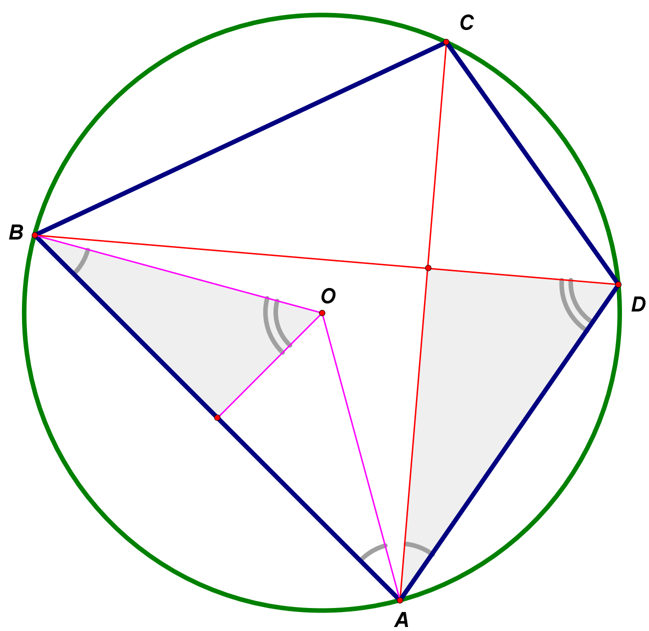
The first Olympiad, 2005

Final round. Solutions

Grade 9

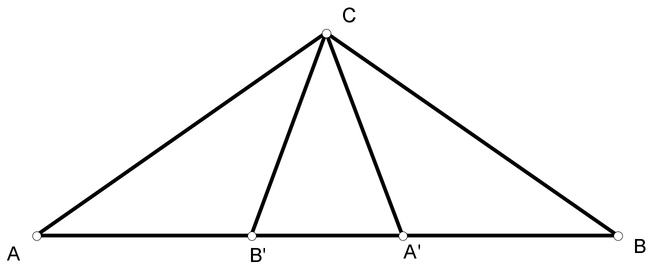
1. (A.A.Zaslavsky) The quadrilateral $ABCD$ is inscribed in a circle with the center O within the quadrilateral. Prove that if $\angle BAO = \angle DAC$, then the diagonals of the quadrilateral are perpendicular to each other.

Solution. Since $\angle ABO = (\pi - \angle AOB)/2 = \pi/2 - \angle ADB$, we have $\angle DAC + \angle ADB = \pi/2$, which is equivalent to the problem statement (see the figure).



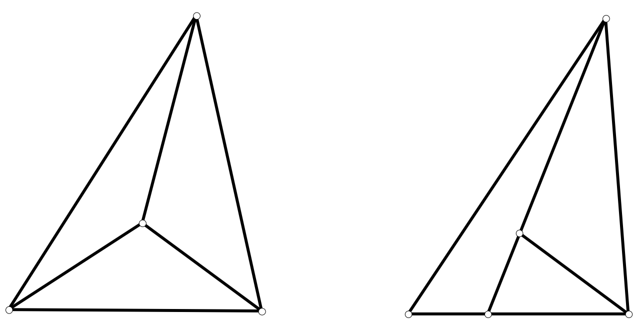
2. (L.A.Yemelyanov) Find all isosceles triangles that cannot be cut into three isosceles triangles having equal lateral sides.

Solution. An acute triangle can be cut into three isosceles triangles with equal lateral sides which are radii of the circumcircle. If the triangle ABC is obtuse (C is the obtuse angle), then let us choose points A', B' on side AB, such that $AB' = B'C = CA' = A'B$ and cut the original triangle into triangles $AB'C$, $A'B'C$ and $A'BC$ (see the figure).



Let us prove that a right triangle ABC ($AC = BC$) cannot be cut in the required manner.

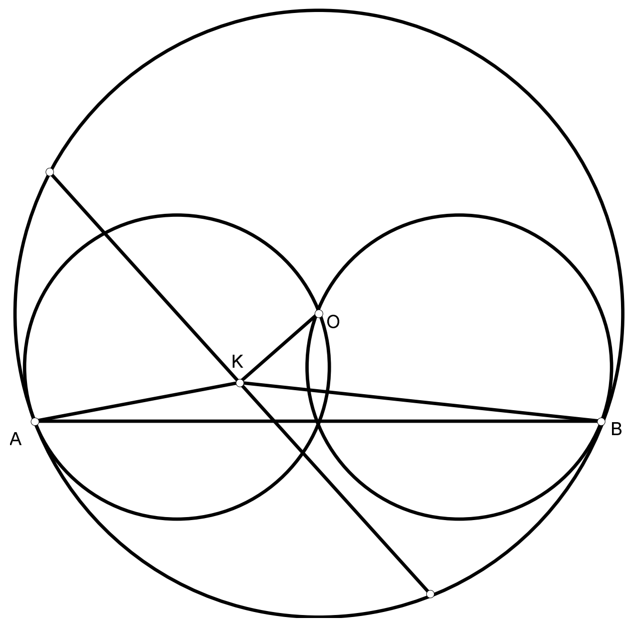
Clearly there are two distinct methods for cutting one triangle into three. The first is to cut along the lines connecting the vertices with a certain point within a triangle. The second is to cut it into two triangles along the line passing through a vertex, and then to repeat this with one of the resulting triangles (see the figure).



