of that kind, then as noted before M being the midpoint of the common chord will also be the midpoint of OO_1. Therefore the locus of interest is the image of the circle formed by the centers, under homothety centered at point O and with factor of $\frac{1}{2}$ (see the figure).

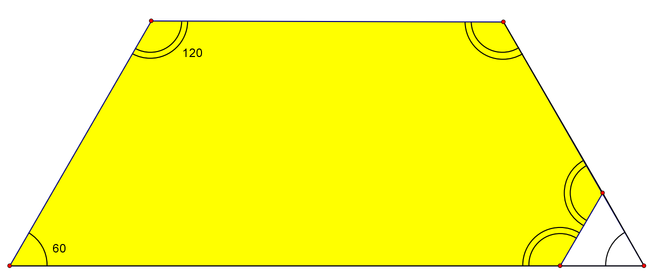


4. (B.Frenkin) For what minimal n there exists a convex n-gon with equal sines of all angles and unequal lengths of all sides?

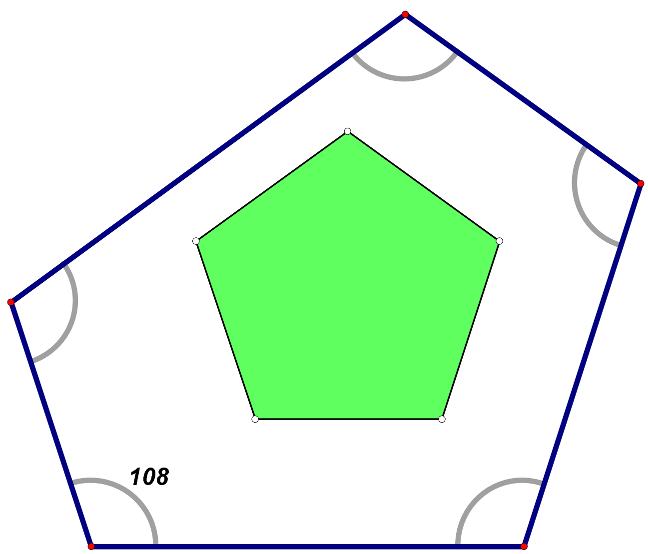
Solution. The minimal value equals five. Obviously, there are no triangles with the property in question. Let us show that there are no quadrilaterals with this property either. This can be done in various ways. For example, as sines of angles are equal, the angles themselves are equal either to ϕ or $180^\circ - \phi$ for a fixed ϕ.

Simple listing easily shows that we deal either with a parallelogram or with an isosceles trapezium. Alternatively, we may use «area method» for the proof. Consider a convex quadrilateral with equal sines of all its angles. Let us denote lengths of its sides as a, b, c ,d. We then calculate its area using formula «half-product of two sides multiplied by the sine of the angle between them» in two ways. In the resulting equation we cancel out half of the sine and arrive at equality $ab + cd = ad + bc$ or $(a - c)(b - d) = 0$. It follows that at least two sides are equal.

In order to construct a pentagon with the properties in question it is sufficient to cut away a small triangle from the larger base of an equilateral trapezium with an angle of 60° (see the figure).

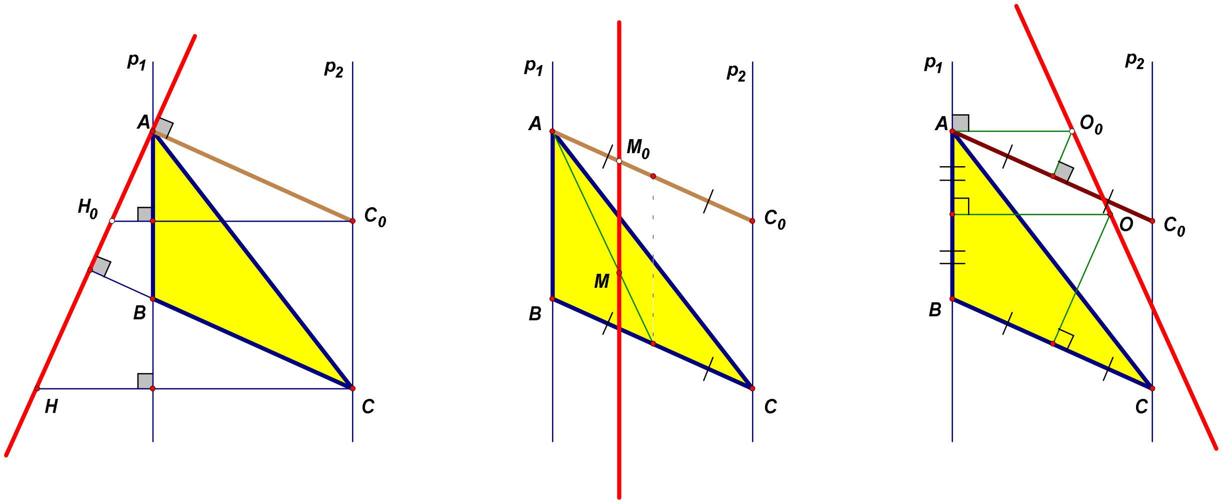


Otherwise, one can use a regular pentagon with angles of $108°$ to construct another pentagon with the sides respectively parallel to the sides of the regular pentagon but unequal between themselves (see the figure).



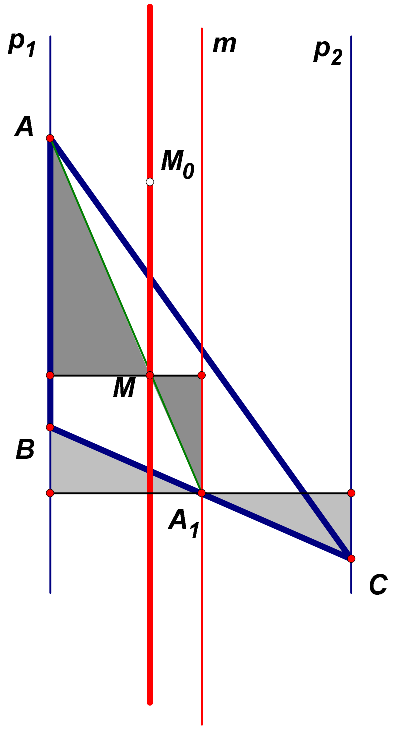
5. (A.Myakishev) Two parallel lines p_1 and p_2 are given. Points A and B belong to p_1, point C belongs to p_2. We move the segment BC parallel to itself and consider all triangles ABC produced in this manner. In these triangles, find the locus of points that are: a) intersections of altitudes; b) intersections of medians; c) circumcenters.

Solution. In all cases the resulting locus is a line with a punctured point which corresponds to the case when the triangle degenerates into a segment (see the figure).



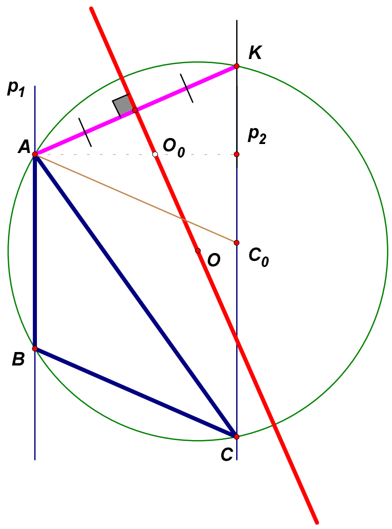
In the first case, evidently, we have the line perpendicular to BC and passing through vertex A. In the second case, the answer is the line parallel to the lines from the problem condition, which splits the segment connecting the given lines, as $1:2$ counting from the first line. Indeed, the segment BC moves with constant speed, so this is true for its midpoint. The point of medians intersection divides the segment linking vertex with the midpoint of BC in constant ratio of $2:1$ and therefore this point also moves with constant speed along some straight line. In the limit case we get point M_0 that splits the segment AC_0 (equal and parallel to BC, but passing through vertex A) in $1:2$ ratio. This is due to the fact that it should split the segment between A and the midpoint of AC_0 in ratio of $2:1$.

Our argument can follow a different path: let us draw a perpendicular to the given lines through A_1, the midpoint of BC, with the endpoints on these lines. We get a pair of equal triangles that share a vertex A_1. This implies that the midpoints lie on the line p_1, equidistant from the given lines. Next, we draw a perpendicular through M with endpoints on p_1 and m. We get a pair of similar triangles with common vertex and with the similarity ratio of 2. It means that the centers of mass lie on the line parallel to p_1 and m, which splits the common perpendicular in ratio of $2:1$. And so on (see the figure).



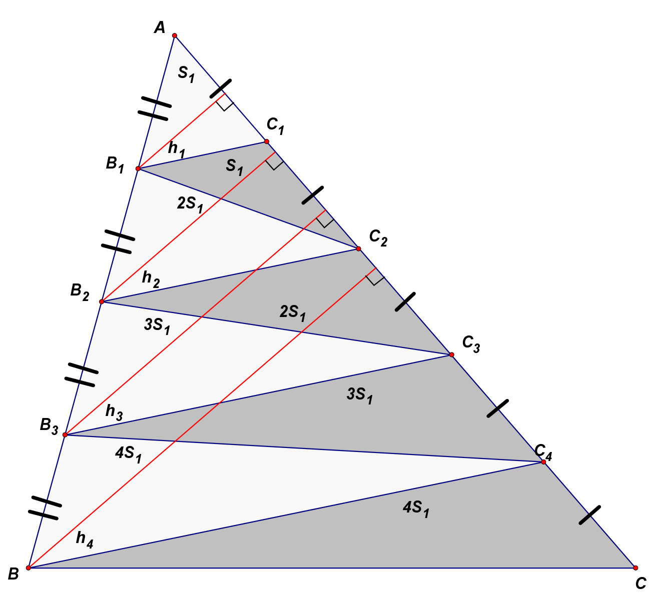
Finally, as the perpendicular bisectors to AC and BC also move at constant speeds, their intersection point (the circumcenter) moves along a line as well. Observe that this very line is the perpendicular bisector to AK, symmetrical to AC_0 (to which the triangle degenerates when point A coincides with point B) about the perpendicular from vertex A.

It is well-known that if a circle can be circumscribed about a trapezium then the latter is isosceles. Immediate consequence is that all the circles circumscribed about triangles $AB'C'$ will have the second intersection with line p_2 at the same point K, so that $AK = BC$. Therefore the centers of these circles must be equidistant from points A and K (see the figure). This reasoning gives us another method of proof that the required locus is a line (with a punctured point).



6. (A.Khachaturyan) The side AB of the triangle ABC has been divided into n equal parts (with points of division $B_0 = A$, B_1, B_2, …, $B_n = B$), while the side AC of this triangle has been divided into $(n+1)$ equal parts (with points of division $C_0 = A$, C_1, C_2, …, $C_{n+1} = C$). Triangles $C_iB_iC_{i+1}$ were colored. What part of the triangle area has been colored?

Solution. Let us show that the colored area equals exactly the half of the total triangle area. For this, drop perpendiculars to side AC from points $B_1, \ldots, B_n$. These perpendiculars are altitudes of triangles $C_iB_iC_{i+1}$ with equal bases. As follows from similarity considerations, $h_i = ih_1$. Hence the same ratio connects the areas of the colored triangles: $S_i = iS_1$. (At the figure the case $n = 4$ is shown.)



