MATHEMATICAL WORLD VOLUME 26

Geometry of Conics

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ГЕОМЕТРИЧЕСКИЕ СВОЙСТВА КРИВЫХ ВТОРОГО ПОРЯДКА

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МЦНМО, МОСКВА, 2007

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Translated by Alex Martsinkovsky

Cover art created by Ivan Velichko

2000 *Mathematics Subject Classification.* Primary 51–02, 51M04.

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Library of Congress Cataloging-in-Publication Data

Akopyan, A. V. (Arseny V.), 1984–
Geometry of conics / A. V. Akopyan, A. A. Zaslavsky.
p. cm. — (Mathematical world ; v. 26)
Includes bibliographical references and index.
ISBN 978-08218-4323-9 (alk. paper)
1. Geometry, Analytic—Plane. 2. Curves, Algebraic. I. Zaslavskii, A. A. (Alexey A.), 1960– II. Title.

QA552.A46 2007
516.2′152—dc22 2007060841

10 9 8 7 6 5 4 3 2 1 12 11 10 09 08 07

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Preface

Curves of second degree, or conics, are traditionally viewed as objects pertaining to analytic geometry and are studied in lower-level courses in engineering colleges. At best, only the optical properties of conics are mentioned among their geometric properties. But those curves also possess a number of other nice properties, a majority of which can be established by methods of elementary geometry well within the reach of high school students. Moreover, conics help solve some geometric problems seemingly unrelated to conics. In this book the reader will find the most interesting facts about curves of order two, including those proved recently.

Chapter 1 deals with the elementary properties of conics. Most of the facts mentioned there are well known. The remaining material is also rather simple, so that the entire chapter does not impose any prerequisites on the reader beyond the standard high school curriculum. Some simple but important results are offered as exercises. We recommend that the reader try to solve them before reaching for the solutions. This should facilitate the understanding of the material later on. Chapter 2 is of an auxiliary nature. It contains some facts from classical geometry needed for understanding the remaining chapters, which are not usually studied in high school. In Chapter 3 we mention projective properties common to all conics. Some of them, such as the theorem on pencils of conics, are quite complicated. Finally, Chapter 4 is devoted to metric properties. As a rule, they concern only special kinds of conics. This is the most complicated chapter of the book, which requires a good understanding of the material in the previous chapters.

The authors are grateful to I. I. Bogdanov and E. Yu. Bun′kova for valuable comments.

Chapter 1

Elementary Properties of Curves of Second Degree

1.1. Definitions

If you stake a goat, it will graze the grass inside the circle that is centered at the stake and has radius the length of the rope. If you use two stakes at the ends of the rope and tie the goat using a sliding ring, the region with grazed grass will look like the one shown in Figure 1.1.

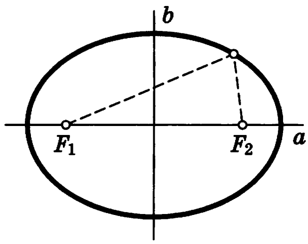

FIGURE 1.1. F_1 and F_2 are the foci; a and b are the major and the minor axes.

For all points on the boundary of that figure, the sum of the distances to the stakes equals the length of the rope. Such a curve is called an *ellipse*, and the points marked by the stakes are called the *foci*.

Clearly, an ellipse looks like an "elongated circle". It obviously has two axes of symmetry. These are the line connecting the foci and the midpoint perpendicular to the segment with endpoints at the foci. These two lines are called the *major* and the *minor axes of the ellipse*. The lengths of their parts inside the ellipse are called the lengths of the major and minor axes. The distance between the foci is called the focal distance.

It is also clear that the length of the rope holding the goat equals the length of the major axis of the elliptical boundary of the grazed region.

Intuitively it is clear that the goat can graze at any point inside the ellipse but it can never get beyond the ellipse. But a purely mathematical reformulation of this is no longer so obvious.

Exercise 1. Prove that the sum of the distances from any point inside the ellipse to the foci is less—and from any point outside the ellipse is greater—than the length of the major axis.

Solution. Denote by F_1 and F_2 the foci of the ellipse, and by X a point. Let Y be the intersection of the ray F_1X and the ellipse. Assume first that X is inside the ellipse. By the triangle inequality, $F_2X < XY + YF_2$, and hence $F_1X + XF_2 < F_1X + XY + YF_2 = F_1Y + F_2Y$ (Figure 1.2).

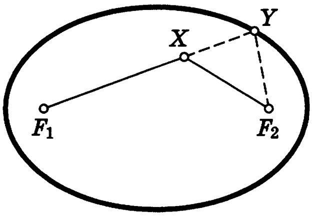

FIGURE 1.2

But $F_1Y + F_2Y$ equals the length of the rope, i.e., the major axis of the ellipse. Using a similar argument when X is outside the ellipse, we have $F_2Y < XY+XF_2$. Therefore $F_1X+XF_2 = F_1Y+YX+XF_2 > F_1Y+F_2Y$.

Ellipses often arise in mechanics. For example, a planet orbiting the Sun moves along an ellipse with the Sun at one of its foci (Kepler's Law).

An ellipse is an example of a *curve of second degree* or a *conic.* Other examples of such curves are *parabolas* and *hyperbolas.*

A *hyperbola* is the set of points for which the absolute value of the difference between the distances to two fixed points, called the *foci,* is constant.

A hyperbola consists of two branches the ends of which approach two lines called the *asymptotes of the hyperbola* (Figure 1.3). A hyperbola with perpendicular asymptotes is said to be *equilateral.*

The line passing through the foci of a hyperbola is an axis of symmetry and is called the *real axis.* The perpendicular line passing through the midpoint between the foci is also an axis of symmetry and is called the *imaginary axis* of the hyperbola.

If a comet is passing by the Sun and the gravitational force exerted by the Sun is too small to keep the comet within the solar system, then its trajectory will be an arc of a hyperbola whose focus will be at the center of the Sun.

A *parabola* is the set of points whose distances to some fixed point and line are constant. That point and line are called, respectively, the *focus* and the *directrix* of the parabola. The line perpendicular to the directrix and passing through the focus is called the *axis of the parabola* (Figure 1.4).

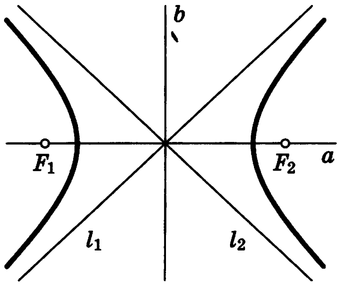

FIGURE 1.3. F_1 and F_2 are the foci, a and b are the real and imaginary axes, and l_1 and l_2 are the asymptotes.

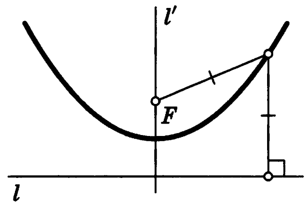

FIGURE 1.4. F is the focus; l and l' are the directrix and the axis of the parabola.

Clearly, it is an axis of symmetry of the parabola.

We remark that a stone thrown at an angle to the horizon will move along a parabola.

In a way, from the geometric point of view, there is only one parabola (just as there is only one circle). More precisely, all the parabolas are similar, i.e., they can be transformed into one another by rotational homotheties.

Consider a family of ellipses with focus at a fixed point and passing through another given point. We send the other focus to infinity along some direction. Then those ellipses will tend to a parabola with the same focus and axis parallel to the chosen direction. A similar experiment works for hyperbolas. Thus the parabola is a limit case of both the ellipse and the hyperbola.

Exercise 2. State and prove, for the parabola and the hyperbola, the results similar to the one in Exercise 1.

Solution. For the points inside the parabola the distance to the focus is less than the distance to the directrix, and for the points outside the parabola the opposite is true (Figure 1.5).

Let Y be the projection of X to the directrix, Z the intersection of XY with the parabola, and F the focus of the parabola. By the definition of the parabola, $FZ = ZY$. If X lies inside the parabola, then $XY = XZ + ZY$. By the triangle inequality, $FX < FZ + ZX = ZY + ZX = XY$. If X and the parabola are on different sides of the directrix, then the assertion

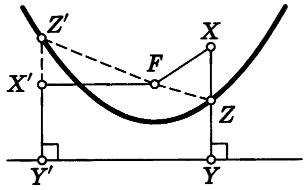

FIGURE 1.5

is obvious. Suppose X' is outside the parabola but on the same side of the directrix. Then $Z'Y' = Z'X' + X'Y'$ and, by the triangle inequality, $FX' + X'Z' > FZ' = Z'Y' = Z'X' + X'Y'$. Therefore $FX' > X'Y'$. □

In the case of a hyperbola the corresponding statement is as follows: let d be the difference of the distances from any point on the hyperbola to the foci F_1 and F_2 and let Γ be the branch of the hyperbola inside which F_1 lies. Then for the points X outside (inside) Γ the quantity $XF_2 - XF_1$ is less (greater) than d.

Suppose X lies inside Γ and let Y be the intersection of the ray F_2X and Γ. We have $F_2X = F_2Y + YX$. By the triangle inequality, $F_1X < F_1Y + YX$; therefore $F_2X - F_1X > (F_2Y + YX) - (F_1Y + YX) = F_2Y - F_1Y = d$.

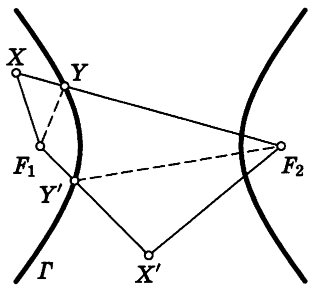

FIGURE 1.6

If X' is outside Γ, let Y' be the point of intersection of F_1X' and Γ. Then $F_1X' = F_1Y' + Y'X'$. By the triangle inequality, $F_2X' < F_2Y' + Y'X'$. Therefore $F_2X' - F_1X' < (F_2Y' + Y'X') - (F_1Y' + Y'X') = F_2Y' - F_1Y' = d$.

We remark (without a proof, for the time being) that the ellipse, the parabola and the hyperbola have the following properties: an arbitrary line intersects each of those curves in at most two points, and, given any point in the plane, there are at most two tangents from that point to the curve. These properties are obvious consequences of the results of 1.5.

Exercise 3. Find the locus of the centers of the circles tangent to two given circles.

Solution. For the sake of definiteness, consider the case when none of the circles with centers O_1, O_2 and radii r_1, r_2 contains the other. If the circle centered at O of radius r is tangent to the two circles on the outside, then

$OO_1 = r + r_1$ and $OO_2 = r + r_2$, and therefore $OO_1 - OO_2 = r_1 - r_2$, i.e., O lies on one of the branches of the hyperbola with foci O_1 and O_2. Similarly, if a circle is tangent to both circles on the inside, then its center lies on the other branch of the same hyperbola. If one of the tangencies is on the inside and the other on the outside, then the absolute value of the difference in distances OO_1 and OO_2 is equal to $r_1 + r_2$, i.e., O sweeps another hyperbola with the same foci. Similarly, if one circle is inside the other, then the desired locus consists of two ellipses with foci O_1 and O_2 and major axes $r_1 + r_2$ and $r_1 - r_2$. The case of intersecting circles is left to the reader.

1.2. Analytic definition and classification of curves of second degree

In the previous section we mentioned the fact that the ellipse, parabola, and hyperbola are particular cases of curves of degree two. Now we make this more precise by showing that, in a sense, there are no other curves of degree two.

Definition. A *curve of second degree* is a set of points whose coordinates in some (and therefore in any) Cartesian coordinate system satisfy a second order equation:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2b_1x + 2b_2y + c = 0. \tag{1}$$

If the left-hand side of (1) is a product of two linear factors, then the curve is the union of two lines (which may coincide). In that case it is said to be *degenerate.* A curve which contains exactly one point (for example, $x^2+y^2 = 0$) is also said to be degenerate.

It is a known result from analytic geometry (see, for example, [**1**]) that for any nondegenerate curve there is a coordinate system in which its equation has a rather simple form. We now describe the main idea behind this result.

First, rotate the coordinate system through an angle ϕ. This means that, in equation (1), the coordinates x and y should be replaced by, respectively, $x\cos\phi + y\sin\phi$ and $-x\sin\phi + y\cos\phi$. Choosing an appropriate ϕ, we can make the coefficient of xy equal to zero. Next we move the origin to (x_0, y_0), i.e., we replace x by $x + x_0$ and y by $y + y_0$. By choosing an appropriate pair (x_0, y_0) we can transform (1) into one of the three canonical forms (I), (II), or (III).