In the first case the triangle AXB can be isosceles only if $AX = BX$, but then another two triangles will not be isosceles. In the second case at least one of triangles resulting from the initial cut needs to be isosceles. Therefore, the initial cut line either is a bisector of the right angle or it links point C and point D on the hypotenuse, such that $AD = AC$. In neither case it is possible to draw a second line to make the required cut.

3. (I.F.Sharygin) Given a circle and points A and B on it. Draw the set of midpoints of the segments one endpoint of which lies on the smaller circle arc AB and the other endpoint lies on the larger one.

Solution. Let K be an arbitrary point within the given circle. A chord with the midpoint K is perpendicular to OK. Therefore it intersects the segment AB iff one of angles OKA, OKB is not acute, while the other one is not obtuse. So, the set of points in question consists of the points within or on the border of one of the disks with diameters OA, OB, and outside or on the border of the other one (see the figure).



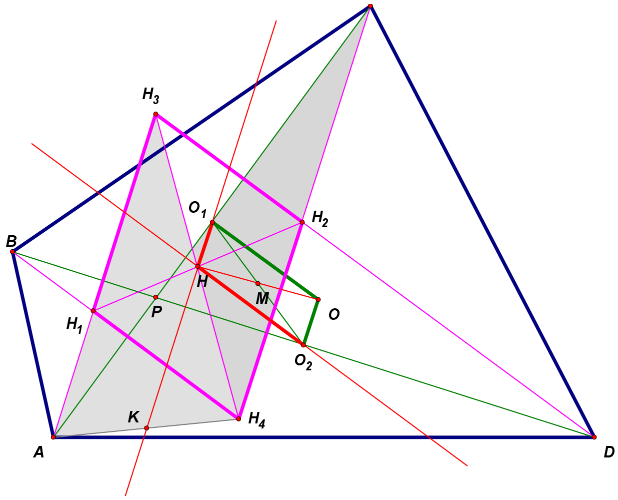
4 (A.G.Myakishev) Let P be the point of intersection for the diagonals of a quadrilateral $ABCD$. Let M be the meet point for the lines linking the midpoints of its opposite sides. Let O be the meet point for the perpendicular bisectors to the diagonals. Let H be the meet point for the lines linking orthocenters of triangles APD and BCP, APB and CPD. Prove that M is the midpoint of OH.

Solution. Let O_1 be the midpoint of AC, and O_2 be the midpoint of BD. It is not hard to show that point M is the midpoint of segment O_1O_2 (clearly M is the mass center in the system $1A, 1B, 1C, 1D$; consider the subsystems $1A, 1C$ and $1B, 1D$ which are equivalent to subsystems $2O_1$, $2O_2$).

Obviously the quadrilateral formed by the orthocenters is a parallelogram with sides belonging to perpendiculars dropped from vertices of the quadrilateral to respective diagonals. So H is the meet point for the diagonals and splits them in halves.

Let us prove that line HO_1 is parallel to OO_2. Or, to put it differently, the former is perpendicular to diagonal BD. Consider the line perpendicular to this diagonal and passing through H. Let us demonstrate that it also passes through the point O_1. Let our line intersect the segment AH_4 at the point K. Then it is a medial line in the triangle AH_3H_4 and therefore K is the midpoint of AH_4. It follows that our line will also be a central line in the triangle AH_4C and as such will pass through O_1.

Following the same line of argument we ascertain that the line HO_2 is parallel to OO_1, i.e. HO_1OO_2 is a parallelogram where the point M is the intersection of its diagonals (see the figure).



It follows from the above that points O, M, H are collinear and $OM = MH$.

5. (B.R.Frenkin) Given a triangle with the following property: for any of its sides it is impossible to construct a triangle from the altitude, the bisector and the median drawn to this side. Prove that one of the angles of the given triangle is greater than $135°$.

Solution. It follows from the problem condition that each median is greater or equal to the sum of the bisector and the altitude from the same vertex. If the angle between some median and the respective altitude does not exceed $60°$, then the median is not greater than the doubled altitude, while the sum of the bisector and the altitude is not less than the doubled altitude, and these equalities are not attained simultaneously. Therefore it follows from the problem condition that the angle between each median and the respective altitude is greater than $60°$. Since the lesser of the angles in the triangle does not exceed $60°$, some altitude is outside of the triangle, which is therefore obtuse.

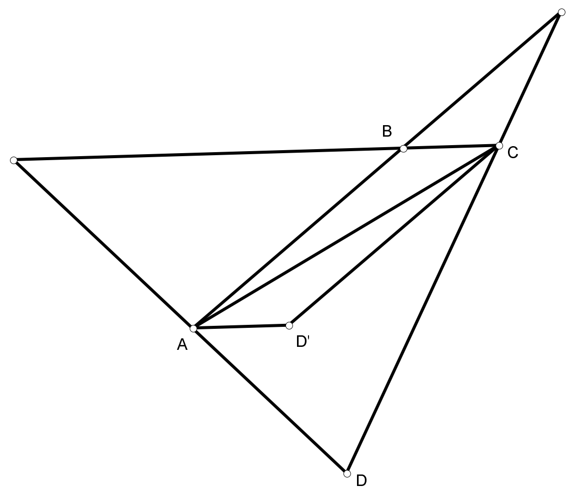
Let A be the vertex of the obtuse angle, B and C the other two vertices, AM the median, AH the altitude, and the point M belongs to the segment BH. As proved above, $\angle AMH < 30°$. It is equal to the sum of angles ABM and BAM. The median from the vertex of the obtuse angle is less than a half of the opposite side. Therefore $\angle ABM < 15°$.

