The first Olympiad, 2005

Final round

Grade 9

1. (A.A.Zaslavsky) The quadrilateral $ABCD$ is inscribed in a circle with the center O within the quadrilateral. Prove that if $\angle BAO = \angle DAC$, then the diagonals of the quadrilateral are perpendicular to each other.

2. (L.A.Yemelyanov) Find all isosceles triangles that cannot be cut into three isosceles triangles having equal lateral sides.

3. (I.F.Sharygin) Given a circle and points A and B on it. Draw the set of midpoints of the segments one endpoint of which lies on the smaller circle arc AB and the other endpoint lies on the larger one.

4. (A.G.Myakishev) Let P be the point of intersection for the diagonals of a quadrilateral $ABCD$. Let M be the meet point for the lines linking the midpoints of its opposite sides. Let O be the meet point for the perpendicular bisectors to the diagonals. Let H be the meet point for the lines linking orthocenters of triangles APD and BCP, APB and CPD. Prove that M is the midpoint of OH.

5. (B.R.Frenkin) Given a triangle with the following property: for any of its sides it is impossible to construct a triangle from the altitude, the bisector and the median drawn to this side. Prove that one of the angles of the given triangle is greater than $135°$.

Grade 10

1. (L.A.Yemelyanov) Given a convex quadrilateral without any parallel sides. For every triple of its vertices, the point is constructed which complements this trio to a parallelogram (one diagonal of which coincides with a diagonal of quadrilateral). Prove that out of four constructed points, exactly one lies within the initial quadrilateral.

2. (A.V.Shapovalov) A triangle can be cut into three similar triangles. Prove that it can be cut into any number of triangles similar to each other.

3. (A.A.Zaslavsky) Two parallel chords AB and CD are drawn in a circle with center O. The circles with diameters AB and CD intersect at point P. Prove that the midpoint of segment OP is equidistant from the lines AB and CD.

4. (Wim Pijls, the Netherlands) Two segments A_1B_1 and A_2B_2 on the plane are given such that $\frac{A_2B_2}{A_1B_1} = k < 1$. The point A_3 is chosen on the segment A_1A_2 and the point A_4 is chosen on the extension of segment A_1A_2 beyond A_2 so that $\frac{A_3A_2}{A_3A_1} = \frac{A_4A_2}{A_4A_1} = k$. Similarly the point B_3 is chosen on the segment B_1B_2 and the point B_4 is chosen on the extension of segment B_1B_2 beyond B_2 so that $\frac{B_3B_2}{B_3B_1} = \frac{B_4B_2}{B_4B_1} = k$. Find the angle between the lines A_3B_3 and A_4B_4.

5. (A.A.Zaslavsky) Two circles of unit radius intersect at the points X and Y. The distance between these points also equals one. From point C on one circle, tangents CA and CB are drawn to the other circle. The line CB has a second intersection with circle one at the point A'. Find the distance AA'.

6. (A.A.Zaslavsky) Let H be the orthocenter in the triangle ABC and X be an arbitrary point. The circle with diameter XH has second intersections with lines AH, BH, CH at points A_1, B_1, C_1, while with lines AX, BX, CX at points A_2, B_2, C_2. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

Grade 11

1. (A.A.Zaslavsky) Let A_1, B_1, C_1 be the midpoints of sides in the regular triangle ABC. Three parallel lines passing through A_1, B_1, C_1, intersect the lines B_1C_1, C_1A_1, A_1B_1 at points A_2, B_2, C_2 respectively. Prove that the lines AA_2, BB_2, CC_2 concur in the point that belongs to the circumcircle of the triangle ABC.

2. (A.G.Myakishev) A convex quadrilateral $ABCD$ is given. The lines BC and AD intersect at point O so that the point B belongs to the segment OC, while the point A belongs to the segment OD. Let I be the incenter of the triangle OAB, J be the center of an excircle of the triangle OCD (tangent to side CD and the extensions of two other sides). Perpendiculars dropped from the midpoint of segment IJ to the lines BC and AD, intersect with respective sides of the quadrilateral (not their extensions) at points X and Y. Prove that the segment XY divides the perimeter of the quadrilateral $ABCD$ in half. In particular, out of all the segments with this property and with the endpoints at BC and AD, the segment XY has the minimal length.

3. (A.A.Zaslavsky) Within the inscribed quadrilateral $ABCD$ there is a point K such that the distances from it to the sides of $ABCD$ are proportional to these sides. Prove that K is the point of intersection of diagonals in $ABCD$.

4. (I.F.Sharygin) In triangle ABC, $\angle A = \alpha$, $BC = a$. The incircle is tangent with lines AB and AC at points M and P. Find the length of the chord dissected from the line MP by the circle with diameter BC.

5. (V.Yu.Protasov) The angle on the plane and point K within it are given. Prove that there exists a point M with the following property: if an arbitrary line passing through K intersects the sides of the angle at points A and B, then MK is the bisector of angle AMB.

6. (I.I.Bogdanov) The sphere inscribed in tetrahedron $ABCD$ is tangent to its faces at points A', B', C', D'. The segments AA' and BB' do intersect, and their meeting point lies on the inscribed sphere. Prove that the segments CC' and DD' also intersect on the inscribed sphere.