A direct calculation shows that the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \geq b > 0, \tag{I}$$

is an *ellipse* centered at the origin, with foci at $(\pm\sqrt{a^2 - b^2}, 0)$ and major and minor semi-axes (i.e., half the lengths of the corresponding axes) equal, respectively, to a and b. In the special case $a = b$, ellipse (I) is a circle.

The curve

$$\text{(II)} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a > 0, \quad b > 0,$$

is a *hyperbola* that intersects its real axis in two points at distance $2a$ from each other. The quantities a and b are called, respectively, the real and the imaginary semi-axes of the hyperbola. The lines $x/y = \pm a/b$ are the asymptotes of the hyperbola and the points $(\pm\sqrt{a^2+b^2}, 0)$ are the foci. When $a = b$ hyperbola (II) is equilateral.

If

$$\text{(III)} \qquad y^2 = 2px, \quad p > 0,$$

the curve is a parabola, whose axis coincides with the x-axis, the focus is at $(p/2, 0)$, and the directrix is given by $x = -p/2$.

The curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$$

is called an *imaginary ellipse*; it contains no real points.

Henceforth, unless stated otherwise, a curve of degree two will always be nondegenerate and not imaginary.

Problem 1. Prove that the equation $y = 1/x$ describes a hyperbola and find its foci.

1.3. The optical property

As is known, if a ray of light is reflected in a mirror, then the reflection angle equals the incidence angle. This is related to the so-called Fermat principle, which states that the light always travels along the shortest path. We shall now prove that the path is indeed the shortest one.

Thus we have a line l and points F_1 and F_2 lying on the same side of it. We want to find a point P on the line such that the sum of the distances from P to F_1 and F_2 is minimal. Reflecting F_2 in l we have a point F_2'. Clearly, $F_2X = F_2'X$ for any point X on l. Thus we need a point P such that the sum of the distances from P to F_1 and F_2' will be the smallest possible. Clearly, the minimum is attained when P lies on the segment F_1F_2' intersecting l. Then the angles in question are obviously equal (Figure 1.7).

Exercise 1. a) When will the absolute value of the difference in distances from P to points F_1 and F_2 lying on different sides of l be maximal?

b) Given two lines l and l' and a point F not on any of those lines, find a point P on l such that the (signed) difference of distances from it to l' and F is maximal.

Solution. a) Let F_2' be the reflection of F_2 in l. Clearly, $F_2X = F_2'X$ for any point X on l. We need a point P such that the difference of distances from P to F_1 and F_2' is maximal. It follows from the triangle inequality that

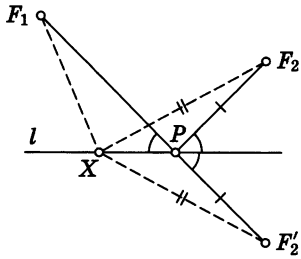

FIGURE 1.7

$|F_1P - F_2'P| < F_1F_2'$ and the maximum is attained if and only if F_1, F_2' and P lie on a straight line. Since the points F_2 and F_2' are the reflections of each other, the angles formed by the lines F_1P and F_2P with l are equal (Figure 1.8).

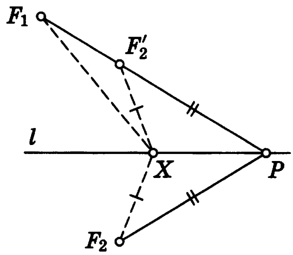

FIGURE 1.8

b) Let F' be the reflection of F in l. Of the two points F and F' choose the one whose (signed) distance to l' is minimal. Let it be F and let d be the distance from F to l'. Then for any point P on l the distance to l' is not greater than $PF + d$. Therefore the difference in question never exceeds d. On the other hand, it is exactly d when P lies on the perpendicular to l' passing through F (Figure 1.9).

We also note that if the line F_1F_2' in a) is parallel to l and the line l' in b) is perpendicular to l, then there is no maximum (it is attained at infinity).

Now we state one of the most important properties of conics, the so-called optical property.

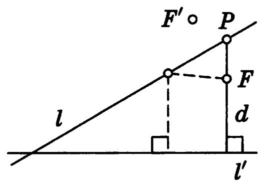

FIGURE 1.9

Theorem 1.1 (The optical property of the ellipse). *Suppose a line l is tangent to an ellipse at a point P. Then l is the bisector of the exterior angle F_1PF_2* (Figure 1.10).

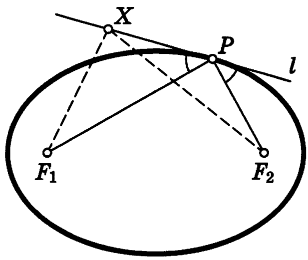

FIGURE 1.10

Proof. Let X be an arbitrary point of l different from P. Since X is outside the ellipse, we have $XF_1 + XF_2 > PF_1 + PF_2$, i.e., of all the points of l the point P has the smallest sum of the distances to F_1 and F_2. This means that the angles formed by the lines PF_1 and PF_2 with l are equal. □

Exercise 2. State and prove the optical property for parabolas and hyperbolas.

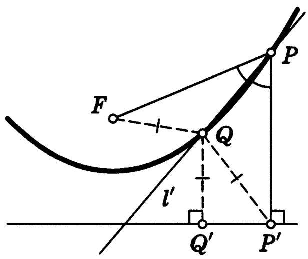

FIGURE 1.11

Solution. For parabolas the optical property is stated as follows. Suppose a line l is tangent to a parabola at a point P. Let P' be the projection of P to the directrix. Then l is the bisector of the angle FPP' (Figure 1.11).

Suppose that the bisector of the angle FPP' (call it l') intersects the parabola in yet another point, say, Q whose projection to the directrix is denoted Q'. By the definition of the parabola, $FQ = QQ'$. On the other hand, triangle FPP' is isosceles, and the bisector of the angle P is the midpoint perpendicular to FP'. Therefore for any point Q on that bisector we have $QP' = QF = QQ'$. But this is impossible because Q' is the only point on the directrix of the parabola where the distance to Q is minimal.

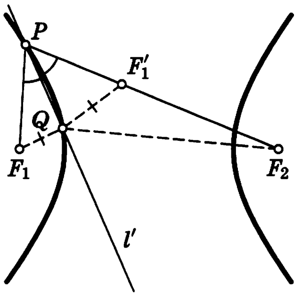

FIGURE 1.12

We now state the optical property for the hyperbola.

If a line l is tangent to a hyperbola at a point P, then l is the bisector of the angle F_1PF_2, where F_1 and F_2 are the foci of the hyperbola (Figure 1.12).

Suppose that the bisector l' of the angle F_1PF_2 intersects the hyperbola at yet another point Q (lying on the same branch with P). For convenience, assume that P lies on the branch closer to F_1. Let F_1' be the reflection of F_1 in l'. Then $F_1Q = QF_1'$, $F_1P = PF_1'$; moreover F_2, F_1' and P lie on a line. Thus, $F_2P - PF_1 = F_2Q - F_1Q$, and therefore $F_2F_1' = F_2P - PF_1' = F_2Q - QF_1'$. But, by the triangle inequality, $F_2F_1' > F_2Q - QF_1'$.

The above results can also be proved by arguments similar to the proof of the optical property of the ellipse. For that, use Exercise 1.

The optical property of the parabola was already known in ancient Greece. For example, Archimedes, by arranging copper plates into a parabolic mirror, managed to set on fire the Roman fleet laying siege to Syracuse.

Exercise 3. Consider the family of confocal conics (these are conics with the same foci). Prove that any hyperbola and any ellipse from that family intersect at right angles (the angle between two curves is by definition the angle between the tangents to them at their point of intersection; see Figure 1.13).

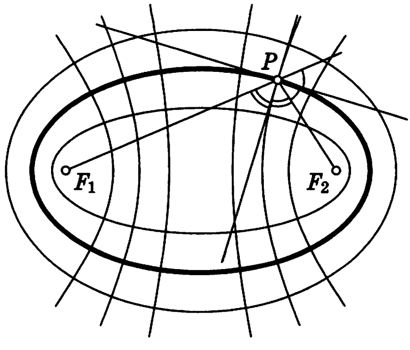

FIGURE 1.13

Solution. Suppose an ellipse and a hyperbola with foci F_1 and F_2 intersect at P. Then their tangents at that point will be the bisectors of the exterior and interior angles F_1PF_2, respectively. Therefore they are perpendicular.

Theorem 1.2. *Suppose the chord PQ contains a focus F_1 of the ellipse and R is the intersection of the tangents to the ellipse at P and Q. Then R is the center of an excircle of the triangle F_2PQ, and F_1 is the tangency point of that circle and the side PQ* (Figure 1.14).

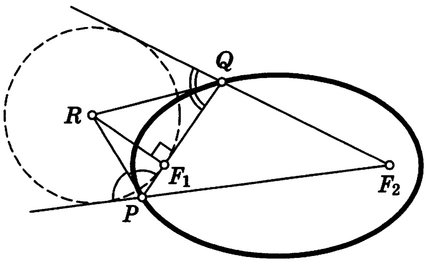

FIGURE 1.14

Proof. By the optical property, PR and QR are the bisectors of the exterior angles of the triangle F_2PQ. Therefore R is the center of an excircle. The tangency point (call it F_1') of the excircle and the corresponding side and the point F_2 cut the perimeter of the triangle into equal parts, i.e., $F_1'P+PF_2 = F_2Q+QF_1'$. But F_1 has this property and there is only one such point. Hence F_1' and F_1 coincide. □

Corollary. *The straight line connecting a focus of an ellipse and the intersection of the tangents to the ellipse at the ends of a chord containing that focus is perpendicular to the chord.*

For the hyperbola, Theorem 1.2 is also true but the excircle should be replaced by the incircle.

1.4. The isogonal property of conics

The optical property yields elementary proofs of some amazing results.

Theorem 1.3. *From any point P outside an ellipse draw two tangents to the ellipse, with tangency points X and Y. Then the angles F_1PX and F_2PY are equal (F_1 and F_2 are the foci of the ellipse).*

Proof. Let F_1', F_2' be the reflections of F_1 and F_2 in PX and PY, respectively (Figure 1.15).

Then $PF_1' = PF_1$ and $PF_2' = PF_2$. Moreover, the points F_1, Y and F_2' lie on a line (because of the optical property). The same is true for the points F_2, X and F_1'. Thus $F_2F_1' = F_2X + XF_1 = F_2Y + YF_1 = F_2'F_1$.

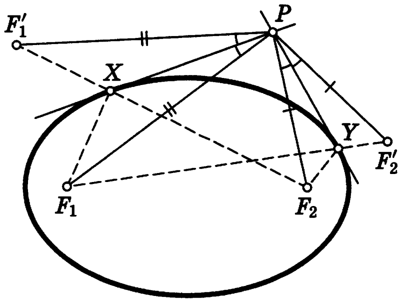

FIGURE 1.15

Thus, the triangles PF_2F_1' and PF_1F_2' are equal (having three equal sides). Therefore

$$\angle F_2PF_1 + 2\angle F_1PX = \angle F_2PF_1' = \angle F_1PF_2' = \angle F_1PF_2 + 2\angle F_2PY.$$

Hence $\angle F_1PX = \angle F_2PY$, which is the desired result.[1] □

Figure 1.16 shows that a similar property holds for the hyperbola.[2]

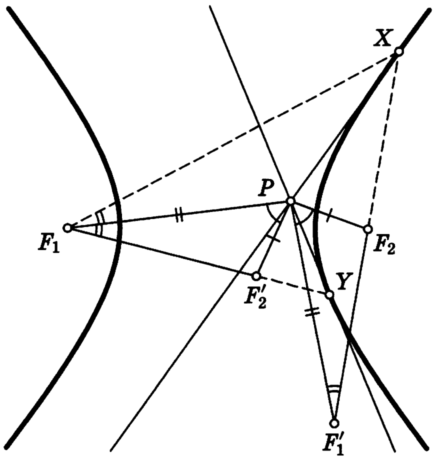

FIGURE 1.16

Suppose now that the ellipse (or hyperbola) with foci F_1 and F_2 is inscribed in triangle ABC. It follows from the above that $\angle BAF_1 = \angle CAF_2$, $\angle ABF_1 = \angle CBF_2$ and $\angle ACF_1 = \angle BCF_2$.