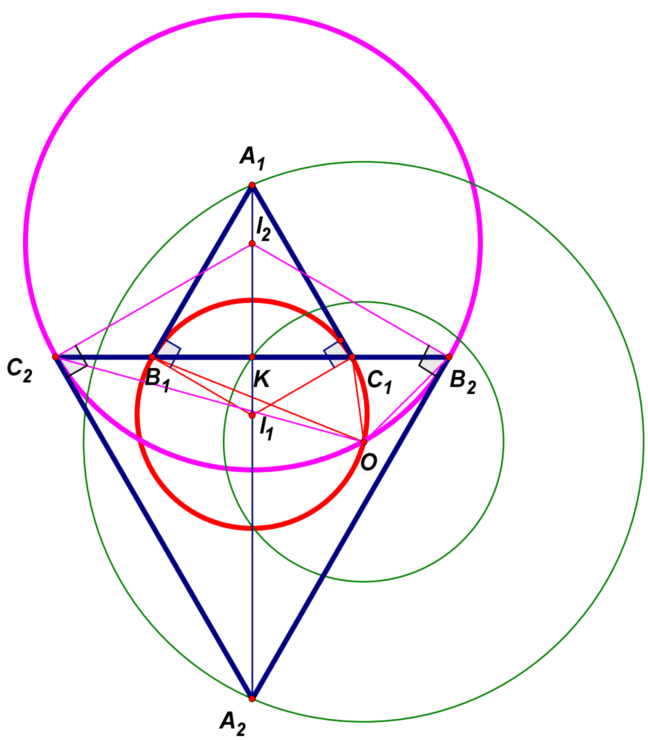
By dropping perpendiculars from points $C_1, \dots, C_n$ to the side AC and arguing as above we obtain the same ratio for the areas of non-colored triangles. As a final observation, the area of the first colored triangle equals the area of the first non-colored one (their bases are equal, while the altitude h_1 is common). We could end the proof differently by mere adding the areas of the colored triangles: $S' = S_1(1 + 2 + \dots + n) = \frac{n(n+1)}{2}S_1$. But clearly $S_1 = \frac{1}{2}h_1\frac{AC}{n+1} = \frac{1}{2}\frac{h_n}{n}\frac{AC}{n+1} = \frac{S_{ABC}}{n(n+1)}$.

Observe that the equality of areas for corresponding pairs of triangles (colored and non-colored) can be obtained with practically no computations. The lines B_iC_i are parallel (by the converse of Thales theorem). So $S_{B_{i-1}B_iC_i} = S_{C_{i-1}B_iC_i}$ (they have a common base B_iC_i, while the vertices are on the line parallel to the base, hence the altitude to the base is common) and $S_{C_{i-1}B_iC_i} = S_{B_iC_iC_{i+1}}$ (vertex B_i is common, while $C_{i-1}C_i = C_iC_{i+1}$ by condition).

7. (V. Protasov) Two circles with radii of 1 and 2 have common center at point O. Vertex A of the regular triangle ABC is on the major circle, whereas the midpoint of BC is on the minor circle. What is the possible measure of angle BOC?

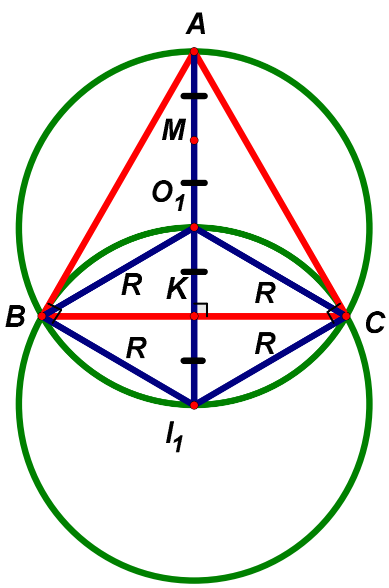
Solution. This angle is equal either to 60° or to 120°.

Method one. In the given configuration the circle passing through vertices B and C of the regular triangle (and touching its sides at these points) will also pass through the common center of the two given circles (see the figure).

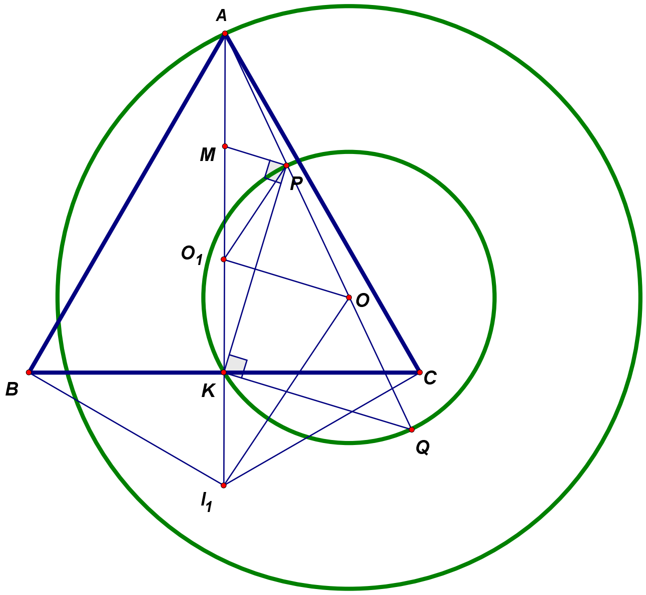


This implies that in the case of "upper" (as in the figure) triangle we have $\angle B_1OC_1 = \frac{1}{2}\angle B_1IC_1 = 60°$, since the inscribed angle is half of the central angle. On the same reason, for the "lower" triangle we will get $\angle B_1OC_1 = 120°$. In order to prove the statement in question, let us use an easily verifiable property of a regular triangle:

Let I_1 be the center of the circle tangent to sides AB and AC of the regular triangle ABC at points B and C respectively, whereas K be the midpoint of side BC. Then point K splits the segment AI_1 in ratio of $3:1$, where I_1K is equal to a half-radius of this circle (see the figure).



Now let us prove the main statement (see the figure).

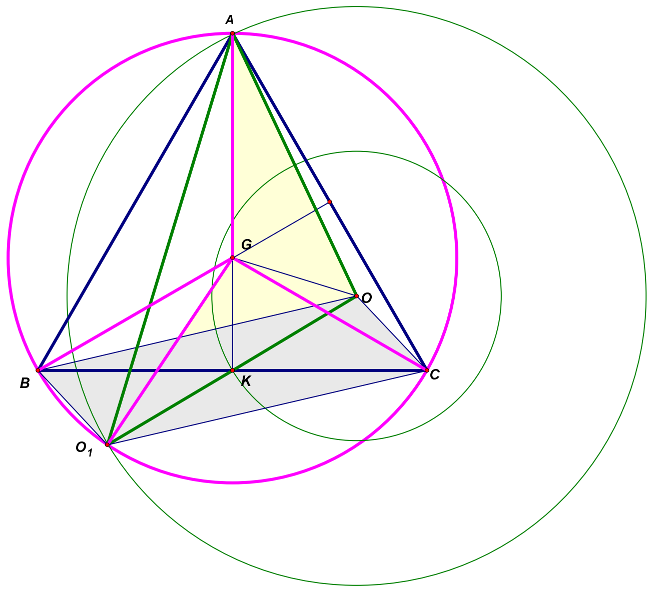


Let us draw the line AO and denote by P and Q the points where it crosses the unit circle. Points M, O_1 and K divide the segment AI_1 into four equal parts, each of which equals $R/2$, where R is the radius of the circle tangent to sides AB and AC at B and C. We want to prove that $I_1O = R$. As it follows from the problem condition, the points P and O split AQ into three equal parts. Therefore from the converse of Thales theorem we get that segments MP and KQ are parallel. However, $\angle PKQ$ is a right angle, as it subtends a diameter, therefore $\angle MPK$ is also a right angle.

The median drawn from the right angle of a right triangle equals half of the hypotenuse. So $O_1P = O_1M = O_1K = \frac{R}{2}$. It remains to observe that the segment O_1P is the medial line of triangle AI_1O and, therefore, equals half of I_1O.

Method two. (Osechkina Maria, city of Perm, Physics and Mathematics School no. 9[1]) Consider the case, for instance, when the points O and A belong to the same half-pane relative to line BC (see the figure).

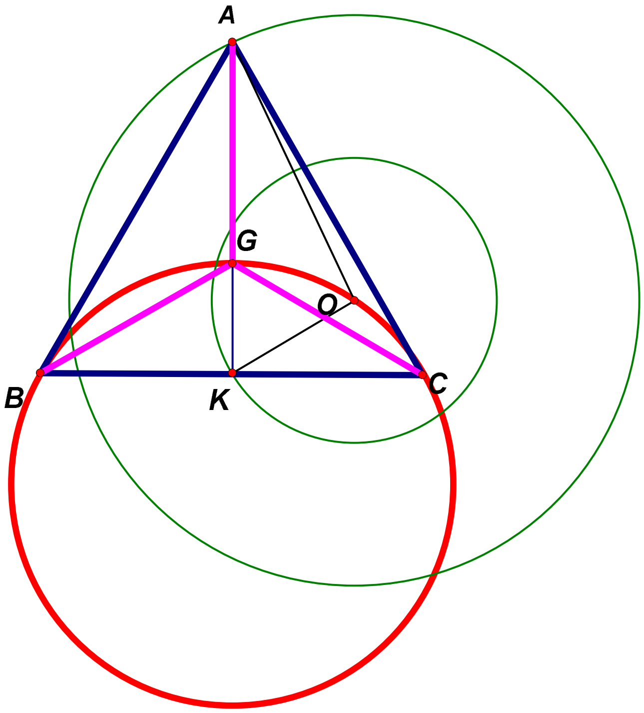
[1] In the sequel, we provide the names of school students who suggested solutions previously unknown to the Jury.



Let K be the midpoint of BC, G be the intersection of medians in triangle ABC. Let us extend the segment OK until it crosses the major circle at point O_1. By the problem condition, $BK = KC$, $OK = KO_1$, so the quadrilateral $BOCO_1$ is a parallelogram. Furthermore, let us note that G will also be the centroid (the point of medians intersection) for triangle AOO_1 since AK is a median of this triangle and $AG : GK = 2 : 1$. So, since the triangle is isosceles, the median OG is also a bisector. Therefore triangles AGO and O_1GO are equal. It follows that $GA = GO_1 = GB = GC$, which means that the points A, B, C, O_1 are concyclic, so $\angle BO_1C = 180^\circ - \angle BAC = 60^\circ$. Considering the second case similarly we arrive at 60°.

Method three (Lysov Mikhail, city of Moscow, Lyceum “The second school”). This is perhaps the most elegant solution. It is based on the following classical theorem of elementary geometry:

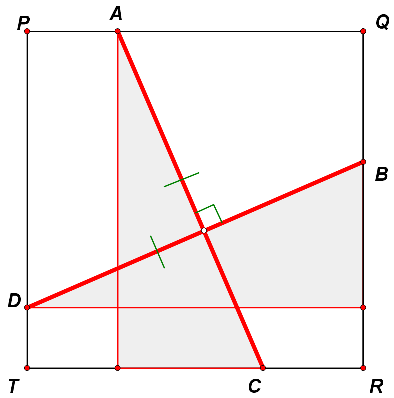
Consider some segment AB on the plane and some positive number λ. Then the locus of points X such that $\frac{AX}{BX} = \lambda$, will be a circle. If P and Q are the points splitting the segment AB at a ratio of λ (internally and externally), then this circle is based on PQ as a diameter. It is called ***the circle of Apollonius*** (see the figure).



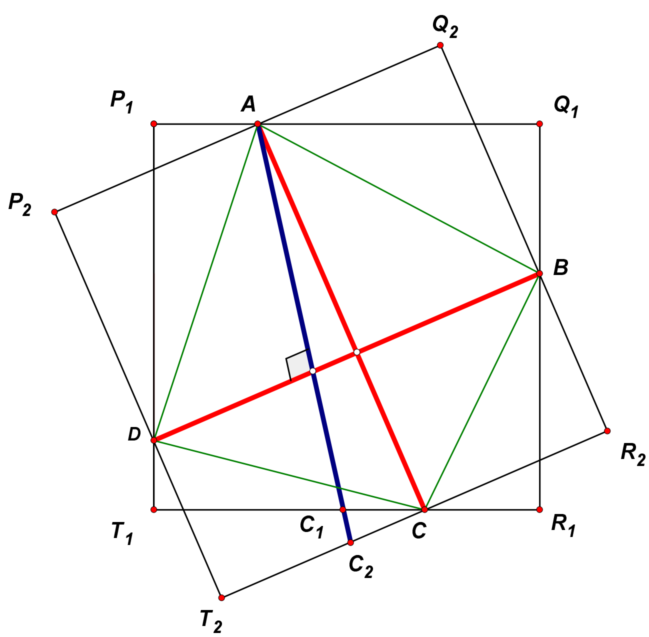
Since it directly follows from the problem condition that $AG/KG = AB/KB = AC/KC = AO/KO = 2$, the points B, G, O, C belong to Apollonius circle for the segment AK, and $\lambda = 2$. It is also clear that $\angle BGC = \angle BOC$ (or $180^\circ - \angle BOC$).