The altitude from vertex B forms an angle greater than $60°$ with the respective median, so with the side BC too. Hence $\angle ACB < 30°$. Thus $\angle BAC > 180° - 15° - 30° = 135°$.

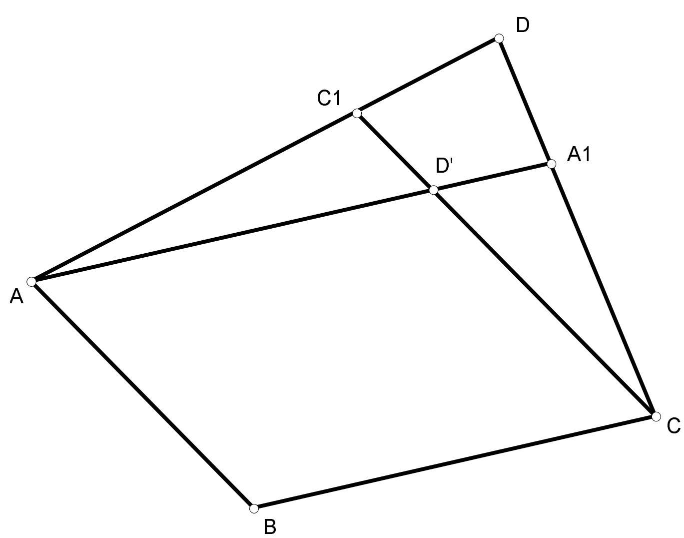
Grade 10

1. (L.A.Yemelyanov) Given a convex quadrilateral without any parallel sides. For every triple of its vertices, the point is constructed which complements this trio to a parallelogram (one diagonal of which coincides with a diagonal of quadrilateral). Prove that out of four constructed points, exactly one lies within the initial quadrilateral.

Solution one. Let the vertex D' of the parallelogram $ABCD'$ lie within the quadrilateral $ABCD$. Then $\angle BCA < \angle CAD$ and $\angle BAC < \angle ACD$. Therefore the points of intersection of opposite sides in $ABCD$ lie on the extensions of segments AB and BC beyond the point B. It is obvious that there can be only one vertex with such property in the quadrilateral (see the figure).



Solution two. Let $ABCD$ be the initial quadrilateral, $ABCD'$ be the parallelogram lying within it. Let the rays CD' and AD' intersect at C_1 and A_1. Then $S_{ABC} = S_{ABD'} = S_{ABC_1} < S_{ABD}$, similarly $S_{ABC} < S_{ACD}$. Then $S_{ABC} < S_{ABD} + S_{ACD} - S_{ABC} = S_{BCD}$, i. e. ABC is the triangle of minimal area formed by three vertices of the quadrilateral. Conversely, if this is the case, then there will be points A_1 and C_1 on the sides, such that $S_{ABC} = S_{ABC_1} = S_{A_1BC}$, and the point of intersection between AA_1 and CC_1 will be the one in question (see the figure).



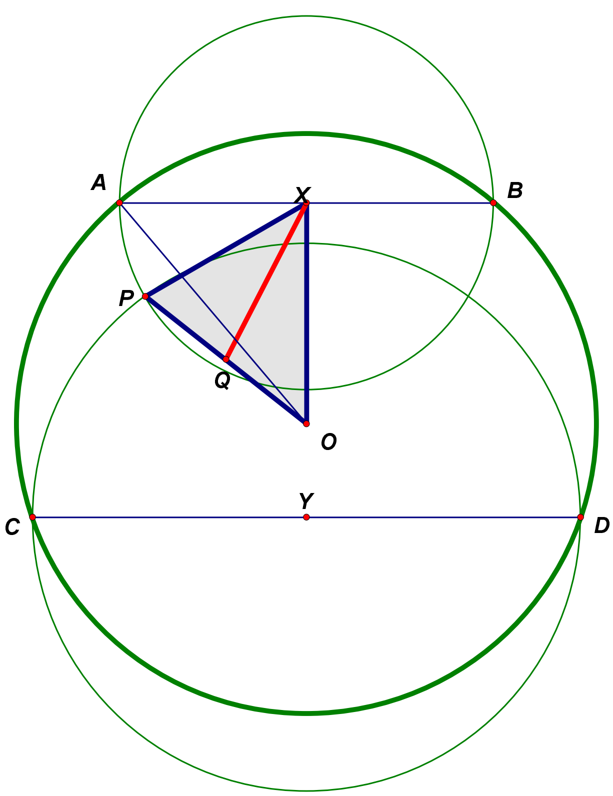
2. (A.V.Shapovalov) A triangle can be cut into three similar triangles. Prove that it can be cut into any number of triangles similar to each other.

Solution. Let the triangle ABC (with angle C being the largest) be cut into three similar triangles by segments AX, BX, CX. Since $\angle AXB > \angle ACB$, the angle AXB can only be equal in the other triangles to angles AXC and BXC. It means that $\angle AXB = \angle AXC = \angle BXC = 120°$. But then $AX = BX = CX$ and the triangle ABC is regular.