We shall show in 2.3 that, in a plane, for any (with rare exceptions) point X there is a unique point Y such that X and Y are the foci of a

[1] We consider the case when F_1 and F_2 are inside the angle $F_1'PF_2'$ and F_1 lies inside the angle F_2PF_1'. In the remaining cases the arguments are similar.

[2] The reader should check two cases: when the tangency points are either on different branches or on the same branch.

conic tangent to each side of a triangle. Such Y is said to be the *isogonal conjugate of* X with respect to the triangle.

The construction used in the proof of Theorem 1.3, allows one to obtain yet another interesting result. Since the triangles PF_2F_1' and $PF_2'F_1$ are equal, the angles $PF_1'F_2$ and PF_1F_2' are also equal. Therefore

$$\angle PF_1X = \angle PF_1'F_2 = \angle PF_1F_2' = \angle PF_1Y.$$

Thus we have proved the following generalization of Theorem 1.2.

Theorem 1.4. *In the notation of Theorem* 1.3, *the line* F_1P *is the bisector of the angle* XF_1Y (Figure 1.17).

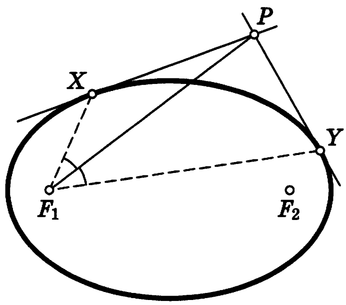

FIGURE 1.17

Theorem 1.5. *The locus of points from which a given ellipse is seen at a right angle* (*i.e., the tangents to the ellipse drawn from such a point are perpendicular*) *is a circle centered at the center of the ellipse* (Figure 1.18).

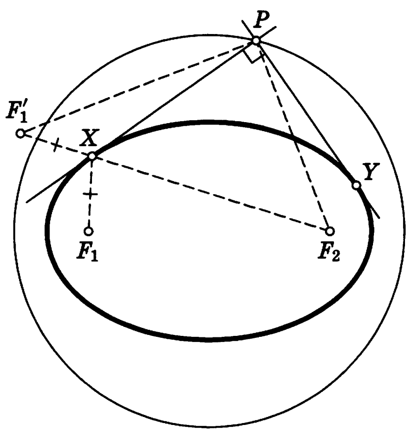

FIGURE 1.18

Proof. Let F_1 and F_2 be the foci of the ellipse and suppose that the tangents to the ellipse at X and Y intersect in P. Reflecting F_1 in PX we have a point F_1'. It follows from Theorem 1.3 that $\angle XPY = \angle F_1'PF_2$ and $F_1'F_2 = F_1X + F_2X$, i.e., the length of the segment $F_1'F_2$ equals the major axis of the ellipse (the length of the rope tying the goat). The angle $F_1'PF_2$ is right if and only if $F_1'P^2 + F_2P^2 = F_1'F_2^2$ (by the Pythagorean theorem). Therefore XPY is a right angle if and only if $F_1P^2 + F_2P^2$ equals the square of the major axis of the ellipse. But it is not difficult to see that this condition defines a circle. Indeed, suppose F_1 has Cartesian coordinates (x_1, y_1), and F_2 has coordinates (x_2, y_2). Then the coordinates of the desired points P satisfy the condition

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 = C,$$

where C is the square of the major axis. But since the coefficients of x^2 and y^2 are equal (to 2) and the coefficient of xy is zero, the set of points satisfying this condition is a circle. By virtue of symmetry, its center is the midpoint of the segment F_1F_2. □

For the hyperbola such a circle does not always exist. When the angle between the asymptotes of the hyperbola is acute, the radius of the circle is imaginary. If the asymptotes are perpendicular, then the circle degenerates into the point which is the center of the hyperbola.

Example. Given points $P_1, \ldots, P_n$ and numbers $k_1, \ldots, k_n$ and C, the locus of points X such that $k_1XP_1^2 + \cdots + k_nXP_n^2 = C$ is a circle, known as the *Fermat–Apollonius circle.* Clearly, it may have an imaginary radius (when?).

Theorem 1.6. *Suppose a string is put on an ellipse α and then pulled tight using a pencil. If the pencil is rotated about the ellipse, it will traverse another ellipse confocal with α* (Figure 1.19).

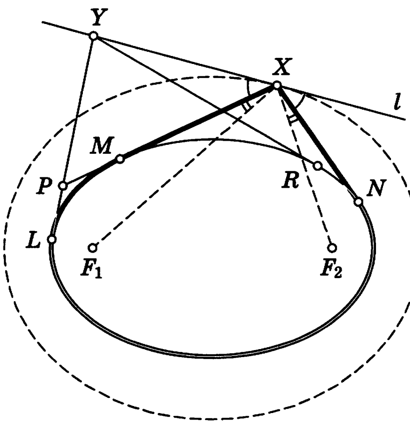

FIGURE 1.19

Proof. Clearly, the new figure (call it α_1) has a smooth boundary. We shall show that at each point X on α_1 the tangent to the new curve coincides with the bisector of the exterior angle F_1XF_2.

Let XM and XN be the tangents to α. Then $\angle F_1XN = \angle F_2XM$, and hence the bisector l of the exterior angle NXM coincides with the bisector of the exterior angle F_1XF_2. Call it l.

Let Y be an arbitrary point on l and YL and YR the tangents to α, as shown in Figure 1.19. We assume that Y lies "to the left" of X; the other case is argued similarly.

Let P be the intersection of the lines XM and YL. It is easy to see that $YN < YR + \smile RN$, and $\smile LM < LP + PM$. Moreover, since l is the exterior bisector of the angle NXP, we have $PX + XN < PY + YN$. Therefore

$$\begin{aligned} MX + XN + \smile NM &< MX + XN + \smile NL + LP + PM \\ &= PX + XN + \smile NL + LP < PY + YN + \smile NL + LP \\ &= LY + YN + \smile NL \\ &< LY + YR + \smile RN + \smile NL = LY + YR + \smile RL \end{aligned}$$

(here the arcs are meant to be the arcs under the string). Therefore Y lies outside α_1. The same is true for any point Y on l. It follows that α_1 contains a single point of l, i.e., the line is tangent. It also follows at once that the obtained curve is convex.

Thus the sum of the distances to the foci F_1 and F_2 does not change with time. Therefore the trajectory of the pencil is an ellipse.

Here is a more rigorous approach to the last claim. Suppose X is outside the ellipse. Put the pencil at X and pull the string around it and around the ellipse. Let $f(X)$ be the length of the string and $g(X) = F_1X + F_2X$ (a point is understood as a pair of its coordinates; thus both f and g depend on a pair of real numbers). One can show that those functions are continuously differentiable and that the vectors $\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and $\operatorname{grad} g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$ are nonzero at each point. Then, by the implicit function theorem, the curve traversed by the pencil with a string of fixed length (i.e., a level curve of f) is smooth (continuously differentiable). It now follows that the curve can be parametrized by a differentiable function $R = R(t)$ (this is again a pair of coordinate functions $x = x(t)$, $y = y(t)$) whose tangent vector is different from zero. As shown before, the tangent vector $\frac{dR}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ of the curve is tangent to a level curve of g, i.e., it is perpendicular to $\operatorname{grad} g(R)$ at $R = R(t)$. Consider the function $g(R(t))$. Its derivative is

$$\frac{dg(R(t))}{dt} = \frac{\partial g}{\partial x}\frac{dx(t)}{dt} + \frac{\partial g}{\partial y}\frac{dy(t)}{dt} \equiv 0$$

(this is the orthogonality condition mentioned above), i.e., $g(R(t))$ is constant. This means that our curve lies on an ellipse with the same foci. Since any ray starting at F_1 must contain a point on our curve, the curve coincides with the ellipse. □

Problem 2. A $2n$-gon is circumscribed about a conic with focus F. Its sides are colored in black and white in an alternating pattern. Prove that the sum of the angles at which the black sides are seen from F equals $180°$.

Problem 3. An ellipse is inscribed in a convex quadrilateral such that its foci lie on the (distinct) diagonals of the quadrilateral. Prove that the products of the opposite sides are equal.

1.5. Curves of second degree as projections of the circle

Given a circle, draw the perpendicular through its center to the plane of the circle and pick a point S on it. The lines connecting S to the points of the circle form a cone. Consider the section of the cone by a plane π intersecting all of its rulings and not perpendicular to its axis of symmetry.

Now inscribe in the cone two spheres touching π at points F_1 and F_2 (Figure 1.20).

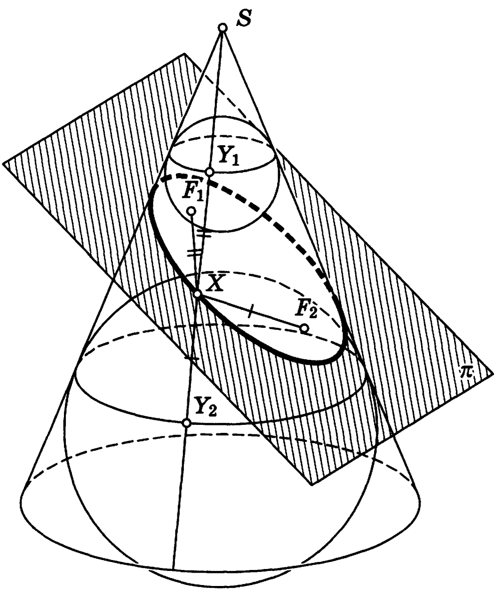

FIGURE 1.20

Let X be an arbitrary point on the intersection of the cone and the plane π. The ruling SX intersects the inscribed spheres at points Y_1 and Y_2. We have $XF_1 = XY_1$ and $XF_2 = XY_2$, since the segments of tangents to a sphere drawn from the same point are equal. Therefore $XF_1 + XF_2 = Y_1Y_2$. But Y_1Y_2 is the segment of the ruling lying between the two planes perpendicular to the axis of the cone, and its length does not depend on the choice of X. Hence the intersection of the cone with π is an ellipse. The ratio of its semiaxes depends on the tilt of the plane and, obviously, can take on any value. Therefore any ellipse can be obtained as a central projection of the circle.

A similar proof shows that if the secant plane is parallel to two rulings of the cone, then the cross-section is a hyperbola (Figure 1.21).

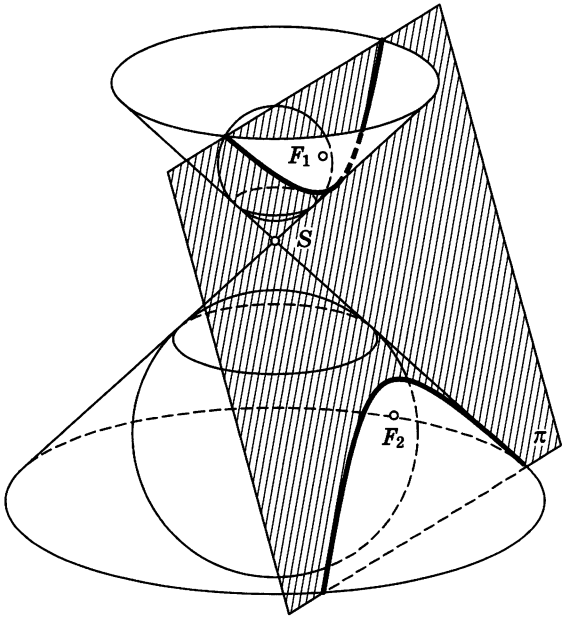

FIGURE 1.21

Finally, consider the case when the secant plane is parallel to one ruling (Figure 1.22).