8. (D. Tereshin) Three rectangles are circumscribed about a convex quadrilateral $ABCD$. Two of these rectangles are squares. Is it true that the third rectangle has to be a square as well? (A rectangle is circumscribed about a quadrilateral $ABCD$ when there is a single vertex of the quadrilateral on each side of the rectangle).

Solution. The third rectangle also has to be a square. The proof is based on the following property of a square: *Let the points A and C lie on a pair of opposite sides of a square, while B and D lie on the other. Then conditions $AC \perp BD$ and $AC = BD$ are equivalent* (see the figure).



This is a direct consequence of congruence between the right triangles shown on the figure. Let us now draw a line from point A of our quadrilateral (inscribed in two squares), that is perpendicular to BD. Mark the points of its intersection with respective sides of the square: C_1 and C_2 (see the figure).



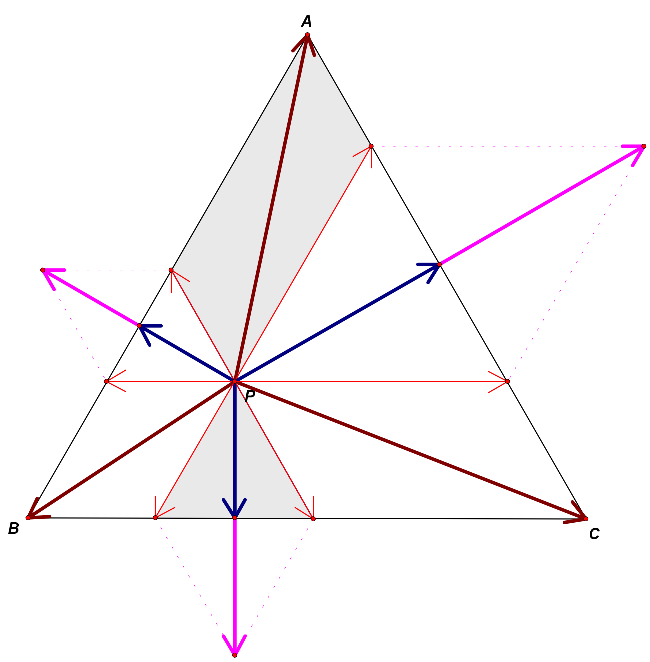
It follows from the above property of the square that $AC_1 = BD$ as well as $AC_2 = BD$, i.e. $AC_1 = AC_2$ and the points C_1, C_2 must concur. But the sides of two squares containing these points have only one common point, that is C. So the perpendicular drawn coincides with AC and therefore the diagonals of quadrilateral $ABCD$ are equal and perpendicular. It is apparent that if a quadrilateral with this property is inscribed into a rectangle then the rectangle is a square.

9. (A. Myakishev) Let O be the center of a regular triangle ABC. From arbitrary point P on the plane the perpendiculars were dropped to the sides of the triangle or their extensions. Let M be the intersection of medians of the triangle with vertices in the feet of the perpendiculars. Prove that M is a midpoint of segment PO.

Solution. In terms of vectors we need to prove that $2\vec{PM} = \vec{PO}$. As it is known, if G is the intersection of medians in a certain triangle ABC, then for an arbitrary point P the following equation holds: $3\vec{PG} = \vec{PA} + \vec{PB} + \vec{PC}$. With this property in mind we can restate the problem as follows:

Let there be a regular triangle ABC and an arbitrary point P. Let us consider vectors $\vec{PA}$, $\vec{PB}$, $\vec{PC}$, as well as three vectors $\vec{n_a(P)}$, $\vec{n_b(P)}$ and $\vec{n_c(P)}$, each of which originates from point P and ends at the foot of the perpendicular dropped from point P to a side of the triangle. Then $2(\vec{n_a(P)} + \vec{n_b(P)} + \vec{n_c(P)}) = \vec{PA} + \vec{PB} + \vec{PC}$.

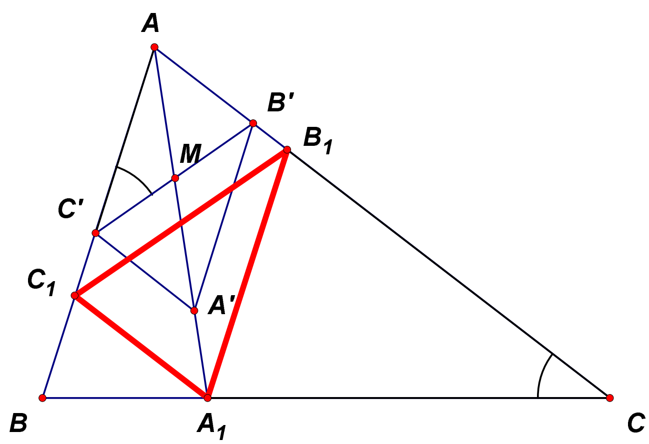
To prove this, let us consider six more vectors each of which lies on the line passing through point P and is parallel to a side of the triangle (see the figure).



The origin of every such vector is at point P while the endpoint is on a certain side of the triangle. (The figure shows P within the triangle). The vectors linking P with vertices as well as the vectors ending at the feet of perpendiculars can be easily expressed through these vectors. It is so because parallel lines split the triangle into regular triangles and parallelograms. As we see, our statement is proved. It is also easy to ascertain that the same argument fits the case when P lies outside of triangle ABC.

10. (T. Yemelyanova) Cut a non-isosceles triangle into four similar triangles not all of which are congruent.

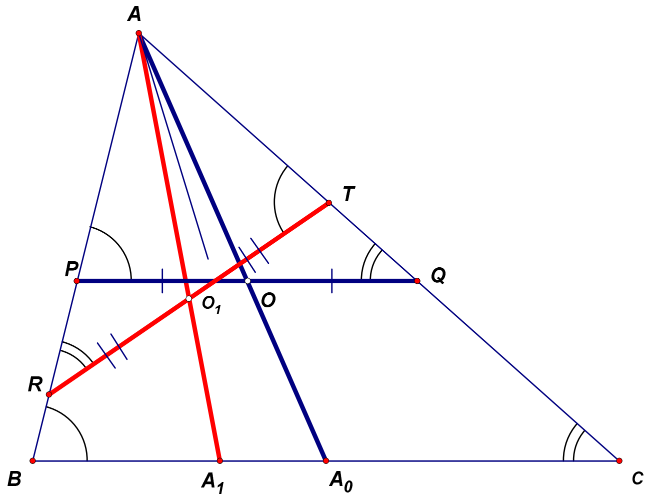
Solution. Let $AB \neq AC$. Draw a segment $B'C'$ such that $\angle AC'B' = \angle ACB$ (see the figure).



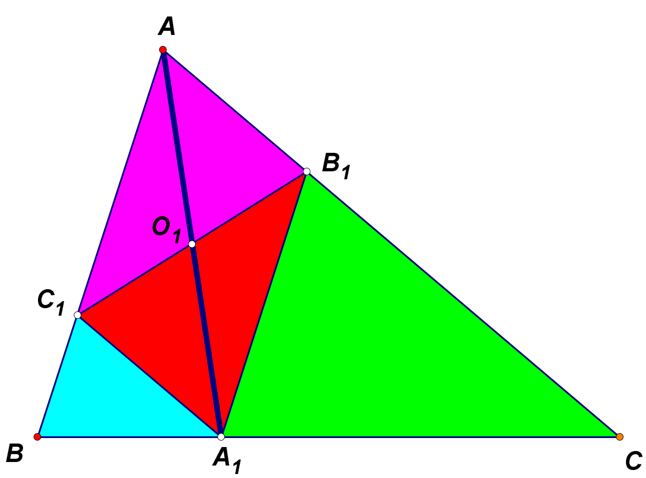
Clearly the triangles ABC and $AB'C'$ are similar, at the same time $B'C'$ is not parallel to BC. Denote the midpoint of $B'C'$ as M and complete triangle $AB'C'$ to a parallelogram $AB'A'C'$. Then find the point A_1 of intersection between AM and BC and draw the parallelogram $AB_1A_1C_1$. The segments A_1C_1, B_1A_1 Pë B_1C_1 make up for the cuts in question.

Note: Essentially, the above solution makes use of the so-called symmedian of the triangle, which is the line symmetrical to the median about the corresponding angle bisector. Let us call *a parallel* (to side BC) of the triangle any segment PQ with endpoints on lines AB and AC, parallel to BC. Then clearly $\angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$. Let us call *an anti-parallel* (to side BC) of the triangle any segment RT with endpoints on lines AB and AC, such that $\angle ART = \angle ACB$ and $\angle ATR = \angle ABC$. (It is not difficult to check that, in particular, the segment formed by the feet of the respective altitudes of the triangle is an anti-parallel). Obviously, the segment is a parallel if and only if the corresponding median splits it in half. Since line symmetry preserves angles and lengths of segments, this statement implies the following

Lemma: a segment is an anti-parallel if and only if the respective symmedian divides it in half (see the figure).



Let us now perform the cuts in question (see the figure).



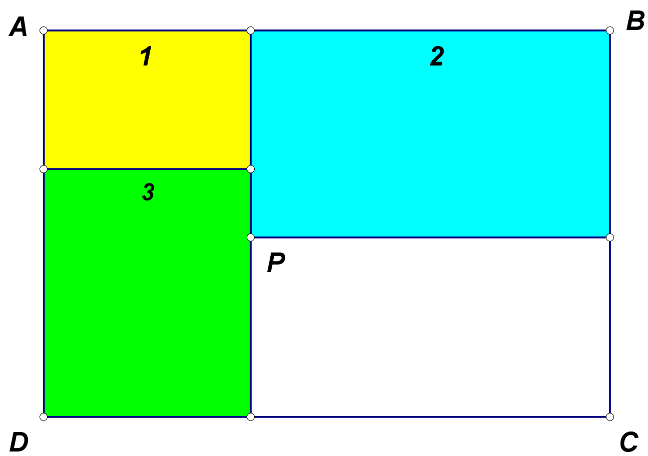
Assume that $AB \neq AC$. Let AA_1 be a symmedian of the triangle, A_1C_1 and A_1B_1 be parallels to sides AC and AB respectively. Since $A_1C_1AB_1$ is a parallelogram, its diagonals are bisected by their intersection point. In other words, the midpoint of C_1B_1 lies on the symmedian and, therefore, according to the lemma, the segment C_1B_1 is an anti-parallel.

It can be easily tested that triangles $A_1B_1C_1$, AB_1C_1, C_1BA_1 and B_1A_1C are similar to triangle ABC and not all of them are congruent. (Indeed, clearly for a non-isosceles triangle the foot of the symmedian A_1 does not coincide with the midpoint of BC. It can even be shown that $BA_1/CA_1 = AB^2/AC^2$ which is another interesting property of a symmedian).

11. (L. Yemelyanov) A square has been cut into n rectangles with sides equal to $a_i \times b_i$, $i = 1, \dots, n$. At what minimal n all numbers in the tuple $a_1, \dots, a_n, b_1 \dots, b_n$ can occur to be different?

