The first Olympiad, 2005

Correspondence round

1. [8] The circle chords AC and BD intersect at point P. Perpendiculars to AC and BD at points C and D respectively intersect at point Q. Prove that the lines AB and PQ are perpendicular.

2. [8,9] Cut the cross composed of five equal squares into three polygons of equal area and perimeter.

3. [8,9] A circle and a point K inside it are given. An arbitrary circle of the same size passes through point K and shares a chord with the initial circle. Find the locus of midpoints of such chords.

4. [9-11] For what minimal n there exists a convex n-gon with equal sines of all angles and unequal lengths of all sides?

5. [8-10] Two parallel lines p_1 and p_2 are given. Points A and B belong to p_1, point C belongs to p_2. We move the segment BC parallel to itself and consider all triangles ABC produced in this manner. In these triangles, find the locus of points that are: a) intersections of altitudes; b) intersections of medians; c) circumcenters.

6. [10,11] The side AB of the triangle ABC has been divided into n equal parts (with points of division $B_0 = A$, B_1, B_2, $\dots$, $B_n = B$), while the side AC of this triangle has been divided into $(n+1)$ equal parts (with points of division $C_0 = A$, C_1, C_2, $\dots$, $C_{n+1} = C$). Triangles $C_iB_iC_{i+1}$ were colored. What part of the triangle area has been colored?

7. [8,9] Two circles with radii of 1 and 2 have common center at point O. Vertex A of the regular triangle ABC is on the major circle, whereas the midpoint of BC is on the minor circle. What is the possible measure of angle BOC?

8. [8,9] Three rectangles are circumscribed about a convex quadrilateral $ABCD$. Two of these rectangles are squares. Is it true that the third rectangle has to be a square as well? (A rectangle is circumscribed about a quadrilateral $ABCD$ when there is a single vertex of the quadrilateral on each side of the rectangle).

9. [9] Let O be the center of a regular triangle ABC. From arbitrary point P on the plane the perpendiculars were dropped to the sides of the triangle or their extensions. Let M be the intersection of medians of the triangle with vertices in the feet of the perpendiculars. Prove that M is a midpoint of segment PO.

10. [8,9] Cut a non-isosceles triangle into four similar triangles not all of which are congruent.

11. [8-10] A square has been cut into n rectangles with sides equal to $a_i \times b_i$, $i = 1, \dots, n$. At what minimal n all numbers in the tuple $a_1, \dots, a_n, b_1 \dots, b_n$ can occur to be different?

12. [8,9] Construct a quadrilateral with the given sides a, b, c and d and the distance l between the midpoints of the diagonals.

13. [9] A triangle ABC and two lines l_1 and l_2 are given. A line parallel to l_1 and intersecting AC at point E, as well as a line parallel to l_2 and intersecting BC at point F are both drawn through an arbitrary point D. Construct the point D such that the segment EF has minimal length.

14. [10,11] Let P be an arbitrary point inside triangle ABC. Denote by A_1, B_1 and C_1 the points of intersection of lines AP, BP and CP with sides BC, CA and AB respectively. Rank areas of the triangles AB_1C_1,A_1BC_1, A_1B_1C and denote the smallest as S_1, the middle one as S_2, and the largest one as S_3. Prove that $\sqrt{S_1S_2} \leq S \leq \sqrt{S_2S_3}$, where S is the area of triangle $A_1B_1C_1$.

15. [11] A circle with its center at the origin is given. Prove that there exists a circle with a shorter radius, which contains equal or greater number of points with integer coordinates.

16. [8,9] In an acute non-regular triangle 4 points were marked: the centers of the incircle and the circumcircle, the center of mass (the intersection point of the medians) and the orthocenter (the intersection point of the altitudes). After that, the original triangle was erased. It turned out to be impossible to determine which point corresponded to which center. Find the angles of the triangle.

17. [11] The incircle of triangle ABC has center I and tangency points P, Q, R with sides BC, CA and AB respectively. With a ruler only, construct the point K where the circle passing through B and C is (internally) tangent with the incircle.

18. [10,11] There are three lines l_1, l_2, l_3 in the plane that form a triangle, as well as a marked point O, the center of its circumcircle. For an arbitrary point X in the plane, let us denote by X_i the point symmetrical to about the line l_i.

a) Prove that for an arbitrary point M the lines linking midpoints of segments O_1O_2 and M_1M_2, O_2O_3 and M_2M_3, O_3O_1 and M_3M_1 concur.

b) Where can their point of intersection lie?

19. [8-11] As is well-known, the Moon rotates around the Earth. Let us assume the Moon and the Earth to be points. Assume also that the Moon travels a circular orbit around the Earth and makes full circle in one month. There is an UFO in the plane of the lunar orbit. The UFO can move in jumps over the Moon and over the Earth: from an old location (point A) it instantly moves to the new one (point A') so that the midpoint of A' is either the Moon, or the Earth. Between the jumps, the UFO is located in space motionless.

a) Determine the minimal number of jumps sufficient for the UFO to reach any random point within lunar orbit from any another random point within lunar orbit.

b) Prove that the UFO is able, within infinite number of jumps, to reach any random point within lunar orbit from any another random point within lunar orbit in any period of time, for example, in one second.

20. [11] Let I be the center of the insphere of tetrahedron $ABCD$; A', B', C', D' be the centers of the circumspheres of tetrahedrons $IBCD$, $ICDA$, $IDBA$, $IABC$ respectively. Prove that the circumsphere of $ABCD$ lies entirely inside the circumsphere of $A'B'C'D'$.

21. [10,11] The planet "Tetraincognito" covered by the "ocean" has a shape of a regular tetrahedron with an edge of 900 kilometers. What area of the ocean will be hit by a "tsunami" after 2 hours of "tetraquake" with the epicenter at

a) the center of a face,

b) the midpoint of an edge, if tsunami propagates at a speed of 300 km/h?

22. [10,11] Perpendiculars were dropped to the faces of a tetrahedron at their mass centers (intersections of medians). Prove that the projections of three perpendiculars onto the fourth face concur.

23. [10,11] Paste over a cube in one layer with five convex pentagons of equal area.

24. [10,11] Given a triangle with all its angles less than ϕ, where $\phi < \frac{2\pi}{3}$. Prove that there is a point in the space such that all sides of the triangle are visible from it at the angle of ϕ.