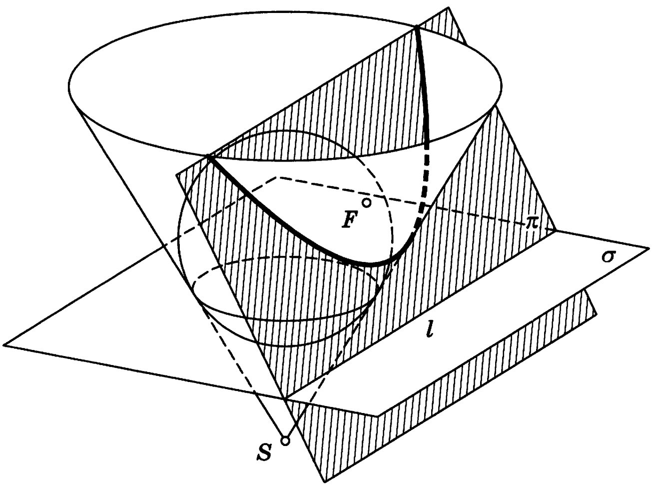

FIGURE 1.22

Inscribe in the cone the sphere tangent to π at a point F. This sphere is tangent to the cone along a circle lying in a plane σ. Let l be the line of intersection of the planes π and σ. For an arbitrary point X in the intersection of the cone and the plane π let Y be the point of intersection of the ruling SX with the plane σ and let Z be the projection of X to l. Then $XF = XY$ since the two segments are tangent to the sphere. On the other hand, Y and Z lie in σ, the angle between XY and σ is equal to the angle between a ruling and a plane perpendicular to its axis, and the angle between XZ and σ is equal to the angle between the planes π and σ. By the choice of π, those angles are equal. Hence $XY = XZ$, since these segments form equal angles with the plane σ. Therefore $XF = XZ$ and X lies on the parabola with focus F and directrix l.

Thus any nondegenerate curve of order two can be obtained as a section of the cone. Because of that, such curves are also called *conic sections* or simply *conics*.

We remark that if the cone is replaced by the cylinder, then the same argument shows that the corresponding section will be an ellipse. Accordingly, the ellipse can be obtained as a parallel projection of the circle.

Exercise 1. Find the locus of the midpoints of the chords of an ellipse which are parallel to a given direction.

Solution. Consider the ellipse as a parallel projection of a circle. Then the parallel chords of the ellipse and their midpoints correspond to parallel chords of the circle and their midpoints, the latter lying on a diameter of the circle. Therefore the locus of the midpoints of parallel chords of the ellipse is also a diameter (i.e., a chord passing through the center).

Exercise 2. Using a straightedge and a compass find the foci of a given ellipse.

Solution. Construct two parallel chords of the ellipse. By the preceding exercise, the line connecting their centers is a diameter of the ellipse. After constructing another diameter, we can find the center O of the ellipse. By the symmetry of the ellipse, a circle centered at O intersects the ellipse at four points forming a rectangle with sides parallel to the axes of the ellipse. Now the foci of the ellipse can be found as the points of intersection of the major axis and the circle centered at the end of the minor axis of radius equal to the major half-axis.

The spheres inscribed in the cone and touching the secant plane are called the *Dandelin spheres*.

1.6. The eccentricity and yet another definition of conics

The construction just described of the Dandelin spheres yields another important property of conics.

Suppose a plane π intersects all the rulings of a circular cone with vertex S. Consider a sphere inscribed in the cone and touching π at a point F_1. As in the parabola case, let σ be the plane containing the tangency points. Let l be the line of intersection of π and σ. Suppose a point X is in the intersection of the cone and the plane π. Let Y be the intersection of the line SX with σ and Z the projection of X to l. We shall show that the ratio of XY and XZ is constant, i.e., does not depend on X.

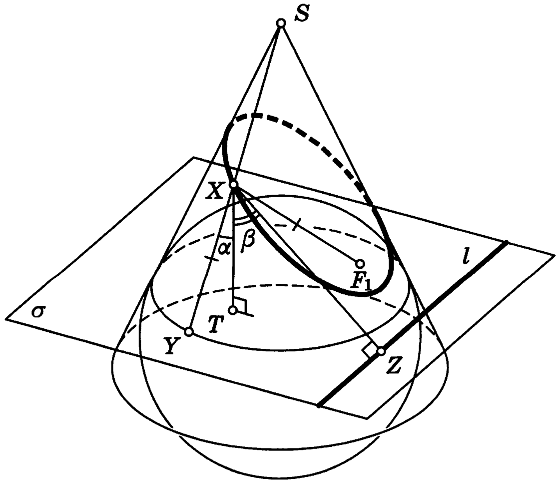

FIGURE 1.23

Let T be the projection of X to σ. The ratio of XT and XY does not depend on X and equals the cosine of the angle between a ruling of the cone and its axis (call that angle α). The ratio of XT and XZ also does not depend on X and equals the cosine of the angle between the plane π and the cone axis (call that angle β). Therefore

$$\frac{XY}{XZ} = \frac{XY}{XT} \cdot \frac{XT}{XZ} = \frac{\cos\beta}{\cos\alpha}.$$

Since XF_1 and XY are equal (as tangents to the sphere passing through X), the ratio of XF_1 and XZ is constant.

Thus for any conic there is a line l such that for any point on the conic the ratio of the distances to the focus and that line is constant. This ratio is called the *eccentricity* of the conic curve, and the lines are called the *directrices.* Both the ellipse and the hyperbola have two directrices (one for each focus).

It is easy to see that this property leads to yet another definition of curves of degree two.

A conic curve with focus F, directrix l (F not on l), and eccentricity ϵ is the set of points where the ratio of distances to F and to l equals ϵ.

If $\epsilon > 1$, then the curve is a hyperbola, if $\epsilon < 1$, it is an ellipse, and when $\epsilon = 1$, it is a parabola.

Problem 4. Prove that the asymptotes of all equilateral hyperbolas with focus F and passing through a point P are tangent to two circles (one circle for each family of the asymptotes).

1.7. Some remarkable properties of the parabola

In this section F denotes the focus of the parabola under consideration.

We begin with a lemma that we will use more than once.

Lemma 1.1. *If the focus of a parabola is reflected in a tangent, then its image will be on the directrix. That image is the projection of the point where the tangent touches the parabola* (Figure 1.24).

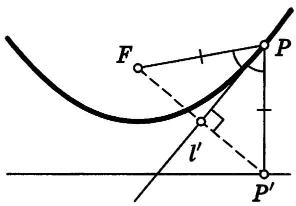

FIGURE 1.24

Proof. Suppose a line l touches the parabola at P and let P' be the projection of P to the directrix. Since the triangle FPP' is isosceles and l is the bisector of the angle P, l is an axis of symmetry of the triangle. Hence the reflection P' of F in l is on the directrix. □

Corollary. *The projections of the focus of the parabola to its tangents lie on the line tangent to the parabola at its vertex.* (Figure 1.25).

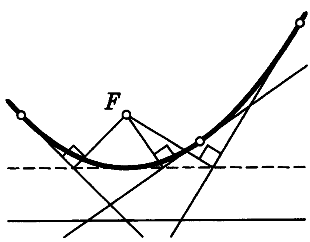

FIGURE 1.25

Lemma 1.2. *Suppose the tangents to the parabola at points X and Y intersect at a point P. Then P is the center of the circumcircle of the triangle $FX'Y'$, where X' and Y' are the projections of X and Y to the directrix of the parabola, and F is the focus of the parabola* (Figure 1.26).

Proof. By Lemma 1.1, these two tangents are midpoint perpendiculars to the segments FX' and FY'. Therefore their point of intersection is the center of the circumcircle of the triangle $FX'Y'$ (Figure 1.26). □

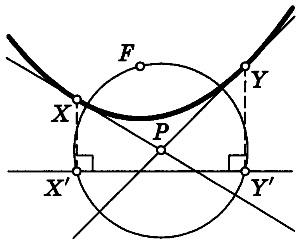

FIGURE 1.26

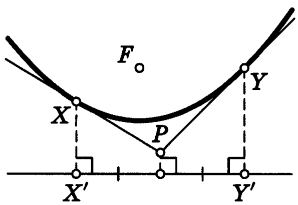

FIGURE 1.27

Corollary. *If PX and PY are tangent to the parabola, then the projection of P to the directrix is the midpoint of the segment with end-points at the projections of X and Y* (Figure 1.27).

The next theorem is similar, with the parabola in place of the ellipse, to Theorems 1.2 and 1.5. What is the set of points where the parabola is seen at a right angle? The answer is given by

Theorem 1.7. *The set of points P where the parabola is seen at a right angle is the directrix of the parabola. Moreover, if PX and PY are tangent to the parabola, then XY contains F and PF is a height of the triangle PXY* (Figure 1.28).

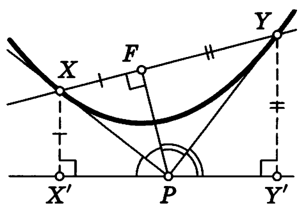

FIGURE 1.28

Proof. Suppose P lies on the directrix, and let X' and Y' be the projections of X and Y to the directrix. Then the triangles PXF and PXX' are equal (since they are symmetric with respect to PX). Hence $\angle PFX = \angle PX'X = 90°$. Similarly, $\angle PFY = \angle PY'Y = 90°$. Moreover, $\angle XPY = \frac{1}{2}(\angle FPX' + \angle FPY') = 90°$. The fact that there are no other points with this property is obvious. □

Since similar assertions are true for the remaining conics, the above theorem seems to be rather natural. However, the first part of the theorem has an unexpected generalization that holds only for parabolas. It will be used later in 3.2 in the proof of Frégier's theorem.

Theorem 1.8. *The set of points from which a parabola is seen at an angle* ϕ *or* $180° - \phi$ *is a hyperbola with focus* F *and directrix* l (Figure 1.29).

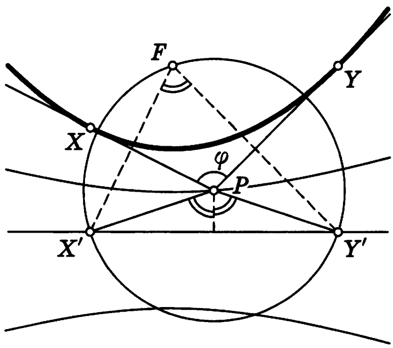

FIGURE 1.29

Proof. Indeed, suppose the tangents PX and PY to the parabola drawn from P form an angle ϕ. We first consider the case when $\phi > 90°$.

Let X' and Y' be the projections of X and Y to the directrix. Clearly, $\angle X'FY' = 180° - \phi$. By Lemma 1.2, P is the center of the circumcircle of the triangle $FX'Y'$. Therefore $\angle X'PY' = 360° - 2\phi$.

Thus the distance from P to the directrix equals $PF\,|\cos(180° - \phi)| = PF\,|\cos\phi|$ and P lies on the hyperbola whose focus and directrix coincide with the focus and directrix of the parabola, and whose eccentricity equals $|\cos\phi|$ (i.e., the angle between the asymptotes equals 2ϕ).

The same is true if the angle between the tangents is $180° - \phi$. Moreover, if the parabola lies inside an acute angle between the tangents, then P is on the "farther" from F branch of the hyperbola, and if it lies inside an obtuse angle, then P is on the "closer" branch. □

For parabolas one can also state a result similar to Theorems 1.3 and 1.4.

Theorem 1.9. *Let* PX *and* PY *be the tangents to the parabola passing through* P*, and let* l *be the line passing through* P *parallel to the axis of the parabola. Then the angle between the lines* PY *and* l *is equal to* $\angle XPF$ *and the triangles* XFP *and* PFY *are similar* (*as a consequence,* FP *is the bisector of the angle* XFY*; see Figure* 1.30).

Proof. Let X' and Y' be the projections of X and Y to the directrix. Then, by Theorem 1.2, the points F, X', and Y' lie on a circle centered at P. Hence $\angle X'Y'F = \frac{1}{2}\angle X'PF = \angle XPF$. On the other hand, the angle between PY

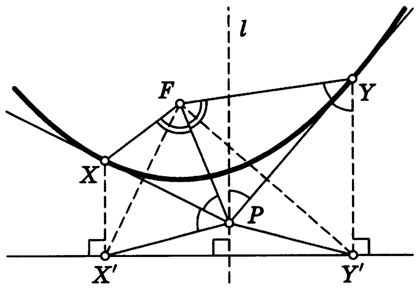

FIGURE 1.30

and l is equal to the angle between $Y'F$ and $X'Y'$ because l is perpendicular to $X'Y'$ (the directrix of the parabola) and $Y'F$ is perpendicular to PY (moreover, PY is the midpoint perpendicular to $Y'F$). This proves the first part of the theorem.

We now prove the second part. Since l is parallel to YY', the angle between PY and l is equal to the angle PYY', which, by the optical property, is equal to the angle FYP. Thus $\angle FYP = \angle XPF$. Similarly, $\angle FXP = \angle YPF$. Therefore the triangles XFP and PFY are similar. □

The next theorem is actually a consequence of Theorem 1.9. But we shall prove it using Simson's line, which will help us find even more interesting properties of the parabola.