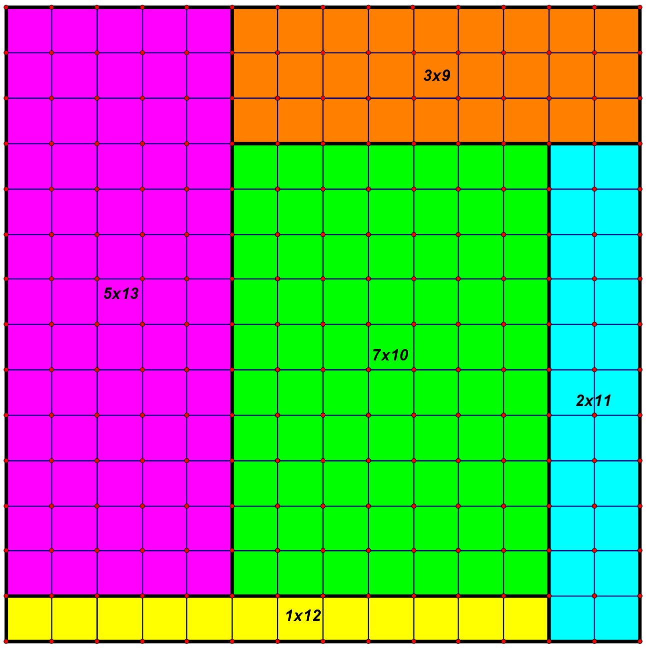
Solution. The minimal value is $n = 5$. Firstly let us show that no rectangle (specifically, no square) can be cut into either 2, or 3, or 4 rectangles with different sides. It is obvious that if a rectangle is cut into 2 rectangles then they will have a common side. Let the rectangle be cut into 3 rectangles. Then one of them contains two vertices of the initial rectangle (as three rectangles have to cover all 4 vertices of the initial rectangle), and our task is reduced to the preceding case (as the remaining part is a rectangle to be split in two).

Finally, let us assume that the rectangle is split into 4 others. We have two possibilities: either one of the resulting rectangles contains two vertices of the initial rectangle (thus reducing the task to splitting the rectangle into three parts), or each of the resulting rectangles contains 1 vertex of the initial rectangle. In the latter case let us consider 2 rectangles containing adjacent vertices (see the figure).



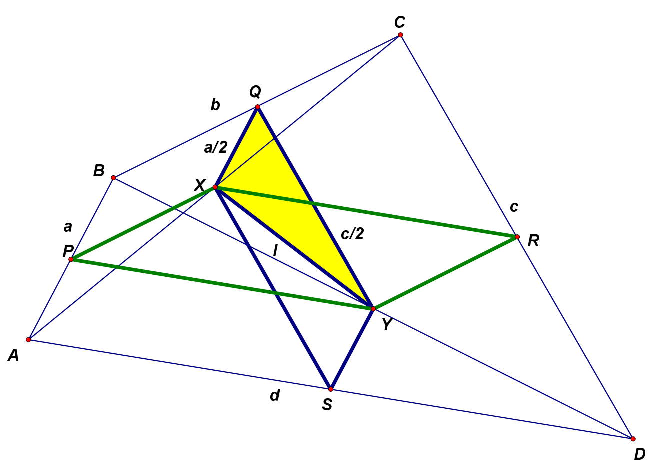
They must be tangent (because, evidently, if there would be a gap, it could not be covered with two rectangles containing the remaining two vertices of the initial rectangle). Let us consider that rectangle of the remaining two, which contains point P. It cannot contain the vertex C, therefore it contains the vertex D which means that it has a common side with the first rectangle.

Now let us present one of the possible ways to cut a square into five different rectangles (see the figure).



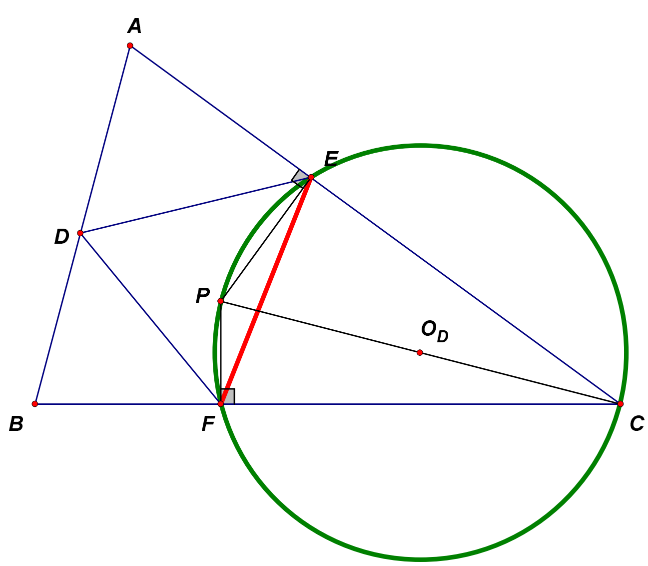
12. (V. Smirnov) Construct a quadrilateral with the given sides a, b, c and d and the distance l between the midpoints of the diagonals.

Solution. Let $ABCD$ be the quadrilateral in question with $AB = a$, $BC = b$, $CD = c$, $DA = d$, and let P, Q, R, S, X, Y be the midpoints of segments AB, BC, CD, DA, AC, BD respectively. As QX, SY are midlines of triangles ABC and ABD, we have $QX = YS = a/2$. Similarly $QY = XC = c/2$. Hence by fixing points X and Y and drawing triangles XYQ and XYS we will find points Q and S. Points P and R can be found similarly. By drawing the lines parallel to QX, PX, QY, PY respectively through P, Q, R, S, we will obtain the quadrilateral in question (see the figure).



13. (A.Zaslavsky) A triangle ABC and two lines l_1 and l_2 are given. A line parallel to l_1 and intersecting AC at point E, as well as a line parallel to l_2 and intersecting BC at point F are both drawn through an arbitrary point D. Construct the point D such that the segment EF has minimal length.

Solution. Let P be the point of intersection for perpendiculars to AC at point E and to BC at point F. When D moves along AB, the sides of quadrilateral $DEPF$ maintain their directions. Since three vertices of the quadrilateral move along straight lines, the fourth vertex also moves along a straight line. Therefore, the midpoint of segment   which is the center of the circumcircle for triangle CEF also moves along a straight line (see the figure). It means that all these circles have a common chord. Thus apart from common point C they share another point Q. Since the chord EF subtends the constant angle C, its length will be minimal provided the radius of the circle circumscribed about EF is minimal. Among all the circles containing the common chord, however, the minimal radius will evidently be in the circle for which this chord CQ is a diameter.

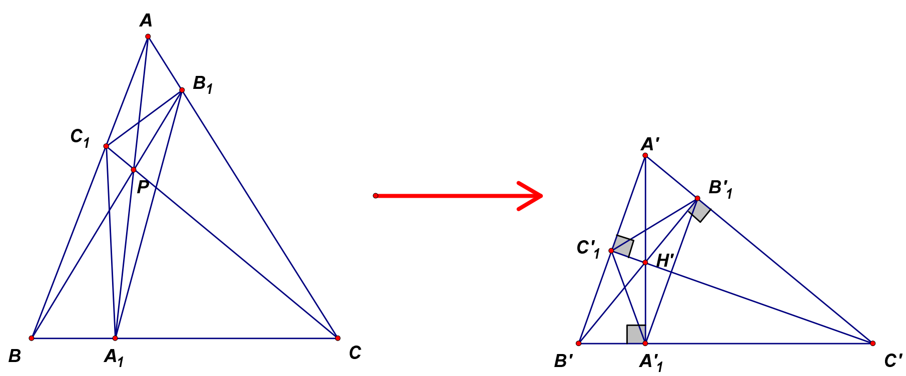


This implies, for instance, the following method for constructing the point D. Draw a line parallel to l_2 through A and find point U of its intersection with BC. Draw a line parallel to l_1 through B and find point V of its intersection with AC. Let Q be the second intersection point of the circumcircles for ACU and BCV, while E is the second point of intersection between line AC and the circle with diameter CQ. Then the line parallel to l_1 and passing through E will intersect AB at the point in question.

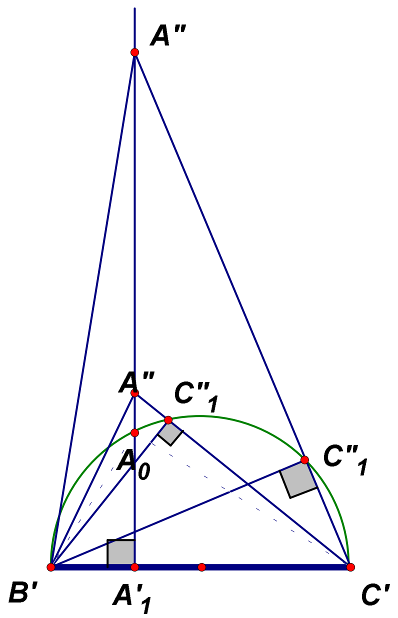
14. (L. Yemelyanov). Let P be an arbitrary point inside triangle ABC. Denote by A_1, B_1 and C_1 the points of intersection of lines AP, BP and CP with sides BC, CA and AB respectively.

Rank areas of the triangles AB_1C_1,A_1BC_1, A_1B_1C and denote the smallest as S_1, the middle one as S_2, and the largest one as S_3. Prove that $\sqrt{S_1S_2} \le S \le \sqrt{S_2S_3}$, where S is the area of triangle $A_1B_1C_1$.

Solution. Method one. Let us call triangle $A_1B_1C_1$ the *cevian* triangle of point P. It turns out that any triangle ABC can be mapped onto some acute triangle $A'B'C'$ by a suitable affine transformation such that point P is mapped to its orthocenter, while the cevian triangle P is mapped to the orthotriangle (i.e. the triangle formed by the feet of altitudes, see the figure).



Indeed, take an arbitrary segment $B'C'$ and mark a point A_1' on it so that $B'A_1'/A_1'C' = BA_1/A_1C$. Next, draw a perpendicular from this point to $B'C'$. Then find point A_0 on this perpendicular, such that $\angle B'A_0C' = \frac{\pi}{2}$ (the point of intersection of the perpendicular with the circle based on $B'C'$ as the diameter), see the figure.



Next, consider point A'' on this perpendicular and drop the altitude $B'B_1''$ to $A''C'$. If A'' is located close to A_0, then the ratio $C'B_1''/B_1''A''$ is very high. If A_0 tends to infinity along the perpendicular, then the ratio tends to zero. By continuity argument, there is a certain point A' on the perpendicular, such that $C'B_1''/B_1''A' = CB_1/B_1A$. The corresponding equality for the third pair of ratios is guaranteed by Ceva theorem. As it is known, for any two triangles ABC and $A'B'C'$ there is a single affine transformation that maps the first triangle to the second triangle. Since an affine transformation maps lines into lines and retains the ratio of side lengths, we have found an affine transformation that maps the cevian triangle to some orthotriangle.

Moreover, an affine transformation also retains the ratio of areas. The above implies that it suffices to prove the problem statement *for an acute triangle and its orthocenter*. Without loss of generality we can assume that areas of triangles AB_1C_1, A_1BC_1, A_1B_1C are respectively equal to S_1, S_2, and S_3. These triangles are similar to the initial one with ratios $\cos A$, $\cos B$, $\cos C$ respectively. Therefore,

$$S_1 \le S_2 \le S_3 \Leftrightarrow \frac{S_1}{S_{ABC}} \le \frac{S_2}{S_{ABC}} \le \frac{S_3}{S_{ABC}} \Leftrightarrow \cos^2 A \le \cos^2 B \le \cos^2 C \Leftrightarrow A \ge B \ge C$$

as all angles are acute, cosines are positive and decreasing. It follows from the latter chain of inequalities that $C \le \frac{\pi}{3} \le A$.