Suppose now that we cut the triangle into two ones along a line passing through the vertex, and one of the resulting triangles is also cut into two. Since the two final triangles are similar, they are right-angled. In other words, a right-angled triangle was cut from the initial triangle at the first cut, and then the remaining triangle was split into two by an altitude. Listing all possible options, it is easy to assure that the initial triangle is either isosceles or right-angled. In both cases it can be cut into any number of similar triangles.

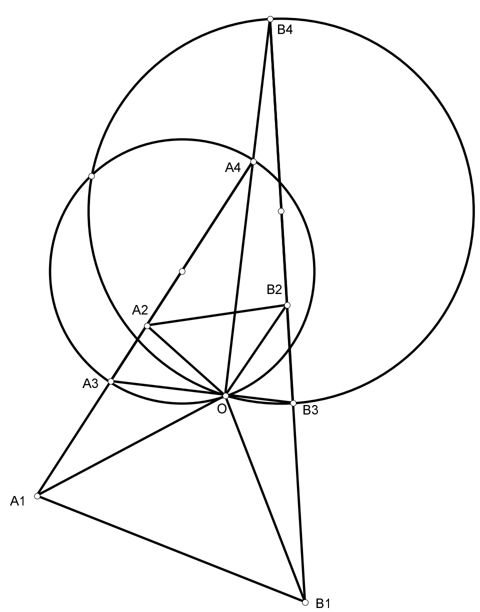
3. (A.A.Zaslavsky) Two parallel chords AB and CD are drawn in a circle with center O. The circles with diameters AB and CD intersect at point P. Prove that the midpoint of segment OP is equidistant from the lines AB and CD.

Solution. Let X, Y be the midpoints of AB and CD, Q be the midpoint of OP. Then $XQ^2 = (2OX^2+2XP^2-OP^2)/4 = (2OX^2+2XA^2-OP^2)/4 = (2R^2-OP^2)/4 = YQ^2$. Thus, Q is equidistant from point X and Y, and therefore from the lines AB and CD (see the figure).



4. (Wim Pijls, the Netherlands) Two segments A_1B_1 and A_2B_2 on the plane are given such that $\frac{A_2B_2}{A_1B_1} = k < 1$. The point A_3 is chosen on the segment A_1A_2 and the point A_4 is chosen on the extension of segment A_1A_2 beyond A_2 so that $\frac{A_3A_2}{A_3A_1} = \frac{A_4A_2}{A_4A_1} = k$. Similarly the point B_3 is chosen on the segment B_1B_2 and the point B_4 is chosen on the extension of segment B_1B_2 beyond B_2 so that $\frac{B_3B_2}{B_3B_1} = \frac{B_4B_2}{B_4B_1} = k$. Find the angle between the lines A_3B_3 and A_4B_4.

Solution one. Let O be the center of similarity that does not preserve orientation and that maps A_1 to A_2 and B_1 to B_2. As the triangles OA_1B_1 and OA_2B_2 are similar, so $\angle A_1OB_1 = \angle B_2OA_2$ and the bisectors of angles A_1OA_2 and B_1OB_2 coincide. Since $OA_2/OA_1 = OB_2/OB_1 = k$, this common bisector crosses the segments A_1A_2 and B_1B_2 at points A_3 and B_3. The perpendicular to that bisector crosses extensions of the above segments at points A_4 and B_4 (see the figure). Therefore, the angle in question is right.



In order to find the point O, let us draw circles with diameters A_3A_4 and B_3B_4 and find their points of intersection. Since the circle with diameter A_3A_4 is the locus of points, for which the ratio of their distances to A_2 and A_1 equals k, the points of intersection will be the centers of two similarities, mapping A_1 into A_2 and B_1 into B_2. One of these similarities preserves orientation, the other changes it.

Solution two. Let $\vec{A_1B_1} = \vec{u}$, $\vec{A_2B_2} = \vec{v}$, by condition $\vec{v}^2 = k^2\vec{u}^2$. Then

$$\vec{A_3B_3} = \vec{A_3A_1} + \vec{A_1B_1} + \vec{B_1B_3} = \frac{1}{1+k}\vec{A_2A_1} + \vec{u} + \frac{1}{1+k}\vec{B_1B_2}; \qquad (*)$$

on the other side,

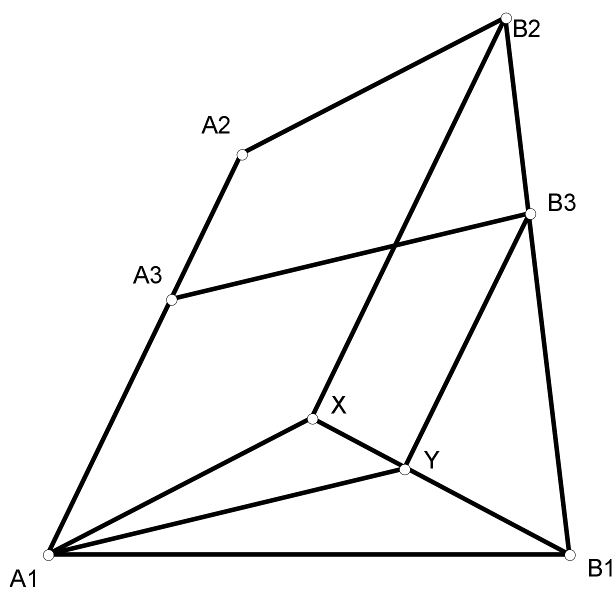
$$\vec{A_3B_3} = \vec{A_3A_2} + \vec{A_2B_2} + \vec{B_2B_3} = \frac{k}{1+k}\vec{A_1A_2} + \vec{v} + \frac{k}{1+k}\vec{B_2B_1}. \qquad (**)$$