Theorem 1.10. *Suppose a triangle ABC is circumscribed about a parabola (i.e., the lines AB, BC, CA are tangent to the parabola). Then the focus of the parabola lies on the circumcircle of the triangle ABC.*

Proof. By the Corollary of Lemma 1.1, the projections of the focus to the sides all lie on a straight line (which is parallel to the directrix and lies at half the distance from the focus). Now we can use Simson's lemma.

Lemma 1.3 (Simson). *The projections of P to the sides of a triangle ABC lie on a line if and only if P lies on the circumcircle of the triangle.*

Proof. Let P_a, P_b and P_c be the projections of P to BC, CA and AB, respectively. We consider the case shown in Figure 1.31; the remaining cases are argued similarly.

The quadrilateral PCP_bP_a is inscribed, hence $\angle PP_bP_a = \angle PCP_a$. Similarly, $\angle PP_bP_c = \angle PAP_c$. The points P_a, P_b and P_c lie on a line if and only if $\angle PP_bP_c = \angle PP_bP_a$ or, equivalently, $\angle PAP_c = \angle PCP_a$. But this means that P lies on the circumcircle of the triangle ABC. The remaining cases are argued similarly.

An identical argument proves the converse. If P lies on the circumcircle of a triangle ABC, then $\angle PAB = \angle PCP_a = \angle PP_bP_a$ (the latter holds since P, C, P_a and P_b lie on a circle). Similarly, $\angle PAB = \angle PP_bP_c$. Therefore P_a, P_b and P_c lie on a straight line. □

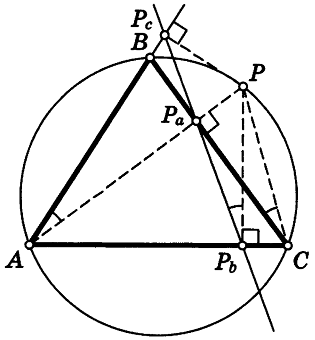

FIGURE 1.31

This proves Theorem 1.10. □

The line just described is called *Simson's line* of P.

Thus with each point on the circumcircle of a triangle ABC we can associate a unique parabola tangent to the sides of the triangle. More precisely, take an arbitrary point P on the circumcircle of the triangle ABC and reflect it in the sides of the triangle. We obtain points P_A, P_B and P_C, lying on a line. The parabola with focus at P and directrix P_AP_C is tangent to all the sides of the triangle (for example, it will touch BC at the point of intersection of BC and the perpendicular to P_AP_C; see Figure 1.32).

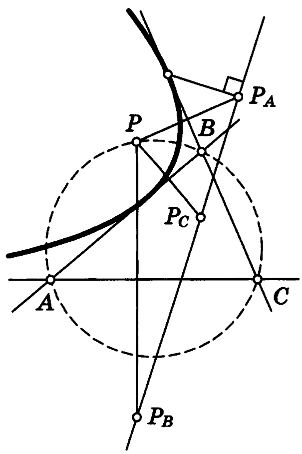

FIGURE 1.32

Simson's line has some interesting properties.

Lemma 1.4. *Suppose a point P lies on the circumcircle of a triangle ABC. Choose a point B' on the circumcircle such that the line PB' is perpendicular to AC. Then BB' is parallel to Simson's line of P* (Figure 1.33).

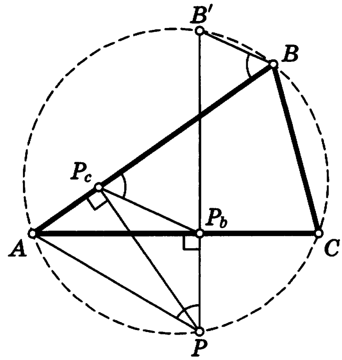

FIGURE 1.33

Proof. Consider the case shown in Figure 1.33; the remaining cases are argued similarly. Let P_c and P_b be the projections of P to the sides AB and AC, respectively. Then $\angle ABB' = \angle APB'$ as the angles subtending the arc AB'. Since quadrilateral AP_cP_bP is inscribed (AP is a diameter of its circumcircle) and the sum of the opposite angles of an inscribed quadrilateral equals 180°, we have $\angle APB' = \angle APP_b = 180° - \angle AP_cP_b = \angle BP_cP_b$. Therefore P_bP_c is parallel to BB'. □

Corollary 1. *When the point P moves along the circle, Simson's line rotates in the opposite direction with velocity one half the rate of change of the arc PA.*

Corollary 2. *Simson's line of P relative to a triangle ABC cuts the segment PH (where H is the orthocenter of the triangle ABC) in half* (Figure 1.34).

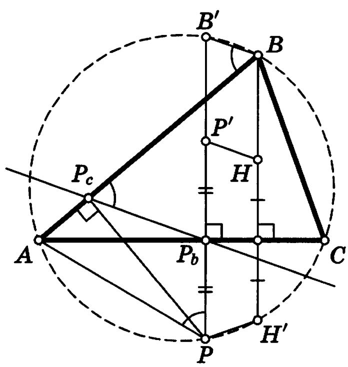

FIGURE 1.34

Proof. It is easy to see that $\angle AHC = 180° - \angle ABC$, and therefore the reflection H' of H in AC lies on the circumcircle of the triangle ABC. Since the lines PB' and BH' are perpendicular to AC, the quadrilateral $PB'BH'$ is a trapezoid; being inscribed, it must be equilateral. Therefore the reflection of PH' in AC (which is a line parallel to the axes of symmetry of the trapezoid) is parallel to BB'. Therefore $P'H$ is parallel to BB', and therefore to Simson's line of P (here P' is the reflection of P in AC). Since P_b (the projection of P to AC) is the midpoint of PP', Simson's line is a midline of the triangle HPP' and therefore cuts HP in half. □

Corollary 2 together with Theorem 1.10 imply the following beautiful result.

Theorem 1.11. *The orthocenter of a triangle circumscribed about a parabola lies on the directrix* (Figure 1.35).

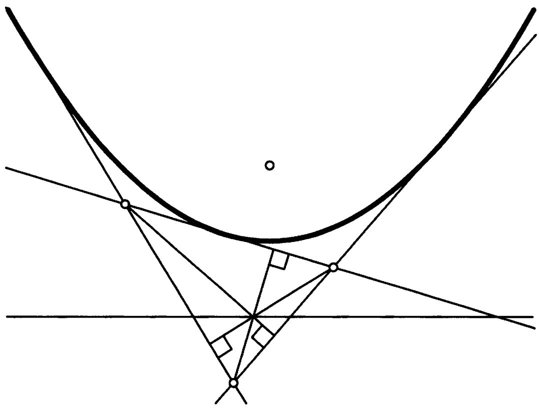

FIGURE 1.35

Problem 5. Suppose a point X moves along a parabola, the normal to the parabola at X (i.e., the perpendicular to the tangent) intersects its axis at a point Y, and Z is the projection of X to the axis. Prove that the length of the segment ZY does not change.

Problem 6. Two travelers move along two straight roads with constant speeds. Prove that the line connecting them is always tangent to some parabola (the roads are not parallel and the travelers pass the intersection at different times).

Problem 7. A parabola is inscribed in an angle PAQ. Find the locus of the midpoints of the segments cut out by the sides of the angle on the tangents to the parabola.

Chapter 2

Some Results from Classical Geometry

2.1. Inversion and Feuerbach's theorem

Inversion in the circle with center O and radius r is the transformation of the plane which sends each point A to the point A' lying on the ray OA and such that $OA' = \frac{r^2}{OA}$. The point O itself is sent to a point at infinity.

Clearly, under such a transformation lines passing through O remain fixed as sets.

Inversion is nice because it transforms circles not passing through the center of inversion into circles, whereas the circles that pass through the center are transformed into lines. The proofs of these statements can be found in [**10**], [**11**], and [**5**].

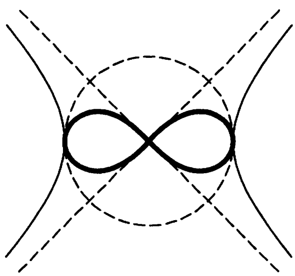

FIGURE 2.1

Even though inversion maps circles to circles, it has a noticeable technical drawback: it does not transform conics into conics. For example, an equilateral hyperbola under inversion in the circle with the same center as the hyperbola transforms into the *Bernoulli lemniscate* (Figure 2.1). However, a little later, using inversion, we will construct the so-called *polar transformation*, which does have this property.

We will use inversion to prove Feuerbach's theorem, thus showing the power of this tool. We will have another encounter with Feuerbach's theorem in 4.1.

First we recall the definition of the nine-point circle, also called the Euler circle.

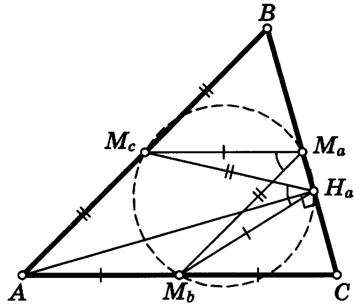

FIGURE 2.2

The *Euler circle* or the *nine-point circle* is the circle passing through the midpoints of the sides of a triangle ABC. It turns out that it also intersects the sides at the feet of the heights. Moreover, if H denotes the orthocenter of the triangle, then the midpoints of the segments AH, BH, and CH also lie on that circle.

Now we shall prove it. Let M_a, M_b, and M_c be the midpoints of the sides and H_a, H_b, and H_c the feet of the heights (Figure 2.2). We show that the angles $M_bM_aM_c$ and $M_bH_aM_c$ are equal. This would imply that H_a lies on the Euler circle.

ACH_a is a right triangle and therefore $M_bH_a = M_bA$. Similarly, $M_cH_a = M_cA$. Since $M_bA = M_aM_c$ and $M_cA = M_aM_b$, triangles $M_bM_aM_c$ and $M_bH_aM_c$ are equal. Therefore the corresponding angles are also equal. Similarly one shows that the points H_b and H_c also lie on the Euler circle.

Notice that the feet of the heights of triangles ABC and ABH coincide and therefore their Euler circles also coincide. Therefore the Euler circle also contains the midpoints of the segments AH and BH. The fact that it also contains the midpoint of the segment HC is established similarly.

Now we can prove Feuerbach's theorem.

Theorem 2.1 (Feuerbach). *The nine-point circle is tangent to the incircle and the excircles of the triangle* (*if the triangle is equilateral, it coincides with the incircle*) (Figure 2.3).

Proof. Let G_a, G_b and G_c be the tangency points of the incircle and the sides of the triangle. Let A_1 be the foot of the bisector of the angle A and C' the reflection of C in AA_1. Let P be the intersection of AA_1 and CC'.

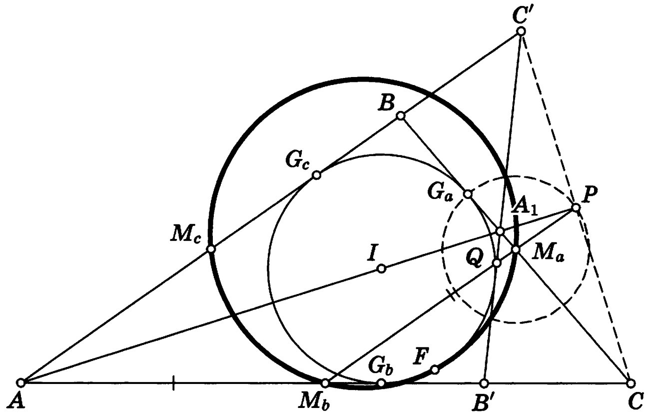

FIGURE 2.3

Notice that P is the midpoint of CC' and therefore P lies on the midline M_aM_b. Also notice that

$$M_aP = |M_aM_b - PM_b| = \frac{1}{2}|AB - AC| = \frac{1}{2}|BG_c - CG_b| = \frac{1}{2}|BG_a - CG_a| = M_aG_a.$$

Since the pairs of triangles M_bM_aC and ABC, ABA_1 and PM_aA_1 are similar, we have

$$\frac{M_aP}{M_aM_b} = \frac{BC'}{BA} = \frac{M_aQ}{M_aP},$$

where Q is the intersection of A_1C' and M_aM_b. Hence, $M_aG_a^2 = M_aP^2 = M_aQ \cdot M_aM_b$. Therefore the inversion with center M_a and radius M_aG_a transforms M_b into a point lying on the line $C'A_1$, the latter being the reflection of BC in the bisector of the angle A. The same is true for M_c. Thus the inversion with center M_a and radius M_aG_a transforms the Euler circle into a line tangent to the incircle and therefore the nine-point circle is also tangent to the incircle.