Let us now prove that $\sqrt{S_1 S_2} \le S$. This is equivalent to $\sqrt{\frac{S_1}{S} \cdot \frac{S_2}{S}} \le 1$. After squaring and dividing the numerator and the denominator by S^2 we get

$$\frac{\cos^2 A \cos^2 B}{(1 - \cos^2 A - \cos^2 B - \cos^2 C)^2} \le 1.$$

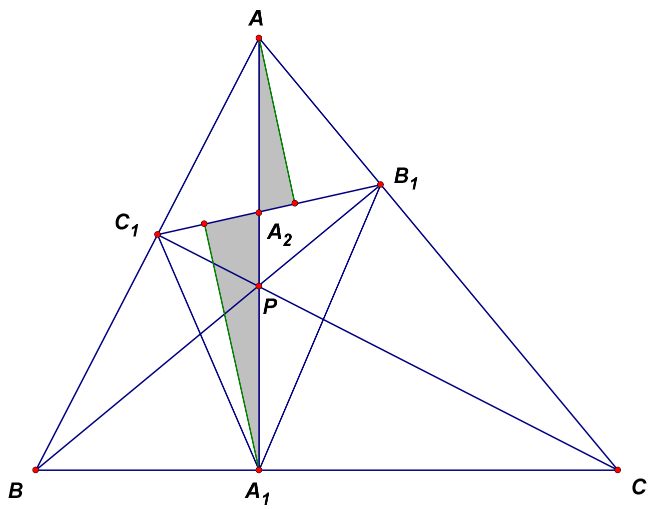
However, it is easy to check that the following equality holds for any triangle: $1 - \cos^2 A - \cos^2 B - \cos^2 C = 2\cos A \cos B \cos C$, therefore our inequality is equivalent to

$$\cos^2 A \cdot \cos^2 B \le 4\cos^2 A \cdot \cos^2 B \cdot \cos^2 C \Leftrightarrow \frac{1}{4} \le \cos^2 C,$$

hence $C \leq \frac{\pi}{3}$. It is proved similarly that $\sqrt{S_2 S_3} \geq S$.

Method two.

Without loss of generality, we can assume the areas of triangles AB_1C_1, A_1BC_1, A_1B_1C to be respectively equal to S_1, S_2, and S_3 (see the figure).



Let the point P have (relative to triangle ABC) *normalized* barycentric coordinates $p : q : r$, i.e. $p + q + r = 1$. Since point P is located inside the triangle, p, q, r are positive. Let us use them to express S_1/S. Denote by A_2 the point of intersection between B_1C_1 and AA_1. Since triangles AB_1C_1 and $A_1B_1C_1$ share a common base, apparently $S_1/S = AA_2/A_1A_2$. Next, it is clear that the coordinates of A_2 are $(2p : q : r)$ (the mass center of the system of $2pA$ and $(q+r)A_1$ is on the line AA_1, whereas the mass center of the system of $(p+q)C_1$ and $(p+r)B_1$ is on the line B_1C_1). Hence by the lever rule we have $AA_2/A_1A_2 = (q+r)/2p = (1-p)/2p$. In the same manner, $S_2/S = (1-q)/2q$ Pë $S_3/S = (1-r)/2r$. Since $S_1 \leq S_2 \leq S_3$, it follows that $p \geq q \geq r$. Considering equation $p + q + r = 1$ we also have $p \geq \frac{1}{3} \geq r$. Let us now prove that $\sqrt{S_1 S_2} \leq S$.

Substution of the previously determined values of ratios gives $(1-p)(1-q) \leq 4pq$, i.e. $r \leq 3pq$. However, $\frac{pq}{r} \geq \frac{1}{3}\frac{q}{r} \geq \frac{1}{3}$. In the same manner it can be proved that $\sqrt{S_2 S_3} \geq S$ (by using inequality $r \leq \frac{1}{3}$).

Note: The ideas fundamental for the above proof can be utilized without mass geometry. For instance, one can introduce ratios $\alpha = BA_1/CA_1$, $\beta = CB_1/AB_1$, $\gamma = AC_1/BC_1$. According to Thales theorem (by introducing respective parallels) one then can express the ratio of areas using those ratios.

Method three (Avksentyev Yevgeny, city of Rostov-on-Don, Gymnasium no. 5). The following nice solution is based on the so-called *Möbius theorem*:

Let P be an arbitrary point inside triangle ABC. Denote as A_1, B_1 and C_1 the points of intersection of lines AP, BP and CP with sides BC, CA, AB respectively. Let areas of *triangles AB_1C_1, A_1BC_1,A_1B_1C and $A_1B_1C_1$ be S_1, S_2, S_3 and S respectively. Then $S^3+(S_1+S_2+S_3)S^2-4S_1S_2S_3=0$.*

(This is not hard to prove using for instance the ratios found in the above argumentation.)

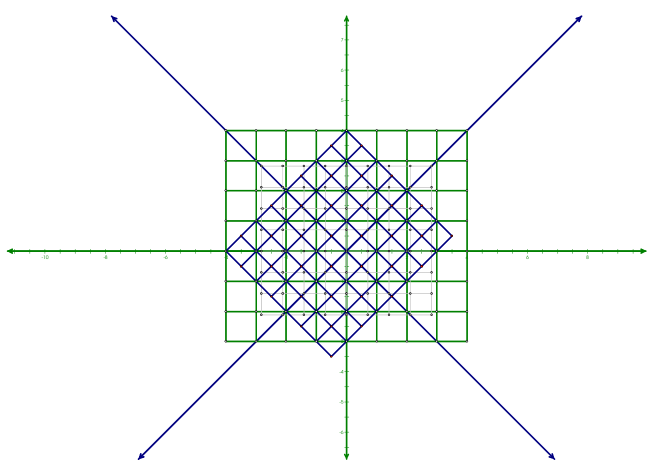
Let us consider the function $\Phi(x)=x^3+(S_1+S_2+S_3)x-4S_1S_2S_3$. By Möbius theorem, $\Phi(S)=0$. Moreover, it is obvious that $\Phi(x)$ is ascending at $(0,\infty)$ (as a sum of two ascending functions). Therefore we are left to demonstrate that $\Phi(\sqrt{S_1S_2})\leq 0\leq \Phi(\sqrt{S_2S_3})$ when $S_1\leq S_2\leq S_3$. However $\Phi(\sqrt{S_1S_2})=S_1S_2(\sqrt{S_1S_2}+S_1+S_2-3S_3)$, while

$$\sqrt{S_1S_2}+S_1+S_2-3S_3\leq \frac{3}{2}(S_1+S_2)-3S_3\leq 0$$

(the geometric mean of two positive values does not exceed their arithmetic mean). The second inequality is proved in the same manner.

15. (A.Zaslavsky) A circle with its center at the origin is given. Prove that there exists a circle with a shorter radius, which contains equal or greater number of points with integer coordinates.

Solution. Consider the rotational homothety with the center at the origin, the ratio of $\frac{1}{\sqrt{2}}$ and the rotation angle of $\frac{\pi}{4}$. If the squared circle radius is an even number, then all integer points map into integer points and we get the circle in question. If the square of the radius is an odd number, then all integer points map into centers of unit squares with vertices in integer points, and the circle in question is obtained by translation along the vector $(\frac{1}{2},\frac{1}{2})$. This is rather obvious visually: in the figure we have the action on integer grid, by contraction first and then by rotation.



From the purely formal standpoint, the point with coordinates (x,y) maps to $(x'=\frac{x-y}{2}, y'=\frac{x+y}{2}$ under the above rotation and dilation. If the squared radius is an even number then x and y are of the same parity, therefore x', y' are integers and $x'^2+y'^2=\frac{x^2+y^2}{2}=R^2/2$. Alternatively, if the squared radius is an odd number then the parity of x and y is different. As a consequence, after the shift by the vector $(\frac{1}{2}, \frac{1}{2})$ we will get an integer point $(x\,y")$, and $(x\frac{1}{2})^2 + (y\frac{1}{2})^2 = \frac{R^2}{2}$.

16. (A.Zaslavsky, B.Frenkin) In an acute non-regular triangle 4 points were marked: the centers of the incircle and the circumcircle, the center of mass (the intersection point of the medians) and the orthocenter (the intersection point of the altitudes). After that, the original triangle was erased. It turned out to be impossible to determine which point corresponded to which center. Find the angles of the triangle.

Solution. The triangle that fits the problem condition is an isosceles triangle with angles at its base equal to $\arccos\frac{1}{4}$.

Let ABC be the triangle in question, and A_1, B_1, C_1 be the midpoints of sides BC, CA, AB respectively. As there is a homothety of triangles ABC and $A_1B_1C_1$ with the center M (with ratio of $-\frac{1}{2}$), and the circumcenter O for triangle ABC is the orthocenter for triangle $A_1B_1C_1$, point M lies on segment OH (H being the orthocenter of triangle ABC), and $HM = 2MO$ (the line containing these three centers is known as Euler line of the triangle ABC).

Therefore if point I (incenter) does not belong to the same line as three other points then we can unambiguously determine the role of each point in the triangle ABC. Observe that this line contains no more than one vertex of the triangle. So we can assume that points A and B do not belong to it. As $\angle OBA = \angle HBC = \frac{\pi}{2} - \angle C$, line BI is the bisector of angle HBO. So, point I lies on the segment OH, and $OI = 2IH$ (otherwise the role of the points is determined unambiguously). According to the property of the bisector we have $BO = 2BH$. Following the same argument we will get $AO = 2AH$. Therefore, $AH = BH = R/2$, where R is the radius of the circumcircle of ABC. Now observe that $AH = 2OA_1$ (and these segments are parallel), which also follows from the homothety shown at the start of the solution. In addition, it is apparent that $OA_1 = R\cos A$. Therefore $AH = 2R\cos A$ and $\cos A = \frac{1}{4}$. Equation $\cos B = \frac{1}{4}$ is proved in the same way.

17. (A.Myakishev) The incircle of triangle ABC has center I and tangency points P, Q, R with sides BC, CA and AB respectively. With a ruler only, construct the point K where the circle passing through B and C is (internally) tangent with the incircle.

Solution. According to the well-known Steiner theorem, *if in the plane there is a fixed circle with the marked center, then it is possible to draw any figure with a ruler alone, that could have been constructed with ruler and compass.* However, application of standard methods, not taking into account peculiarities of the construction in question, requires a considerable number of “steps”. Of course, it is desired to use the minimal number of lines. It turns out that just four lines do suffice! Firstly observe that if $AB = AC$ then the construction is obvious (K matches the point diametrically opposite to point P), so let us consider the case when $AB \neq AC$.

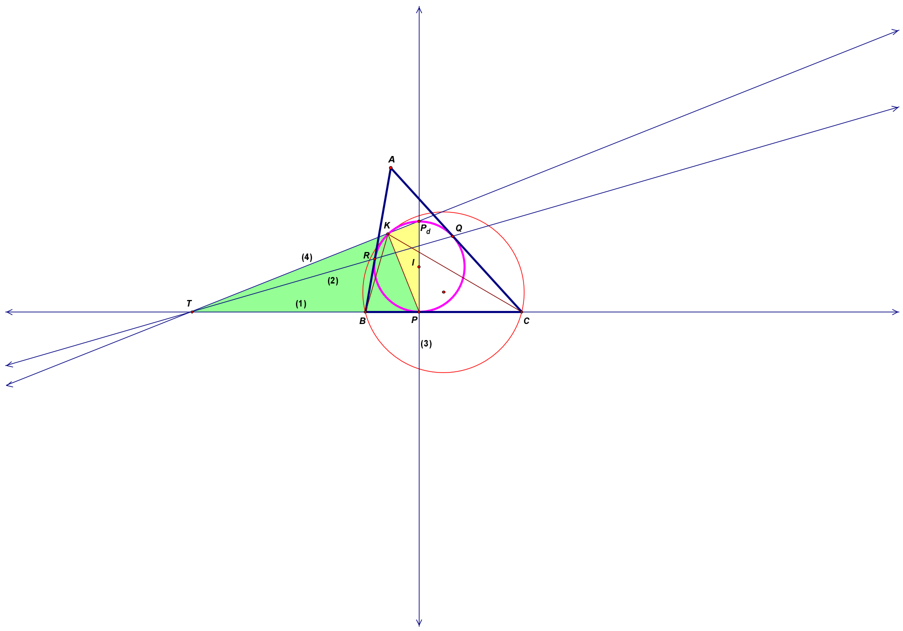
Construction algorithm:

1. Draw a line BC.

2. Draw a line QR and mark point T of its intersection with line BC.

3. Mark point P_d that is diametrically opposite to point P.

4. Draw the line P_dT and mark point K as the second point of intersection of this line and the incircle. Point K is the point in question (see the figure).