Multiplying $(*)$ by $\frac{k}{1+k}$, and $(**)$ by $\frac{1}{1+k}$ and summing up the resulting equations, we get $\vec{A_3B_3} = \frac{k}{1+k}\vec{u} + \frac{1}{1+k}\vec{v}$. By analogy we have $A_4B_4 = \frac{1}{1-k}\vec{v} - \frac{k}{1-k}\vec{u}$. Then

$$\left(\vec{A_3B_3}, \vec{A_4B_4}\right) = \frac{k\vec{u}+\vec{v}, \vec{v}-k\vec{u})}{(1+k)(1-k)} = \frac{k^2\vec{u}^2 - \vec{v}^2}{1-k^2} = 0,$$

i.e. the vectors are orthogonal.

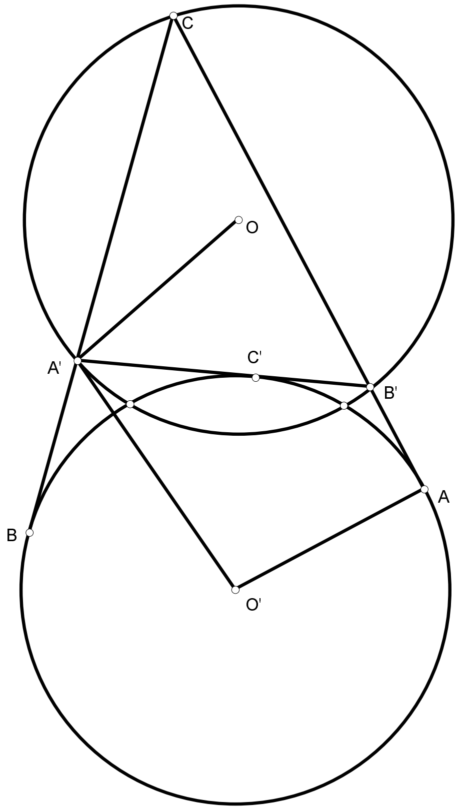
Solution three (S.Safin). Construct parallelogram $A_1A_2B_2X$ and draw the bisector A_1Y in triangle A_1XB_1. Since $\frac{B_1Y}{XY} = \frac{A_1B_1}{A_1X} = k$, we have $B_3Y \parallel B_2X$ and $B_3Y = kB_2X = A_1A_3$. Therefore, $A_1A_3B_3Y$ is a parallelogram, i.e. $A_3B_3 \parallel A_1Y$ (see the figure).



Similarly A_4B_4 is parallel to the external bisector of the angle XA_1B_1, which means that the lines A_3B_3 and A_4B_4 are perpendicular.

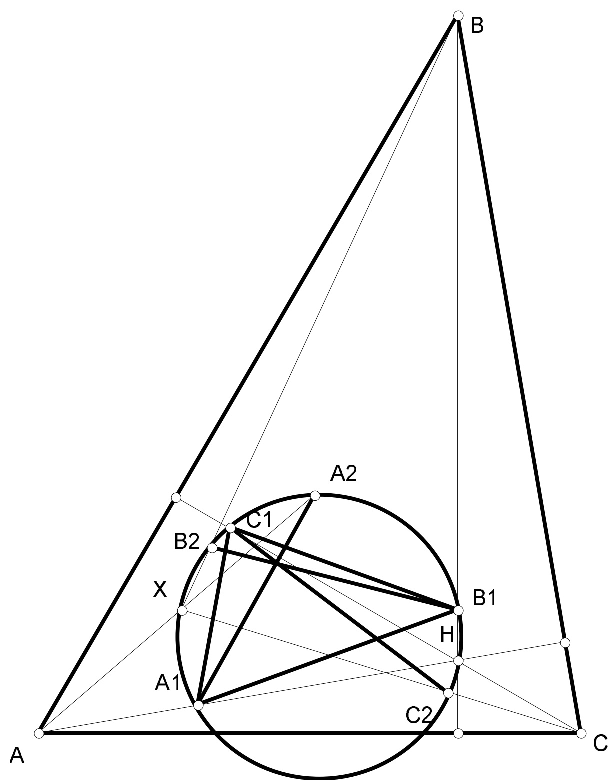
5. (A.A.Zaslavsky) Two circles of unit radius intersect at the points X and Y. The distance between these points also equals one. From point C on one circle, tangents CA and CB are drawn to the other circle. The line CB has a second intersection with circle one at the point A'. Find the distance AA'.

Solution. Let O be the center of the circle to which point C belongs. Let O' be the center of the other circle. Since $OO' = \sqrt{3}$, the line $A'B'$ is tangent to the second circle at point C'. Therefore, $\angle A'O'A = \angle AO'C' + \frac{1}{2}\angle C'O'B = 2\angle ABC' + \angle C'AB = \angle CB'A' + \frac{1}{2}\angle CA'B'$, $\angle O'A'O = \angle O'A'B' + \angle B'A'O = \frac{\pi}{2} - \angle C'O'A' + \frac{\pi}{2} - \angle BCA = \pi - \angle BCA - \frac{1}{2}\angle CA'B' = \angle CB'A' + \frac{1}{2}\angle CA'B'$. Since $O'A = OA'$, $AO'A'O$ is an isosceles trapezium, and $AA' = OO' = \sqrt{3}$ (see the figure).