For excircles, Feuerbach's theorem is proved similarly. □

The tangency point of the incircle and the Euler circle is called the *Feuerbach point* (denoted F). Sometimes the tangency points of the excircles and the Euler circle are also called the Feuerbach points (denoted F_a, F_b and F_c).

2.2. Basic facts about projective transformations

A transformation of the plane is said to be projective if it preserves lines. Under such a transformation parallel lines need not remain parallel. However, if we mean the usual plane, parallel lines will transform into parallel lines because the transformation is one-to-one. For that reason one adds the so-called *line at infinity*. The points of that line, also referred to as points at infinity, are viewed as the intersections of parallel lines and each point at infinity is regarded as belonging to all the lines with the same direction. The plane completed this way is called the *projective plane.*

Definition. A transformation of the projective plane that sends each line (including the line at infinity) to a line is said to be *projective.*

It follows from the definition that projective transformations form a group (in other words, the composition of two projective transformations is a projective transformation). Notice that among the subgroups of this group are the groups of *affine* transformations preserving parallel lines (these can also be defined as the transformations preserving the line at infinity), as well as the groups of similarities and of motions.

A projective transformation can be visualized as follows. Suppose a drawing on a glass plate is projected from a point source of light to a wall. Then the drawing may look quite distorted but the lines on the glass will transform into lines on the wall. Moreover the plane passing through the source of light parallel to the wall will intersect the plate along a line. The points of that line will not project to the wall and we may consider the image of that line to be the line at infinity on the wall. Similarly, the plane parallel to the glass plate intersects the wall along a line which may be considered as the image of the line at infinity on the plate.

One can show that this example is universal, i.e., any projective transformation is a composition of a central projection and a motion of the space sending the plane of the projection to the original plane. Therefore by virtue of the results proved in 1.5, projective transformations send conics into conics. Indeed, any projective transformation is a composition of two transformations such that the first of them sends the conic into a circle. The second transformation will send the circle only into a conic. This shows that projective transformations are well suited for work with conics.

Notice that the hyperbola intersects the line at infinity at two points. Those points define the directions of the asymptotes of the hyperbola. The parabola is tangent to the line at infinity at the point which defines the direction of the axis of the parabola. Finally, the ellipse has no common points with the line at infinity.

We now state some of the main properties of projective transformations. Some of them will be explained rather than proved. Detailed proofs can be found in [**10**], [**11**] and [**12**].

1. All quadrilaterals are projectively equivalent. More precisely, for any two quadruples of points in general position A, B, C, D and A', B', C', D', there is a unique projective transformation sending A to A', B to B', C to C' and D to D'.

It suffices to check that any quadruple can be sent by a projective transformation into a square and that such a transformation is unique.

We can first send the intersections of the pairs of lines AB and CD and also AD and BC to infinity. Then our four points turn into the vertices of a parallelogram. Using an affine transformation, we then turn that parallelogram into a square.

Now connect the points A, B, C, D by straight lines. Then do the same with all the intersections of those lines and mark the intersections of the new lines. Keep repeating this procedure. The images of all the marked points are uniquely determined and they approximate any point in the plane. Therefore the desired transformation is unique.

Next we want to show that any five points in general position determine a unique conic. Transform four of them into the square whose vertices have coordinates ± 1. It is easy to check that the conics passing through these vertices are of the form $ax^2 + (1-a)y^2 = 1$. It is now obvious that exactly one curve of this form may pass through a given point in the plane.

Later on, we will show that there is a unique conic tangent to five given lines in general position.

2. Projective transformations preserve the *cross-ratios* of points on a line. This means that if points A, B, C and D lying on a line are sent to points A', B', C' and D', then

$$(AB;CD) = \frac{AC \cdot BD}{AD \cdot BC} = (A'B';C'D').$$

Notice that the lengths of the segments are signed.

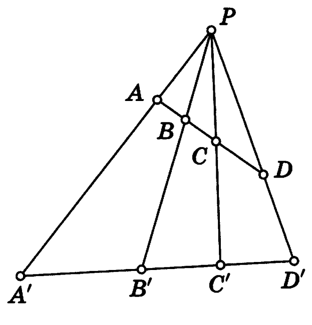

FIGURE 2.4

Let us prove this. As was shown before, any projective transformation may be viewed as a central projection. Let P be the center of that projection (Figure 2.4). Then

$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{S_{\triangle ACP} \cdot S_{\triangle BDP}}{S_{\triangle ADP} \cdot S_{\triangle BCP}},$$

since the area of each of those triangles equals half the product of the length of the corresponding segment by the distance from P to the line containing all those points. On the other hand, the area of each triangle equals half the product of the sides by the sine of the angle between them (for convenience, denote the angle between PA and PB by α, between PB and PC by β, and between PC and PD by γ). Hence

$$\frac{S_{\triangle ACP}\cdot S_{\triangle BDP}}{S_{\triangle ADP}\cdot S_{\triangle BCP}}=\frac{(AP\cdot CP\cdot\sin(\alpha+\beta)\cdot(BP\cdot DP\cdot\sin(\beta+\gamma))}{(AP\cdot DP\cdot\sin(\alpha+\beta+\gamma))\cdot(BP\cdot CP\cdot\sin\beta)}$$
$$=\frac{\sin(\alpha+\beta)\cdot\sin(\beta+\gamma)}{\sin(\alpha+\beta+\gamma)\cdot\sin\beta}.$$

Since this ratio does not depend on the line on which our points lie, we have

$$(A'B';C'D')=\frac{\sin(\alpha+\beta)\cdot\sin(\beta+\gamma)}{\sin(\alpha+\beta+\gamma)\cdot\sin\beta}=(AB;CD).$$

This property allows us to define the cross-ratio of four lines intersecting at a single point as the cross-ratio of their intersections with an arbitrary line. Clearly, the latter is also preserved under projective transformations.

The cross-ratio of four lines a, b, c and d will be denoted by $(ab;cd)$.

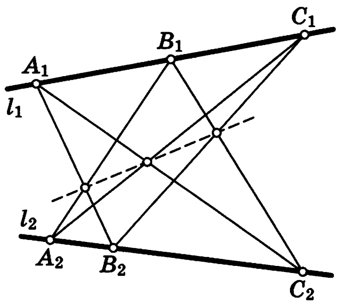

FIGURE 2.5

The preservation of the cross-ratio implies that if the images of three points on the line are known, then the images of the remaining points are uniquely determined. In particular, a projective transformation fixing three points on a line fixes the entire line.

3. Pappus' theorem.

Theorem 2.2. *If points A_1, B_1, C_1 lie on a line l_1 and A_2, B_2, C_2 lie on a line l_2, then the intersections of the lines A_1B_2 and A_2B_1, B_1C_2 and B_2C_1, C_1A_2 and C_2A_1 lie on a straight line* (Figure 2.5).

To see this, just move the intersections of the pairs A_1B_2 and A_2B_1, B_1C_2 and B_2C_1 to infinity and use Thales' theorem.

4. Desargues' theorem.

Theorem 2.3. *The lines A_1A_2, B_1B_2, C_1C_2 connecting the corresponding vertices of triangles $A_1B_1C_1$ and $A_2B_2C_2$ intersect in one point if and only if the intersections of lines A_1B_1 and A_2B_2, B_1C_1 and B_2C_2, C_1A_1 and C_2A_2 lie on one line* (Figure 2.6).

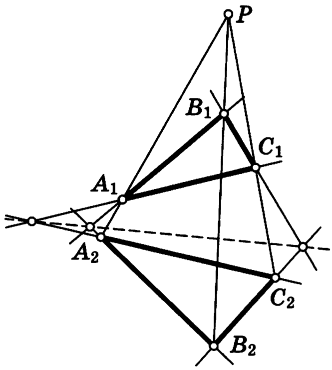

FIGURE 2.6

Here use a projective transformation to move the intersections of pairs of lines A_1B_1 and A_2B_2, B_1C_1 and B_2C_2 to infinity and again use Thales' theorem.

As a rule, projective transformations do not preserve circles. However, the following is true.

5. Given a circle and a point C inside it, there is a projective transformation sending the circle into a circle and C into the center of the new circle.

This fact will be established a little later using polar correspondence.

6. Given a circle and a nonintersecting line l, there is projective transformation sending the circle to a circle, and the line l to the line at infinity.

Using a projective transformation send l to the line at infinity. Then the circle can only transform into an ellipse, since the image does not intersect the line at infinity. Now use an affine transformation to send the ellipse into a circle (such a transformation does obviously exist).

7. Pascal's theorem.

Theorem 2.4. *The intersection points of the opposite sides of an inscribed hexagon are all on one straight line* (Figure 2.7).

Proof. Let $ABCDEF$ be the inscribed hexagon. Using a projective transformation, move the intersections of the pairs of lines AB and DE, BC and EF to infinity. Then $AB \parallel DE$ and $BC \parallel EF$; we need to show that $CD \parallel FA$. But this is not difficult. As the angles ABC and DEF have parallel sides, they are equal. Therefore the arcs AC and DF are also equal. But this means that the lines AF and CD are parallel. □

8. Brianchon's theorem.

Theorem 2.5. *The principal diagonals of a circumscribed hexagon meet at one point* (Figure 2.8).

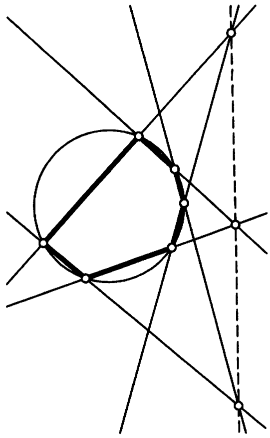

FIGURE 2.7

Proof. Move the intersection of two diagonals to the center of the circle. We need to show that the third diagonal passes through the center.

Thus let the hexagon $ABCDEF$ be circumscribed about a circle centered at O and suppose that the diagonals AD and BE pass through O. Let $A_1, B_1, \ldots, F_1$ be the tangency points of the circle and the sides $AB, BC, \ldots,$ FA, respectively. It is easy to see that $\angle E_1OC_1 = \angle F_1OB_1 = 2\angle AOB$ and that $\angle E_1OF = \angle FOF_1$ and $\angle B_1OC = \angle COC_1$. Hence

$$\angle FOF_1 + \angle F_1OB_1 + \angle B_1OC = \angle E_1OF + \angle E_1OC_1 + \angle COC_1 = \frac{360^\circ}{2} = 180^\circ.$$

Therefore F, O and C all lie on a straight line. □

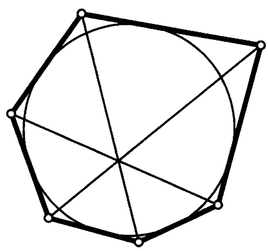

FIGURE 2.8

Notice that the theorems of Pascal and Brianchon remain true if the hexagon degenerates into a pentagon or a quadrilateral. Later on, we will use this observation on several occasions.

In the proof of Pascal's theorem we needed the assumption that the corresponding line does not intersect the circle, and in the proof of Brianchon's theorem—that the intersection of the diagonals lies inside the circle. It turns out that these two theorems are always true, i.e., the order of the points and the lines in those theorems can be arbitrary.

It is also important to mention that these are purely projective theorems. Hence they are also true for conics. In Chapter 3 we formulate and prove these theorems in a general form.

9. Suppose points A, B, C and D lie on a circle. The inscribed angle theorem implies that for all points X on that circle the cross-ratio of lines XA, XB, XC and XD is the same. Let us call it the cross-ratio of A, B, C, and D. Clearly, if a projective transformation sends the circle to a circle, then it preserves the cross-ratio of the points. The converse is also true: a cross-ratio preserving transformation of a circle can be extended to a projective transformation of the entire plane.

Projective transformations are closely related to transformations that interchange lines and points.