Proof: Clearly the point T will split the segment BC in the same ratio as the point P (by theorems of Ceva and Menelaus). Assume the circle in question has been constructed. Then KP is the bisector of the angle BKC (by the well-known *lemma of Archimedes: let the line cross a given circle at points B and C; consider an arbitrary circle tangent to the given circle at point K and tangent to BC at point P; then the line KP crosses one of BC arcs at its midpoint*).

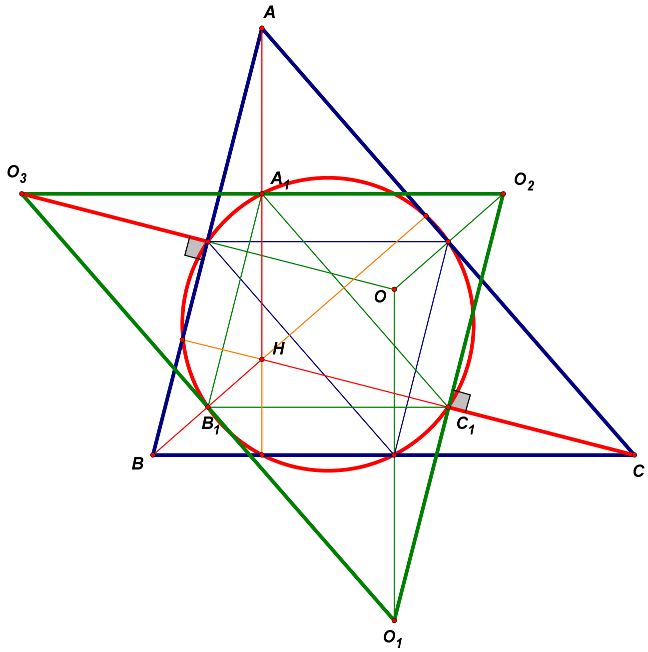
By the bisector property, $BP/CP = KB/KC = \lambda \neq 1$. Therefore the point lies on *the circle of Apollonius* (see solution to the problem 7, method 3) for the segment BC with the ratio of λ. Since PT is its diameter, $\angle TKP = 90°$ or equivalently $\angle PKP_d = 90°$. This argument justifies our construction.

18. (V. Protasov) There are three lines l_1, l_2, l_3 in the plane that form a triangle, as well as a marked point O, the center of its circumcircle. For an arbitrary point X in the plane, let us denote by X_i the point symmetrical to about the line l_i.

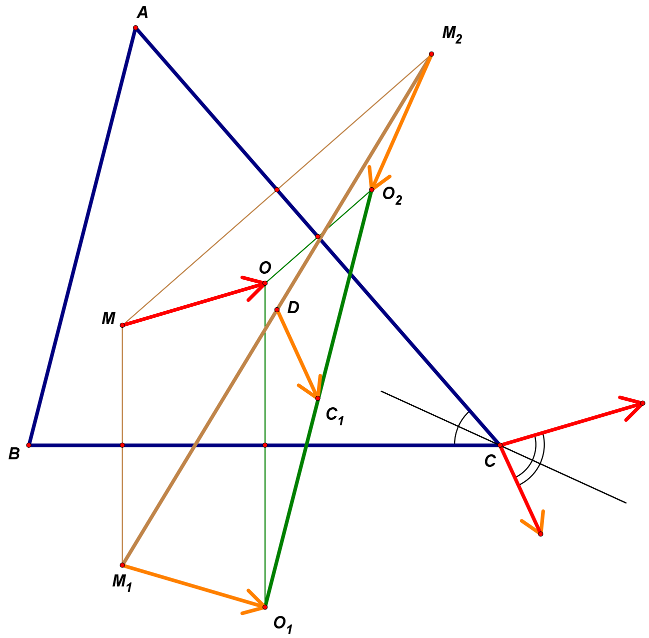
a) Prove that for an arbitrary point M the lines linking midpoints of segments O_1O_2 and M_1M_2, O_2O_3 and M_2M_3, O_3O_1 and M_3M_1 concur.

b) Where can their point of intersection lie?

Solution. Method one. Let us show that these lines intersect at the point belonging to *Euler circle*. (Reminder: Euler circle for a triangle ABC is the circle circumscribed about its medial triangle, i.e. passing through midpoints of its sides. Feet of the altitudes and the midpoints of segments linking the orthocenter with the vertices also lie on this circle.) Let ABC be the triangle formed by lines l_i, let H be its orthocenter. Then the midpoints of O_1O_2, O_2O_3, O_3O_1 coincide with the midpoints of segments AH, BH, CH (in the sequel, called A_1, B_1, C_1) and, therefore, lie on Euler circle of the triangle ABC. The sides of triangle $O_1O_2O_3$ are in fact parallel to the midlines of triangle ABC and are twice longer because they are linked with the latter by the homothety with center O and ratio 2. Therefore, triangle $O_1O_2O_3$ is centrally symmetrical to ABC. So the line passing through C and midpoint of O_1O_2, is parallel to the line passing through O_3 and the midpoint of AB, i.e. coincides with an altitude of triangle ABC, whereas H is the homothety center for ABC and $A_1B_1C_1$ (see the figure).



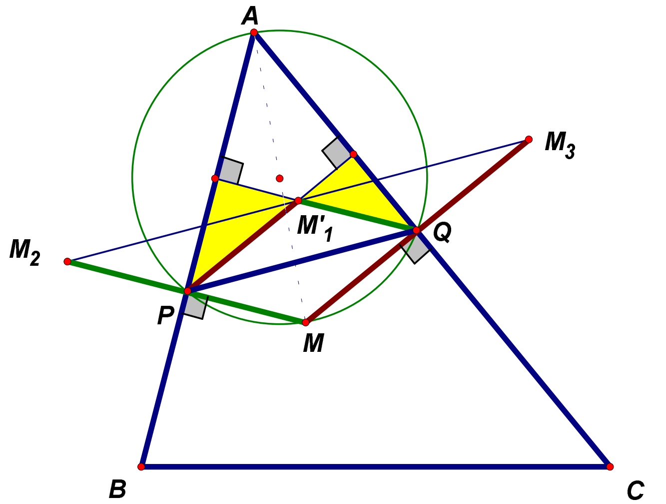
Let M be an arbitrary point, whereas D be the midpoint of M_1M_2. Then $\vec{DC_1} = (\vec{DO_1} + \vec{DO_2})/2$ and, since $\vec{M_1O_1}$ and $\vec{M_2O_2}$ interchange under the rotation around point C by angle $2C$, vector $\vec{DC_1}$ forms an angle equal to C with each of them. Furthermore $\vec{M_1O_1}$ and $\vec{M_2O_2}$ map to $\vec{MO}$ under the symmetry about CB and CA, respectively. Therefore $\vec{DC_1}$ and $\vec{MO}$ form equal angles with the bisector of C (this implies also equal angles with the bisector of C_1 in triangle $A_1B_1C_1$), see the figure.



Arguing similarly for two other midpoints we conclude that the lines linking A_1, B_1, C_1 with the midpoints of sides of triangle $M_1M_2M_3$ are symmetrical about bisectors of triangle $A_1B_1C_1$ to the lines intersecting A_1, B_1, C_1 and parallel to OM. To complete, we shall use a classical theorem of plane geometry: *a triple of lines drawn from the vertices of a triangle concurs at a point belonging to the circumcircle of the triangle iff the lines, symmetrical to the given lines about the bisectors of respective angles, are parallel.* (A rather simple proof uses simple calculation of angles).

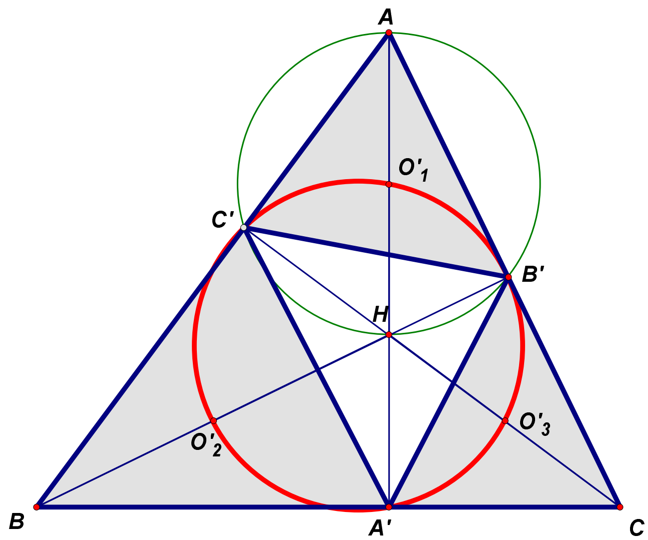
According to this theorem, the triple of lines in our problem intersects on the circumcircle of triangle $A_1B_1C_1$, i.e. on Euler circle of the initial triangle.

Method two. (Avksentiev Yevgeniy, city of Rostov-on-Don, Gymnasuim no. 5). To start with, let ABC be the triangle formed by lines l_i; H be its orthocenter and A', B', C' be the feet of altitudes dropped to BC, CA, AB respectively. Let us now give the following definition: Let there be two similar shapes Ψ_1 and Ψ_2 as well as some similarity transformation $\mathcal{H}$, mapping one shape to another. Let us say that two shapes Φ_1 and Φ_2 are *equally located relative to* Ψ_1 *and* Ψ_2 if the transformation $\mathcal{H}$ also maps Φ_1 into Φ_2. Now let us prove that points M_1' (the midpoint of M_3M_2) and M are equally located relative to triangles $AB'C'$ and ABC (as it is known, these triangles are similar with a ratio of $\frac{1}{\cos A}$, and this similarity can be defined as a composition of an axial symmetry about the bisector of angle A and a homothety with center at A, see note to solution of problem 10). It suffices to show that $AM_1' = AM/\cos A$ (the ratio of distances from the similarity center to these points equals the similarity ratio) and that the ratio of distances from point M_1' to AB and AC is reversely proportional to the ratio of distances from M to the same sides (i.e. line AM_1' maps into line AM under symmetry about the bisector of angle A), see the figure.



Since $M_1'Q$ is the midline of triangle M_2MM_3, it is perpendicular to AB. By the same logic $M_1'P$ is perpendicular to AC, therefore M_1' is the orthocenter of triangle APQ, and hence $AM_1' = 2\rho\cos A$, where ρ is the radius of the circumcircle of APQ (as it was shown in the solution to problem 16). It is evident that $\rho = AM_1'/2$. Equality of the inversed ratios of distances to the sides follows from the similarity of the triangles shaded in the picture.

In the same manner we prove that M_2' and M are equally located relative to $A'BC'$ and ABC, while M_3' and M relative to $A'B'C$ and ABC. Now, if we choose for M the point O, the center of the circumcircle of ABC, then evidently points O_1', O_2', O_3' will be the midpoints of segments linking orthocenter H of triangle ABC with its vertices (since the lines linking a vertex of the triangle with H and O are symmetrical about the respective bisector: the fact we have already encountered in the solution for problem 16; therefore, for instance, point O_1' lies on the line AH, furthermore $AO = R$ and $AH = 2R\cos A$, therefore $AO_1' = AH/2$ and so on). It follows from the proven equal location that the lines $O_1'M_1'$, $O_2'M_2'$ and $O_3'M_3'$ are equally located relative to triangles $AB'C'$, $A'BC'$ and $A'B'C$ (see the figure).