6. (A.A.Zaslavsky) Let H be the orthocenter in the triangle ABC and X be an arbitrary point. The circle with diameter XH has second intersections with lines AH, BH, CH at points A_1, B_1, C_1, while with lines AX, BX, CX at points A_2, B_2, C_2. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

Solution. For definiteness, let us consider the case when the points are located on the circle in the following order: $A_1B_2C_1A_2B_1C_2$. Let $XH = d$. Then $A_1B_2 = d\sin\angle A_1HB_2 = d\sin\angle XBC$, because HA_1 is perpendicular to BC, whereas HB_2 is perpendicular to BX. Therefore, $\frac{A_1B_2\cdot C_1A_2\cdot B_1C_2}{A_2B_1\cdot C_2A_1\cdot B_2C_1} = \frac{\sin\angle XBC\sin\angle XCA\sin\angle XAB}{\sin\angle XAC\sin\angle XCB\sin\angle XBA} = 1$, which is equivalent to the problem statement (see the figure).

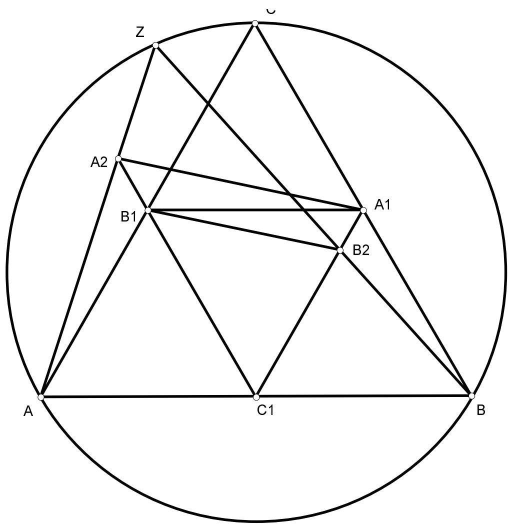


Remark. Apparently triangle $A_1B_1C_1$ is similar to the triangle ABC, while the common point of the lines corresponds to the point isogonally conjugated with X.

Grade 11

1. (A.A.Zaslavsky) Let A_1, B_1, C_1 be the midpoints of sides in the regular triangle ABC. Three parallel lines passing through A_1, B_1, C_1, intersect the lines B_1C_1, C_1A_1, A_1B_1 at points A_2, B_2, C_2 respectively. Prove that the lines AA_2, BB_2, CC_2 concur in the point that belongs to the circumcircle of the triangle ABC.

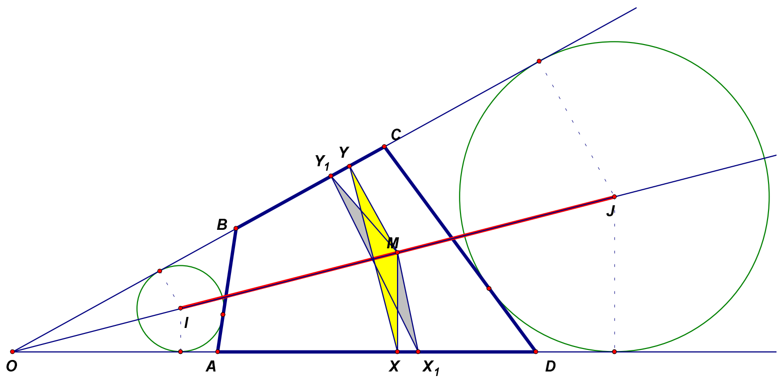
Solution. Let Z be the common point of AA_2 and BB_2. Since the points B and B_1 are symmetrical about the line A_1C_1, we have $\angle ABZ = \angle C_1BB_2 = \angle B_2B_1C_1$. Similarly $\angle BAZ = \angle A_2A_1C_1$. Since the lines AA_2 and BB_2 are parallel, we have $\angle A_2A_1C_2 = \angle B_1B_2C$, therefore, $\angle AZB = \angle ACB$ and the points A, B, C, Z are concyclic. This implies the problem statement (see the figure).



2 (A.G.Myakishev) A convex quadrilateral $ABCD$ is given. The lines BC and AD intersect at point O so that the point B belongs to the segment OC, while the point A belongs to the segment OD. Let I be the incenter of the triangle OAB, J be the center of an excircle of the triangle OCD (tangent to side CD and the extensions of two other sides). Perpendiculars dropped from the midpoint of segment IJ to the lines BC and AD, intersect with respective sides of the quadrilateral (not their extensions) at points X and Y. Prove that the segment XY divides the perimeter of the quadrilateral $ABCD$ in half. In particular, out of all the segments with this property and with the endpoints at BC and AD, the segment XY has the minimal length.

Solution. Since tangential segments drawn from the same point are equal, it is not hard to show that the segment $X'Y'$ with its endpoints on the sides AD and BC divides the perimeter in half iff $OX'+OY'=l$, where l is a constant value equal to the doubled length of the segment of the respective tangent plus half-perimeter of the quadrilateral.