Definition. The *polar correspondence* with respect to the circle with center O and radius r associates to each point A of the plane, different from O, the line a perpendicular to OA and cutting the ray OA at the point inverse to A with respect to the circle. The line a is called the *polar* of A and the point A is called the *pole* of the line a. The polar of O is defined as the line at infinity, and the polar of a point at infinity is defined as the diameter perpendicular to the parallel lines passing through that point.

Now we mention some important properties of the polar correspondence.

1. If a point B lies on the polar a of a point A, then its polar b passes through A.

Proof. Suppose A' and B' are the inverses of A and B with respect to our circle. Then the triangle $OA'B$ is obviously similar to the triangle $OB'A$, and therefore $AB'O$ is a right angle, i.e., A lies on b (Figure 2.9). □

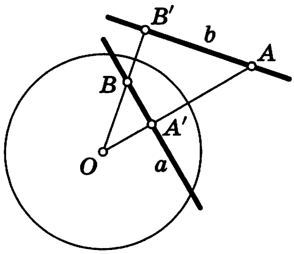

FIGURE 2.9

This implies that the pole of any line is the intersection of the polars of all of its points and, dually, the polar of a point is the locus of the poles of all the lines passing through the point.

2. The polar of a point A lying outside the circle will be the line connecting the tangency points of the circle and its tangents passing through A (the tangency points are the poles of the corresponding tangents). It now follows that, despite the metric properties mentioned in the definition, the polar correspondence is a projective notion, i.e., if a projective transformation preserves the given circle and sends A to A', then the polar a of A transforms into the polar a' of A'. This yields the following result.

The duality principle. *If an assertion from projective geometry is true, then the assertion obtained from it by interchanging the terms* (point) $\leftrightarrow$ (line), (to lie on a line) $\leftrightarrow$ (to pass through a point), (to lie on a circle) $\leftrightarrow$ (to be tangent to a circle) *is also true. As examples of the duality principle we mention the theorems of Pascal and Brianchon, the direct and the converse assertions in Desargues' theorem, etc.*

Using the duality principle we can also prove property 5. It suffices, using property 6, to move the polar of C to infinity. Then, obviously, C will transform into the center of the circle.

3. The line connecting the intersections of the opposite sides of an inscribed (circumscribed) quadrilateral is the polar of the intersection of its diagonals.

This follows from property 1 of polar correspondences and Newton's theorem: The diagonals of a circumscribed quadrilateral pass through the point of intersection of the lines connecting the tangency points of the opposite sides and the inscribed circle. This theorem is a special case of Brianchon's theorem.

4. The cross-ratio of four points on a line equals the cross-ratio of their polars.

Proof. Let A, B, C and D be the four points. Then the cross-ratio of this quadruple equals the cross-ratio of the lines OA, OB, OC and OD, which in turn equals the cross-ratio of the lines OA', OB', OC' and OD', where A', B', C', D' are the projections of O to the polars of A, B, C and D, respectively. Let P be the pole of the line AB. Then A', B', C', D', O and P lie on a circle (with diameter OP). Therefore $(PA', PB'; PC', PD') = (OA', OB'; OC', OD') = (A, B; C, D)$. But PA', PB', PC' and PD' are the polars of A, B, C and D. □

Problem 8. Points A_1, B_1 and C_1 lie on the sides of a triangle ABC so that AA_1, BB_1 and CC_1 intersect at a point P. Let C' be the intersection of A_1B_1 and AB. Points A' and B' are defined similarly. Prove that A', B' and C' lie on a straight line.

The constructed line is called the *trilinear polar* of P with respect to the triangle ABC, whereas P is called the *trilinear pole* of the line.

Problem 9. A line intersects a hyperbola at points P and Q and its asymptotes at points X and Y. Prove that the segments PX and QY are equal (Figure 2.10).

Problem 10. Two parallel lines intersect a parabola at points A, B and C, D respectively. Prove that the line connecting the midpoints of those segments is parallel to the axis of the parabola (Figure 2.11).

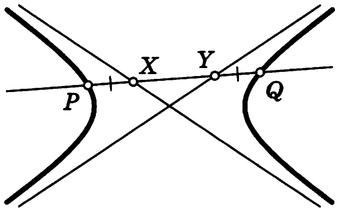

FIGURE 2.10

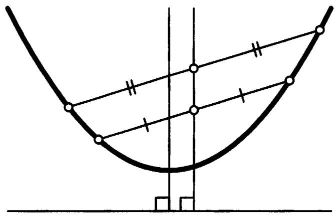

FIGURE 2.11

Problem 11. Given a circle, choose a point C inside (outside) it. Draw four chords (secants) A_iB_i, $i = 1, \dots, 4$ through C. Let D be the intersection of the lines A_1A_2 and A_3A_4, and E the intersection of the lines B_1B_2 and B_3B_4. Prove that C, D and E lie on a straight line (Figure 2.12).

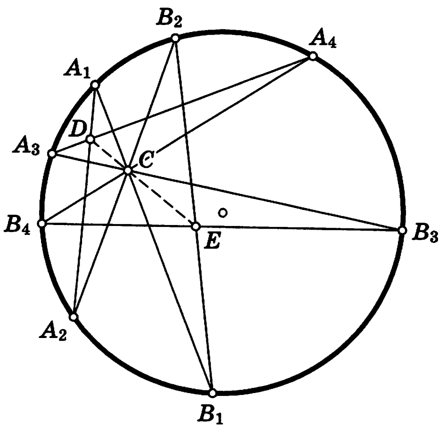

FIGURE 2.12

Problem 12. Circles that are tangent to a pair of conjugate diameters[1] of an ellipse and have their centers lying on that ellipse are of equal radii.

Problem 13. The ends of a segment BC are sliding along two lines intersecting at a point A in such a way that the length of the segment remains

[1]Two diameters of an ellipse are conjugate if each is parallel to the tangents to the ellipse at the ends of the other.

constant. Prove that the trajectory of any fixed point P on BC is an ellipse (Figure 2.13).

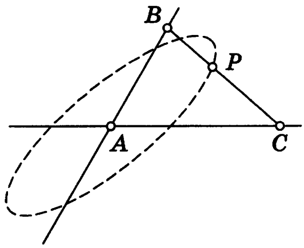

FIGURE 2.13

Problem 14. The sides of a triangle ABC contain six points: A_1, A_2 on BC, B_1, B_2 on AC and C_1, C_2 on AB. Prove that these six points lie on a conic if and only if

$$\frac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} \cdot \frac{CB_1 \cdot CB_2}{AB_1 \cdot AB_2} \cdot \frac{AC_1 \cdot AC_2}{BC_1 \cdot BC_2} = 1.$$

Here the ratios are signed. For each expression, the positive direction is the one from a vertex to the other vertex of the underlying side.

2.3. Some facts from the geometry of the triangle

In this section we consider some very useful but not very well known properties of the triangle. These are mainly related to isogonal and isotomic conjugation but we shall also mention some other beautiful results not directly related to the subject of this book.

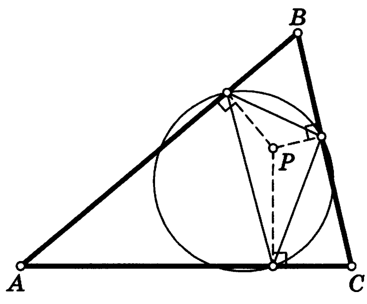

FIGURE 2.14

Definition. The *pedal triangle* of a point P with respect to a triangle ABC is formed by the vertices which are the projections of P to the sides of ABC. The circumcircle of the pedal triangle is called the *pedal circle* of P with respect to the triangle ABC (Figure 2.14).

Theorem 2.6. *The pedal triangle degenerates (i.e., the projections lie on a line)* $\Leftrightarrow$ *P lies on the circumcircle of the triangle ABC.*

(This is just a reformulation of Simson's lemma.)

Definition. The *Ceva triangle* of a point P with respect to a triangle ABC is formed by the vertices which are the intersections of the lines AP and BC, BP and AC, CP and AB. The circumcircle of a Ceva triangle is called the *Ceva circle* of P with respect to the triangle ABC (Figure 2.15).

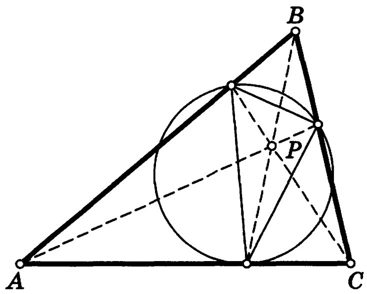

FIGURE 2.15

We shall discuss properties of Ceva triangles later. At the moment we define the *circumcevian triangle.*

Definition. The *circumcevian triangle* of a point P with respect to a triangle ABC is formed by the vertices which are the second intersections of the lines AP, BP, CP with the circumcircle of the triangle ABC (Figure 2.16).

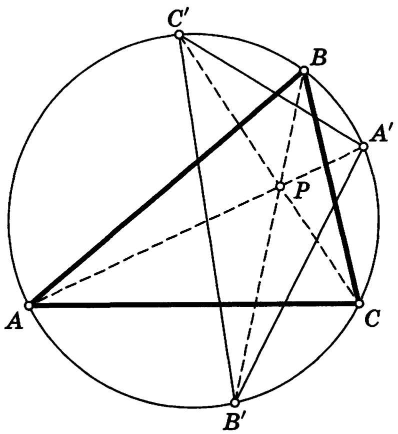

FIGURE 2.16

Lemma 2.5. *The pedal and the circumcevian triangles of P with respect to a triangle ABC are similar and have the same orientation.*

Proof. We consider the case shown in Figure 2.17. The remaining cases are proved similarly.

The points P_A, P_B, P_C are the vertices of the pedal triangle, and A', B', C' are the vertices of the circumcevian triangle. We have $\angle AA'C' = \angle ACC' = \angle P_BP_AP$. The latter equality holds since the quadrilateral PP_ACP_B is inscribed. Similarly, $\angle AA'B' = \angle P_CP_AP$. Therefore $\angle C'A'B' = \angle P_BP_AP_C$. Similarly, $\angle A'B'C' = \angle P_AP_BP_C$ and $\angle A'C'B' = \angle P_AP_CP_B$. But this means that the triangles $A'B'C'$ and $P_AP_BP_C$ are similar. □

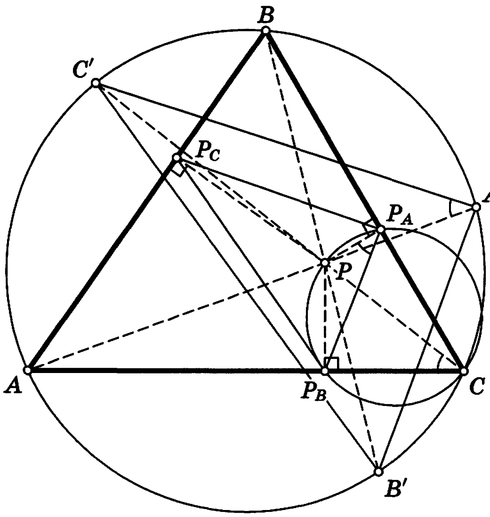

FIGURE 2.17

Theorem 2.7. *The circumcevian triangles of the points inverse to each other with respect to the circumcircle of the triangle are similar and have different orientations.*

First we prove a lemma.

Lemma 2.6. *Suppose points P and Q are inverse to each other with respect to a circle ω centered at O and the segment PQ intersects ω at a point R. Then for any point A on ω, RA is the bisector of the angle PAQ.*

Proof. Since P and Q are inverse to each other, the triangles OAP and OQA are similar (Figure 2.18), and therefore $\angle OQA = \angle OAP$. Since O is the center of ω, the triangle AOR is isosceles and therefore $\angle OAR = \angle ORA$. Thus

$$\angle PAR = \angle OAR - \angle OAP = \angle ORA - \angle OQA = \angle RAQ,$$

as desired. □

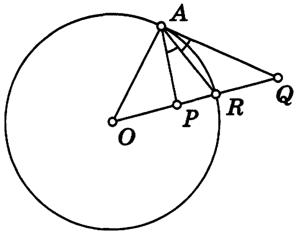

FIGURE 2.18

This lemma implies that ω is the locus of points at which the ratio of the distances to P and Q is constant (and different from 1). The circle ω is called the *Apollonius circle* of the segment PQ. We shall say more about it later on.