Additionally, it is evident (since respective elements of triangles map into respective ones under similarity, and in particular circumcenters do) that the lines $O_1'A'$, $O_2'B'$ and $O_3'C'$ are equally located relative to the same triangles, and all these points are located on the Euler circle of triangle ABC. Finally, we can conclude that the angles between pairs $O_1'M_1'$ and $O_1'A'$, $O_2'M_2'$ and $O_2'B'$, $O_3'M_3'$ and $O_3'C'$ are equal.

We have thus proved that the lines $O_1'M_1'$, $O_2'M_2'$ and $O_3'M_3'$ intersect in the same point located on Euler circle of the original triangle.

19. (A. Tarasov) As is well-known, the Moon rotates around the Earth. Let us assume the Moon and the Earth to be points. Assume also that the Moon travels a circular orbit around the Earth and makes full circle in one month. There is an UFO in the plane of the lunar orbit. The UFO can move in jumps over the Moon and over the Earth: from an old location (point A) it instantly moves to the new one (point A') so that the midpoint of A' is either the Moon, or the Earth. Between the jumps, the UFO is located in space motionless.

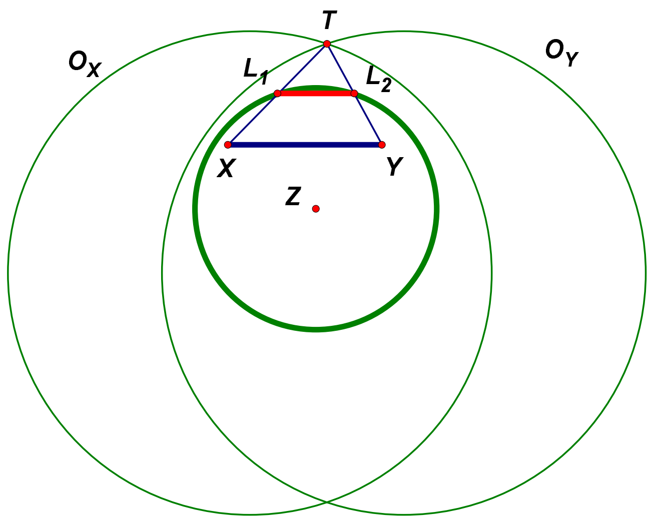
a) Determine the minimal number of jumps sufficient for the UFO to reach any random point within lunar orbit from any another random point within lunar orbit.

b) Prove that the UFO is able, within infinite number of jumps, to reach any random point within lunar orbit from any another random point within lunar orbit in any period of time, for example, in one second.

Solution. a) It is possible to reach any point within lunar orbit from any other point within lunar orbit in two jumps. For that, the UFO needs to jump via the Moon. The first jump is when the Moon is at point L_1. The second jump is when the Moon is at L_2. In these two jumps the UFO will move in total along the vector $2\vec{L_1L_2}$ (since the compoisition of two central symmetries is a parallel shift by the doubled vector from the first center to the second one, see the figure).

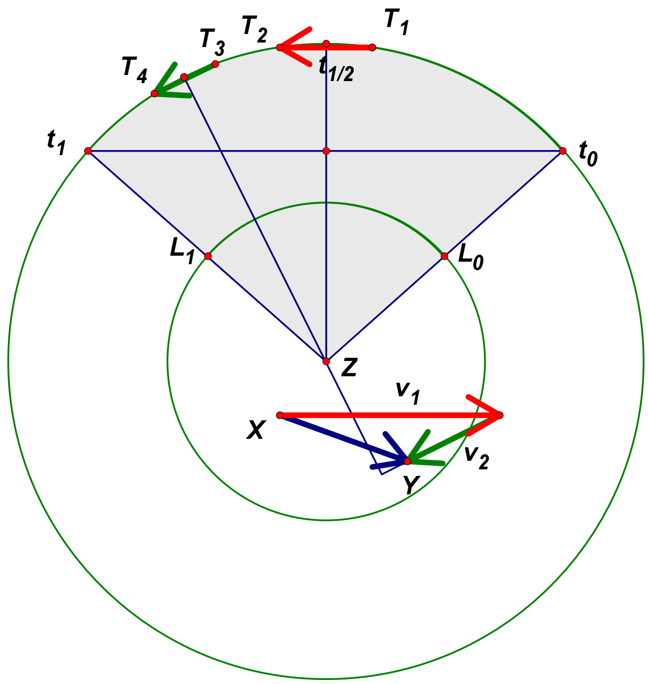
For any two points X and Y within the orbit it is possible to find a chord L_1L_2 such that $\vec{XY} = 2\vec{L_1L_2}$. This chord can be constructed, for instance, by drawing a diameter parallel to XY, and choosing a segment on it of length $XY/2$, such that its midpoint is the circle center. Then drop the perpendiculars from the endpoints of the segment. These perpendiculars will cut off the chord in question.

Also it is possible to consider homotheties with centers at points X and Y and factor 2. Images of the lunar orbit must intersect, since if the lunar orbit radius is R and the centers of images are O_x, O_y, we have $O_xO_y < XY + 2(XZ + YZ) < 4R$ (see the figure).



b) Let the initial location of the Moon be at L_0, the final one at L_1. Consider a pair of jumps, via the Earth first and then via the Moon, as a twin jump. The UFO moves along the vector $2\vec{ZL}$ in this case. The end of this vector, point T, will lie on a circle arc t_0t_1 with center Z and radius twice as long as the radius of the Moon orbit. Let us denote such vectors simply as $\vec{T}$.

As the jump happens instantly then at any moment the UFO can make an *integer* number of jumps $k\vec{T}$ (in order to jump along the vector $-\vec{T}$, the UFO needs first to jump via the Moon and then via the Earth). Now, in order to arrive to point Y from point X, we need to represent the vector $\vec{XY}$ as a finite sum of vectors consisting of summands $k_i\vec{T}_i$, $k_i \in Z$, where T_i is a certain set of points on circle arc t_0t_1 located consecutively one after another (see the figure).



First, represent $\vec{XY}$ as a sum $\vec{v_1}+\vec{v_2}$, so that both of these vectors are perpendicular to certain radii of our sector. This can be done assuming $\vec{v_1}=\lambda(\vec{t_1}-\vec{t_0})$, $\vec{v_2}=\vec{XY}-\vec{v_1}$. Since $\vec{v_1}$ is perpendicular to the "centerline" radius for all values of λ, and $\vec{v_2}$ in case of increasing the absolute value of λ tends to $-\vec{v_1}$, the vector $\vec{v_2}$ for sufficiently big λ will also be perpendicular to a certain radius. Obviously, there exist integers m_1 and m_2 with a rather big absolute value as well as certain points on the circle arc T_1, T_2, T_3, T_4 located consecutively and such that $\vec{v_1}=m_1(\vec{T_2}-\vec{T_1})$ and $\vec{v_2}=m_2(\vec{T_3}-\vec{T_4})$. The representation in question is obtained.

20. (A.Zaslavsky) Let I be the center of the insphere of tetrahedron $ABCD$; A', B', C', D' be the centers of the circumspheres of tetrahedrons $IBCD$, $ICDA$, $IDBA$, $IABC$ respectively. Prove that the circumsphere of $ABCD$ lies entirely inside the circumsphere of $A'B'C'D'$.

Solution: Let R, r be the radii of the circumscribed and inscribed spheres of $ABCD$; O be the center of the circumsphere of $ABCD$; L be the circumcenter of triangle ABC; H be the projection of I to the plane ABC. The problem condition implies that O and D' lie on the perpendicular to the plane ABC, that passes through L. Therefore the lines OD' and IH are parallel. Furthermore $D'A=D'I$ (as radii of the circumsphere of $IABC$), $OA=R$, $IH=r$.

Let us apply the cosine law twice, to triangles $AD'O$ and $OD'I$:

$$R^2=D'A^2+D'O^2-2D'A\cdot D'O\cos\angle AD'O,$$

$$OI^2=D'I^2+D'O^2-2D'I\cdot D'O\cos\angle ID'O.$$

Subtracting the second equation from the first one, we get:

$$R^2-OI^2=2D'O(D'A\cos\angle AD'O-D'I\cos\angle ID'O).$$

So, $D'O=(R^2-OI^2)/2r$. It can be similarly proved that points A', B', C' are at the same distance from O. Therefore spheres $ABCD$ and $A'B'C'D'$ are *concentrical* and $D'O=\rho$ is the radius of the circumsphere of $A'B'C'D'$. Let us prove that $\rho > R \Leftrightarrow \frac{R^2-OI^2}{2r}$. For this, draw a plane DOI. It intersects the circumscribed and inscribed spheres by circles with centers O, I and radii R, r, while the tetrahedron intersects them by a certain triangle. The vertex D of this triangle lies on the larger circle, while at least one of the two remaining vertices lies within this circle. In addition, the smaller circle lies entirely within this triangle and within the larger circle.

Therefore, if one draws through D chords DX_1 and DY_1 of the larger circle tangent to the smaller circle, then the latter lies strictly inside triangle DX_1Y_1. Now "blow up" the smaller circle, maintaining the center and increasing the radius. Continuity considerations imply that at a certain moment the "overblown" circle (of some radius r') will be inscribed in triangle $DX'Y'$ formed by a pair of tangential lines with vertex in D. The same triangle will be inscribed in the bigger circle. Therefore it satisfies the classical relation which expresses the distance between the centers of inscribed and circumscribed circles via their radii (so called *Euler's formula*):

$$OI' = R^2 - 2Rr'.$$

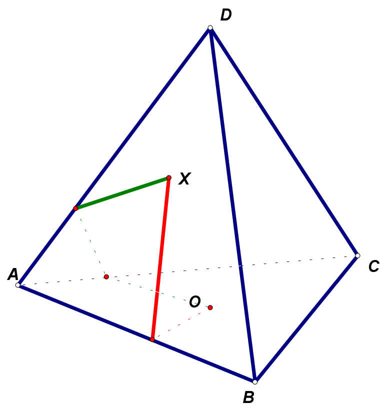
Therefore $r' = (R^2 - OI^2)/2R$. It is also evident that $r' > r$. The problem is solved.

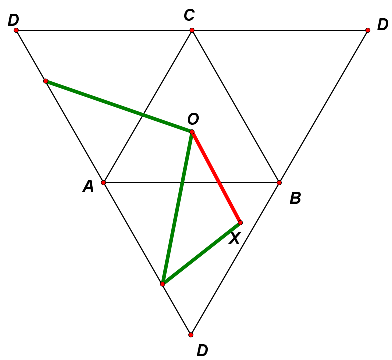
21. (N. Dolbilin) The planet "Tetraincognito" covered by the "ocean" has a shape of a regular tetrahedron with an edge of 900 kilometers. What area of the ocean will be hit by a "tsunami" after 2 hours of "tetraquake" with the epicenter at

a) the center of a face,

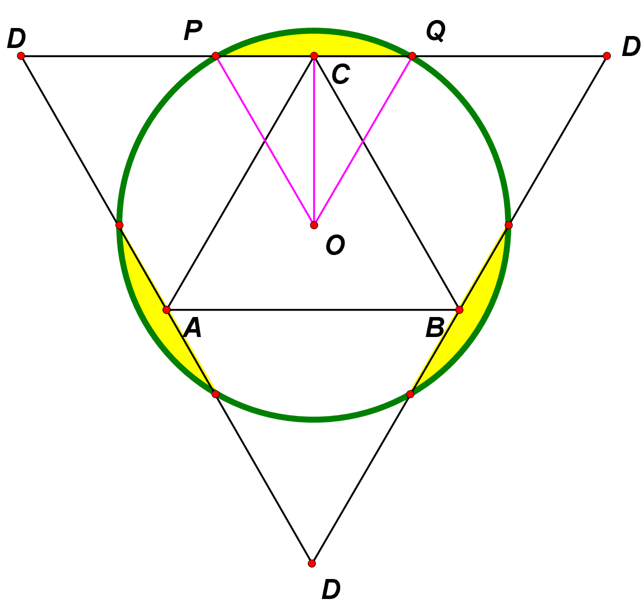
b) the midpoint of an edge, if tsunami propagates at a speed of 300 km/h?