Let M be the midpoint of IJ. It is easy to show that $OX+OY=l$. Then the triangles MXX' and MYY' are equal, hence the triangles MXY and $MX'Y'$ are similar by two angles. Therefore, $X'Y'$ is minimal when MX' is minimal, i.e. when X' coincides with X (see the figure).



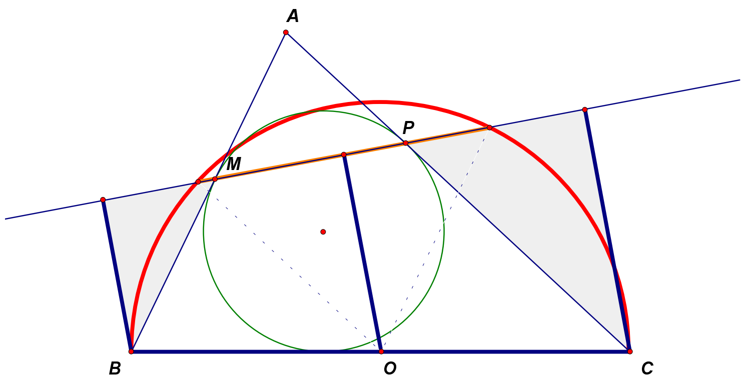
3. (A.A.Zaslavsky) Within the inscribed quadrilateral $ABCD$ there is a point K such that the distances from it to the sides of $ABCD$ are proportional to these sides. Prove that K is the point of intersection of diagonals in $ABCD$.

Solution one. Let U be the intersection of tangent lines to the circle $ABCD$ at points A and C, while X, Y be projections of U to AB and BC. Then $UX/UY = \sin\angle UAX/\sin\angle UCY = \sin\angle BCA/\sin\angle BAC = AB/BC$, i.e. K belongs to the line UB. Similarly K belongs to the line UD, and if these lines do not coincide then $K = U$. In the same manner it is proved that if the lines AV and CV do not coincide where V is the meet point of the lines tangent at points B and D, then $K = V$, which is impossible. Let us assume that points B, D, U belong to the same line. Then $AB/AD = AU/UD = CU/UD = BC/CD$ and the points A, C, V also belong to the same line. Therefore, K is the point of intersection between AC and BD.

Solution two. The set of points with distances to the lines AB and CD proportional to respective sides, is the line passing through the point of intersection of AB and CD. Since $ABCD$ is inscribed, the triangles LAB and LCD (where L is the meet point of the diagonals) are similar, i.e. L belongs to the indicated line. By analogy, L lies on the second such line and therefore coincides with K.

4. (I.F.Sharygin) In triangle ABC, $\angle A = \alpha$, $BC = a$. The incircle is tangent with lines AB and AC at points M and P. Find the length of the chord dissected from the line MP by the circle with diameter BC.

Solution one. The distance from the circle center to the chord is equal to the half-sum of distances from points B and C to the line MP, i.e. $\frac{1}{2}(BM\sin\angle AMP + CP\sin\angle APM) = \frac{1}{2}(BM + CP)\cos\frac{\alpha}{2} = \frac{a}{2}\cos\frac{\alpha}{2}$ (see the figure).

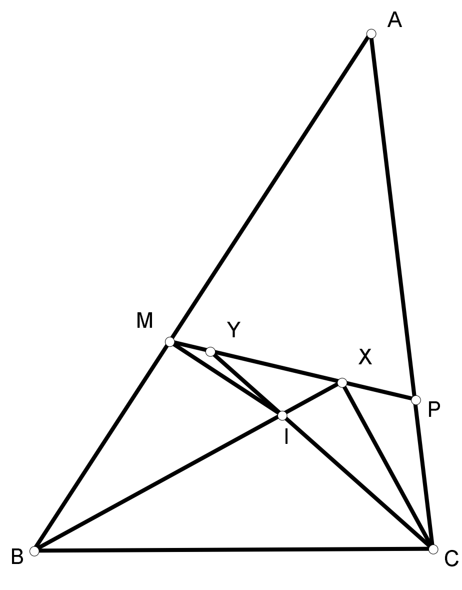


Therefore, the length of the chord is equal to $a \sin \frac{\alpha}{2}$.

Solution two. Let I be the incenter of the triangle, X and Y be the points of intersection of lines BI, CI with line MP. Then $\angle MXB = \angle AMP - \angle MBX = \frac{\angle B}{2}$. It follows that the triangles BXM and BCI are similar, i.e.

$$\frac{BX}{BC} = \frac{BM}{BI} = \cos \frac{\angle A}{2}.$$

Hence the angle BXC is right (see the figure).