Solution: a) Consider an unfolding in the shape of a regular triangle. Let us prove that the shortest route from the center of the triangle to any its point will be a segment on this unfolding. Let O be the center of face ABC, X be a point on face ABD, and let some route from O to X first cross the edge AC. If we extend this route on the unfolding, we get to some point on the edge AD. However there is a symmetrical way to this point via the edge AB, which is a straightforward way to X (see the figure).





Therefore the area hit by tsunami will be the difference between the area of the disc with radius of 600 km and the triple area of the segment (see the figure).



We have

$$\cos\angle POC = \frac{OC}{OP} = \frac{900}{600\sqrt{3}} = \frac{\sqrt{3}}{2},$$

therefore, $\angle POQ = \frac{\pi}{3}$.

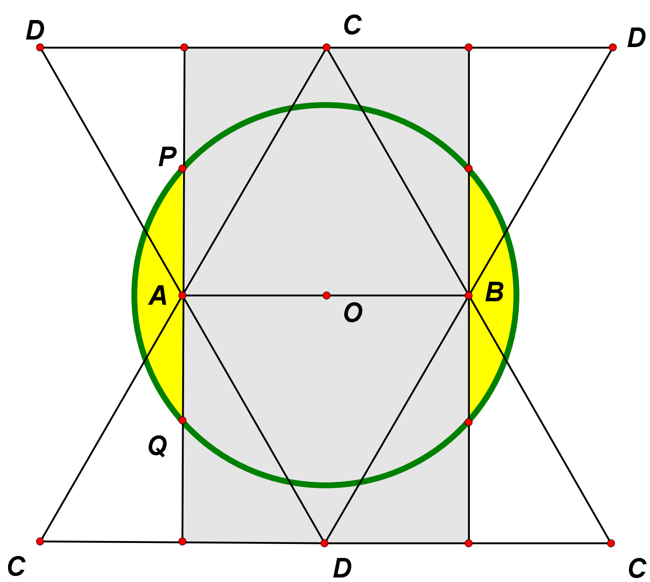
The area of the segment is the difference between the areas of the sector and the triangle:

$$S_{seg} = \frac{1}{2}\frac{\pi}{3}600^2.$$

Therefore the area hit by tsunami is

$$\pi \cdot 600^2 - \frac{3}{2} \cdot 600^2(\frac{\pi}{3} - \frac{\sqrt{3}}{2}) = 180000\pi + 270000\sqrt{3}.$$

b) Considering the "double" unfolding of the tetrahedron and following the argument from the previous part we assure that the shortest ways lie within the shaded rectangle (see the figure).



The area hit by tsunami is the difference of the area of the disc and the double area of the segment:

$$\angle POA = \arccos\frac{OA}{OP} = \arccos\frac{3}{4}.$$

$$PQ = 2PO\sin\angle POA = 300\sqrt{7}.$$

$$S_{seg} = 2 \cdot 180000 \arccos \frac{3}{4} - 67500\sqrt{7}.$$

$$S = \pi \cdot 600^2 - 720000 \arccos \frac{3}{4} + 135000\sqrt{7} = 720000 \arcsin 34 + 135000\sqrt{7}.$$

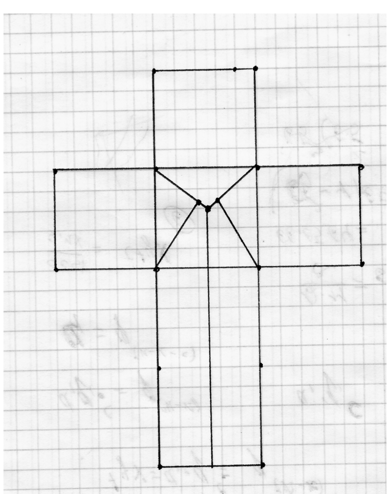
22. (V. Boss) Perpendiculars were dropped to the faces of a tetrahedron at their mass centers (intersections of medians). Prove that the projections of three perpendiculars onto the fourth face concur.

Solution. Let $ABCD$ be the tetrahedron given, A', B', C', D' be the centroids (centers of mass) for the faces BCD, CDA, DAB, ABC. It turns out that the faces of the tetrahedron formed by centroids are parallel to the respective faces of the original tetrahedron. So, for example, the plane ABC is parallel to the plane $A'B'C'$ and so on.

Indeed, let P and Q be the midpoints of AC and AB. As the centroid divides a median in the ratio of $2:1$, the converse of Thales theorem implies $B'C' \parallel PQ$. But $PQ \parallel BC$ as the medial line, therefore $B'C' \parallel BC$. In the same manner $A'C' \parallel AC$, and, by property of parallelism for two planes, the faces are parallel. Therefore perpendiculars dropped from the points A', B', C', D' to the respective faces of $ABCD$ are the altitudes of the tetrahedron $A'B'C'D'$. By the three perpendicular theorem, their projections to the plane of the face $A'B'C'$ will be the altitudes of this face and therefore concur. But then their projections on the parallel plane ABC also concur.

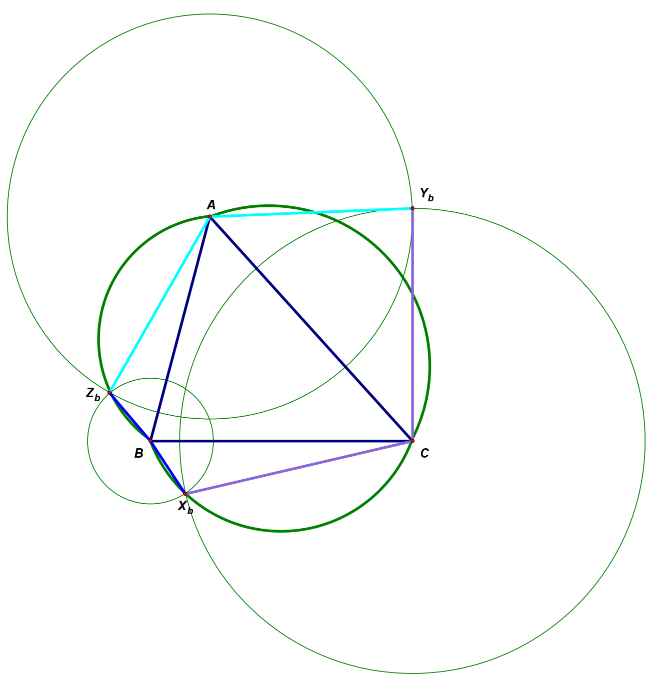
23. (L. and T. Yemelyanov) Paste over a cube in one layer with five convex pentagons of equal area.

Solution. For instance, this can be done if we consider the following unfolding of a cube:

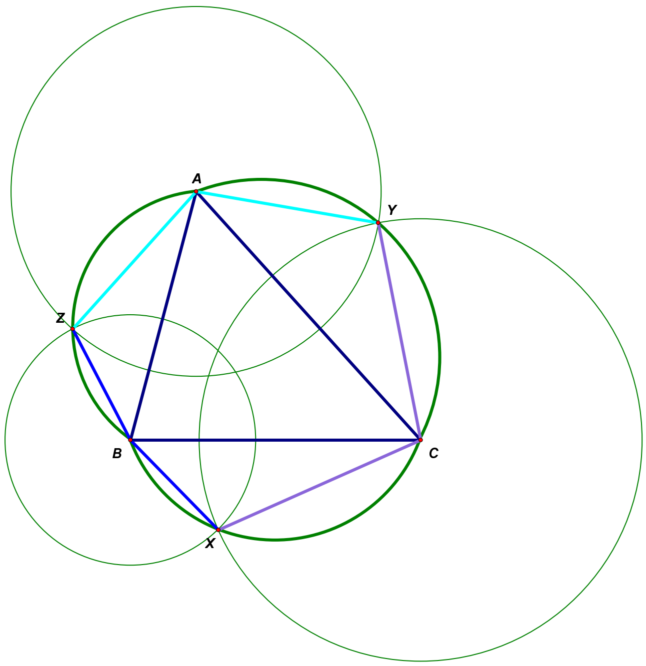


24. (V. Senderov) Given a triangle with all its angles less than ϕ, where $\phi < \frac{2\pi}{3}$. Prove that there is a point in the space such that all sides of the triangle are visible from it at the angle of ϕ.

Solution. Method one. Let ABC be the triangle in question. Construct an outward arc on each of its sides so that the angle measure of each of these arcs equals ϕ. Let us show that the arcs BC, CA, AB contain points X, Y, Z respectively, such that $AZ = AY$, $BZ = BX$, $CX = CY$. Let AC be the longest side of the triangle, AB the shortest. Choose an arbitrary point Z on the arc AB, find point X on the arc BC, such that $BX = BZ$ (is determined unambiguously as $AB \leq BC$), and find point Y on the other side from B about the line such that $AY = AZ$, $CY = CX$. For $Z = B$, we have $AY = AB$, $CY = CB$. Therefore, $\angle AYC = \angle B < \phi$ and Y lies outside the segment drawn on AC (see the figure).

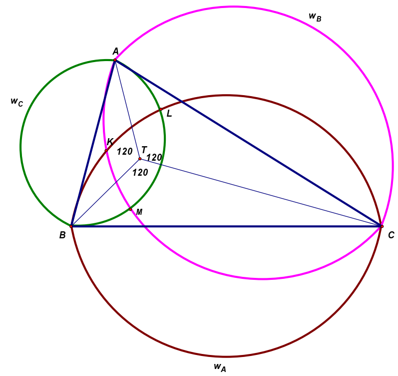


In the case $Z = A$ the point Y does not exist because $AC \geq BC$. It follows that at a certain intermediate position of the point Z, the point Y belongs to the arc AC (see the figure).



We are left to prove that a tetrahedron can be constructed from triangles ABC, ABZ, BCX, ACY. I.e. at least for one vertex (A, B or C) the angle of triangle ABC is less than the sum of the angles of two other triangles, adjacent tj the same vertex. But if this is not the case, then $\angle A + \angle B + \angle C \geq 3\pi - 3\phi > 3\pi - 2\pi = \pi$, a contradiction. (We made use of a well-known theorem of stereometry: three plane angles with a common vertex form a trihedral angle iff either of them is smaller then the sum of the other two).

Method two (Pechonkin Nikolay, city of Moscow, school no. 192). For each of segments AB, BC and CA let us determine the set of points in the plane, from which the segments in question are visible at angle ϕ. We get 6 circle archs. For BC let this set be ω_a, for AC be ω_b, and for AB be ω_c. Points K, L, M are intersection points for these sets (see the figure). Evidently, there is a region shared by all three regions and having two circle arcs as its borders. (For instance, the Fermat-Toricelli point from which all sides of the triangle are visible at angle $\frac{2\pi}{3}$ belongs to this region).



Obviously M, L, K lie in the regions resttricted respectively by ω_a, ω_b and ω_c. Next, the set of points in space, from which the segment BC is visible at angle ϕ is the surface obtained by rotation of ω_a about BC. Let us denote it as F_a. Similarly we obtain two other surfaces: F_b, F_c. The intersection of F_a and F_b is a certain continuous curve passing through C and K. Moreover, K lies within the body limited by F_c, while C is outside that body. Hence the line of intersection between F_a and F_b will also intersect F_c.