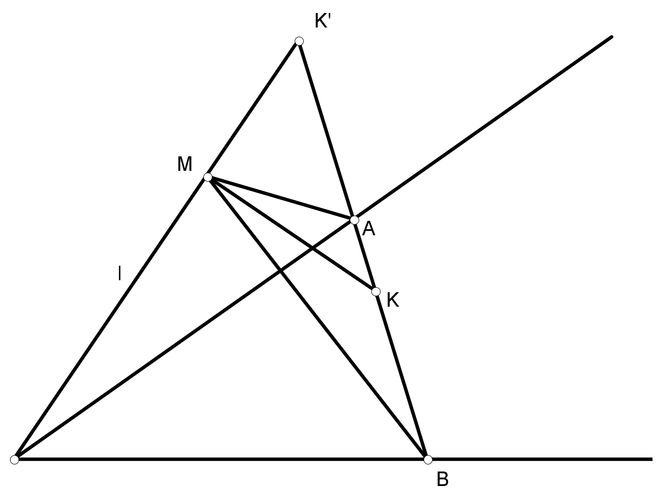


Similarly the angle BYC is right. Therefore the chord in question equals

$$XY = BC \sin \angle XCY = a \sin \frac{\alpha}{2}.$$

5. (V.Yu.Protasov) The angle on the plane and point K within it are given. Prove that there exists a point M with the following property: if an arbitrary line passing through K intersects the sides of the angle at points A and B, then MK is the bisector of angle AMB.

Solution one. On an arbitrary line passing through K and intersecting the angle sides at points A and B, choose point K' such that $AK'/BK' = AK/BK$. Since all points K' belong to the line l passing through the vertex of the angle, all circles with diameter KK' pass through the projection M of K to l. At the same time, the following equation always holds: $AM/BM = AK/BK$, i.e. M is the point in question (see the figure).

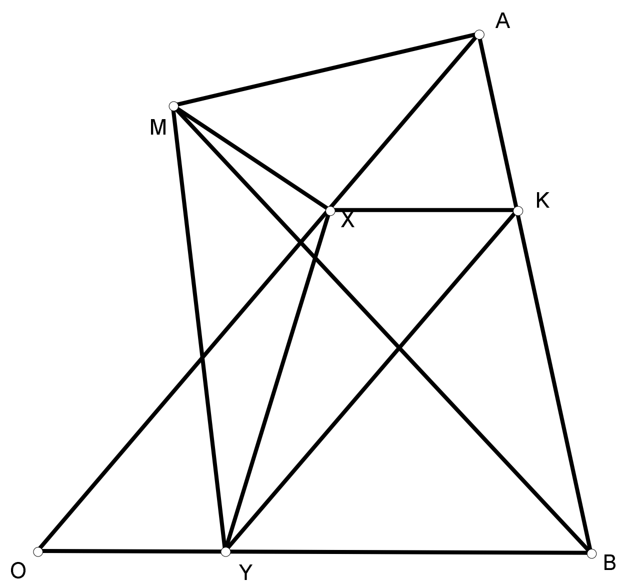


Solution two (R.Devyatov). Let O be the vertex of the angle. Construct a parallelogram $KXOY$ with two sides on angle sides. Let M be the point symmetrical to K about XY. Let us prove that M is the point in question.

Let the line passing through K intersect lines OX and OY at points A and B. Observe that $MX = KX$, $MY = KY$, $\triangle MXY = \triangle KXY = \triangle OYX$, so $MOYX$ is an isosceles trapezium and $\angle MXO = \angle MYO$. It means that $\angle MXA = 180^\circ - \angle MXO = 180^\circ - \angle MYO = \angle BYM$. Now the triangles AXK and KYB are similar, as their sides are respectively parallel, therefore $KX/XA = BY/YK$. From this we get

$$\frac{MX}{XA} = \frac{KX}{XA} = \frac{BY}{YK} = \frac{BY}{YM}.$$

From the above and the equality of angles MXA and BYM we get that the triangles MXA and BYM are similar (see the figure).



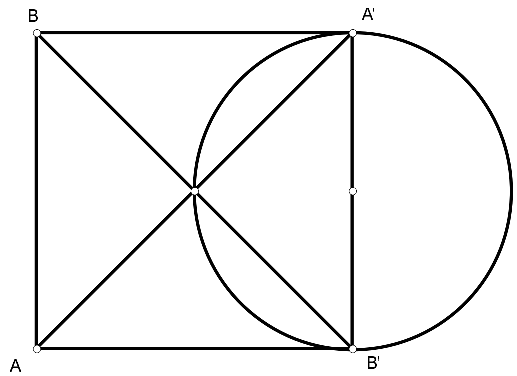
Now, from two proven similarities we get

$$\frac{MA}{BM} = \frac{MX}{BY} = \frac{KX}{BY} = \frac{AK}{KB},$$

which means that MK is a bisector of triangle AMB.

6. (I.I.Bogdanov) The sphere inscribed in tetrahedron $ABCD$ is tangent to its faces at points A', B', C', D'. The segments AA' and BB' do intersect, and their meeting point lies on the inscribed sphere. Prove that the segments CC' and DD' also intersect on the inscribed sphere.

Solution. Since the segments AA' and BB' intersect, the lines AB and $A'B'$ either also intersect or are parallel. Let us denote their intersection point (possibly an infinite one) as P. Since P lies outside the dihedral angle at the edge CD, the plane CDP does not intersect the insphere. Thus there exists a projective transformation that fixes the sphere and maps this plane into the infinite one. Under this transformation, the segment $A'B'$ will become the diameter of the sphere, whereas AB will be parallel to it. As the common point of AA' and BB' lies on the sphere, the distance from its center to AB equals its doubled radius. (In the figure, the projection to the plane $ABA'B'$ is shown).



Thus the angle between the planes ABC and ABD is equal to $60°$, the large circle arc linking C' and D' is equal to $120°$, and the lines passing through C', D' and parallel to ABC, ABD, intersect on the sphere (at the figure, the projection to the plane perpendicular to AB is shown).