Proof of the theorem. Suppose P and Q are inverse to each other with respect to the circumcircle (call it ω) of the triangle ABC and that PQ intersects ω in R. Let A' and A'' be the second intersection points of AP and AQ, respectively, with ω. By Lemma 2.6, the line AR is the bisector of the angle PAQ and therefore R cuts the arc $A'A''$ in half, i.e., A' and A'' are symmetric with respect to OP. The same is true for the pairs B' and B'', C' and C'', which are the vertices of the circumcevian triangles of P and Q with respect to the triangle ABC. Thus the symmetry with respect to PQ interchanges the triangles, and therefore they are similar and have opposite orientations. □

Corollary. *The pedal triangles of inverse points are similar and have different orientations.*

One can show that for any triangle XYZ there is a unique point such that the pedal triangle of that point with respect to a given triangle ABC is similar to the triangle XYZ with a fixed order of the vertices.

In the proof of the preceding theorem we had two similar triangles and a point P inside them such that the angles formed by the sides and the Ceva lines of P are equal but in a sense interchanged (see Figure 2.17).

Thus we have the so-called *isogonal conjugation* with respect to a triangle.

Let ABC be an arbitrary triangle and P a point different from the vertices of the triangle. Reflect the lines connecting the vertices of the triangle with P in the bisectors of the corresponding angles of the triangle. It turns out that these three lines always intersect in a single point (or are parallel, i.e., intersect in a single point of the projective plane), which we denote P' (Figure 2.19). The point P' is called the *isogonal conjugate* of P with respect to triangle ABC; the transformation sending each point of the projective plane to its isogonal conjugate is called the *isogonal conjugation.*

That the above concept is well defined was almost proved in Theorem 2.7. Indeed, given a triangle ABC and a point P, let $A'B'C'$ be the circumcevian

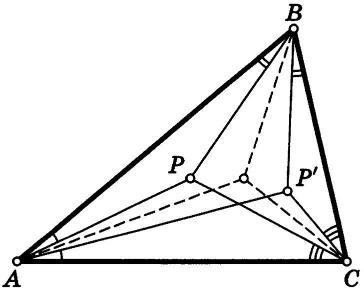

FIGURE 2.19

triangle of P with respect to the triangle ABC. Then ABC is the circumcevian triangle of P with respect to the triangle $A'B'C'$ and therefore is similar to the pedal triangle of P with respect to $A'B'C'$. Therefore the image of P under the similarity transforming the pedal triangle of P into the triangle ABC will be P', which is exactly the isogonal conjugate.

Now we mention several elementary properties of the isogonal conjugation.

1. If P does not lie on the sidelines of the triangle, then P' is determined uniquely and the isogonal conjugate of P' will be P. Such two points are said to be *isogonally conjugate.*

2. The isogonal conjugate of a point on a sideline of the triangle is the vertex of the triangle opposite to the respective side.

3. The isogonal conjugation leaves exactly four points of the plane fixed—these are the centers of the incircle and of the three excircles of the triangle.

4. If P lies on the circumcircle of the triangle ABC, then the isogonal conjugate of P is the point on the line at infinity in the direction perpendicular to Simson's line of P with respect to ABC (i.e., the line passing through the projections of P to the sides of the triangle ABC).

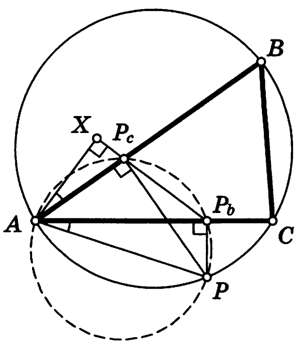

FIGURE 2.20

The first three properties are obvious. To prove the fourth, consider the case shown in Figure 2.20; the remaining cases are argued similarly. Suppose

P lies on the circumcircle and P_b and P_c are the projections of P to the sides AC and AB, respectively. Let X be the intersection of Simson's line of P and the line a which is the reflection of AP in the bisector of $\angle A$. The quadrilateral APP_bP_c is inscribed and therefore $\angle AP_cP_b = 180° - \angle APP_b = 180° - (90° - \angle PAP_b) = 90° + \angle PAP_b = 90° + XAP_c$. But since an exterior angle equals the sum of the two interior angles of the triangle, $\angle AXP_c = 90°$. A similar proof shows that the reflections of PB and PC in the bisectors of the corresponding angles are perpendicular to P_bP_c.

The existence proof just given for isogonal conjugation does not readily indicate any of its properties. We shall now give another method for constructing isogonally conjugate points, which would immediately lead us to some nice properties of this transformation.

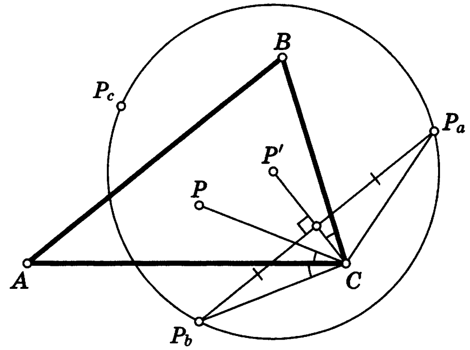

FIGURE 2.21

Suppose a point P lies inside a triangle ABC and let P_a be its reflection in the side BC; points P_b and P_c are defined similarly (Figure 2.21). Let P' be the center of the circumcircle of the triangle $P_aP_bP_c$. The point C is equidistant from P_a and P_b; therefore the line CP' is the midpoint perpendicular to the segment P_aP_b. Hence $\angle P_aCP' = \frac{1}{2}\angle P_aCP_b = \angle C$. But then $\angle BCP' = \angle P_aCP' - \angle BCP_a = \angle C - \angle BCP = \angle ACP$. A similar argument shows that $\angle ABP' = \angle CBP$ and $\angle BAP' = \angle CAP$. But this means that P' is isogonally conjugate to P with respect to ABC.

If P is outside the triangle, then the argument is identical, but if P lies on the circumcircle of the triangle ABC, then the triangle $P_aP_bP_c$ is degenerate. In this case the center of the circumcircle of the triangle $P_aP_bP_c$ is not defined (although it would be natural to view the line P_aP_b as the circumcircle, the center being the point on the line at infinity in the direction perpendicular to P_aP_b).

The second method for constructing isogonally conjugate points shows that the center of the pedal circle of P is the midpoint of the segment PP' and its radius is half of the segment $P'P_a$, since the pedal circle of P is the circle obtained from the circumcircle of the triangle $P_aP_bP_c$ by the homothety with center P and coefficient $\frac{1}{2}$.

This also implies the following theorem.

Theorem 2.8. *The pedal circles of two points coincide if and only if the points are isogonally conjugate.*

Proof. Indeed, if P and P' are isogonally conjugate, then the pedal circle for each of them is the circle centered at the midpoint of PP' of radius $\frac{P'P_a}{2} = \frac{PP'_a}{2}$, where P_a and P'_a are the reflections of P and P' in the side BC of the triangle ABC.

Now we prove the converse. If the pedal circles of P and Q coincide, then, by the above, they coincide with the pedal circle of the isogonal conjugate P' of P. By the Dirichlet principle, two out of three vertices of the pedal triangle of Q are common with the pedal triangle of either P or P'. Therefore Q coincides with one of those points, because the projections of a point to two lines completely determine the point.

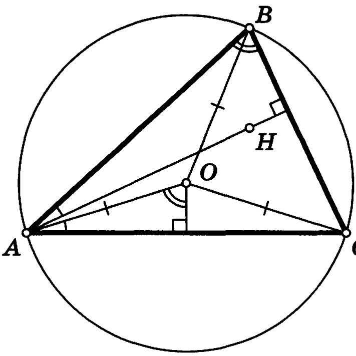

FIGURE 2.22

As a direct consequence of this theorem, we see that the orthocenter H of the triangle ABC is isogonally conjugate to the center O of the circumcircle. Indeed, the pedal circles of H and O coincide with the nine-point circle. Of course, this can be proved directly by examining the angles. Consider the case shown in Figure 2.22; the remaining cases can be argued similarly. We have $\angle BAH = 90° - \angle B$, but $\angle AOC = 2\angle B$, hence $\angle OAC = \frac{1}{2}(180° - 2\angle B) = 90° - \angle B$. It now follows that $\angle BAH = \angle OAC$, but this means that the reflection of AH in the bisector of the angle A is the line AO. For the other two angles the proof is similar. □

Let K_a, K_b and K_c be the intersections of the lines BC and $P'P_a$, AC and $P'P_b$, AB and $P'P_c$, respectively. Clearly, $\angle PK_aB = \angle P'K_aC$. Therefore the conic with foci P and P' and the sum (or the modulus of the difference, in the case of a hyperbola) of distances to the foci equal to $P'P_a$ is tangent to the line BC. Similarly, the conic is tangent to the other two sides of the triangle since the distances $P'P_a = P'P_b = P'P_c$ equal twice the radius of the pedal circle of P. In Figure 2.23, the shaded regions consist of the points where the corresponding conics are hyperbolas, and in the clear regions the conics are ellipses. The points of the circumcircle correspond to parabolas.

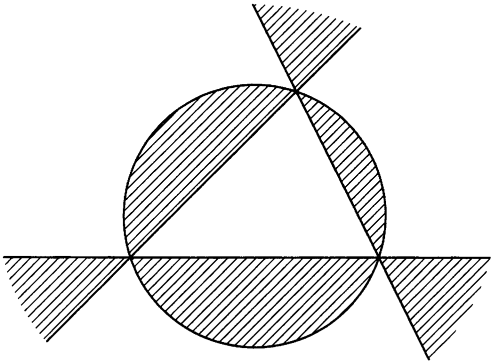

FIGURE 2.23

If P and Q are isogonally conjugate in a polygon (i.e., for any vertex X of this polygon the lines XP and XQ are symmetric with respect to the bisector of the angle X), then there is a conic tangent to all the sides of the polygon with foci at those points. The converse is also true, i.e., if a conic is inscribed in a polygon, then its foci are isogonally conjugate with respect to that polygon. Similarly one can show that in this case the pedal circles of P and Q coincide (in addition to the fact that they exist!).

In the next section we shall show that for any five lines there is a unique conic tangent to those lines. Hence a pentagon contains only one pair of isogonally conjugate points. It is easy to see that for a quadrilateral such points form a curve (actually, it will be a cubic, i.e., a curve of order three), and for hexagons (and polygons with a larger number of sides) such points, in general, do not exist.

Using isogonal conjugation one can easily prove Pascal's theorem in a rather general form.

Theorem 2.9 (Pascal). *Suppose points A, B, C, D, E and F lie on a conic. Then the intersections of the lines AB and DE, BC and EF, CD and FA lie on a line.*

Proof. We consider only one case of the relative positions of the points on the circle (or a conic). The other cases are treated similarly.

Using a projective transformation we can transform the conic into a circle. We then have the following configuration (Figure 2.24).

The points A, B, C, D, E and F lie on a circle. Suppose the lines AB and DE intersect at X, the lines BC and EF at Y, and AF and CD at Z. We want to show that X, Y and Z lie on a straight line.

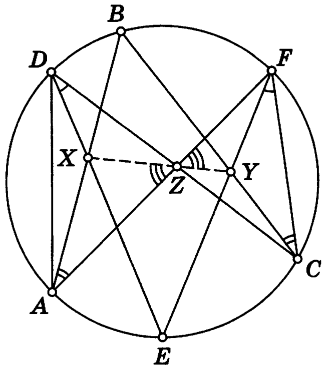

FIGURE 2.24

The angles BAF and BCF are equal since they subtend the same arc. Similarly, the angles CDE and CFE are equal. Moreover, the triangles AZD and CZF are similar. Consider the similarity transforming triangle AZD into triangle CZF. Then X will transform into X', which is isogonally conjugate to Y with respect to the triangle CZF (in view of the equality of the above angles). Therefore $\angle AZX = \angle CZX' = \angle FZY$, but this means that X, Z and Y lie on a line. □

Now we describe a few more pairs of isogonally conjugate points.

1. The centroid and the Lemoine point. The lines symmetric to the medians with respect to the bisectors of the corresponding angles, are called the *symmedians*. The point of intersection of the symmedians is obviously isogonally conjugate to the intersection point of the medians and is called the *Lemoine point.*

We now list several basic properties of the Lemoine point, which are somewhat similar to the properties of the centroid.

1a. *Suppose the tangents to the circumcircle of a triangle ABC at B and C intersect at A_1. Then AA_1 is the symmedian of the triangle ABC* (Figure 2.25).

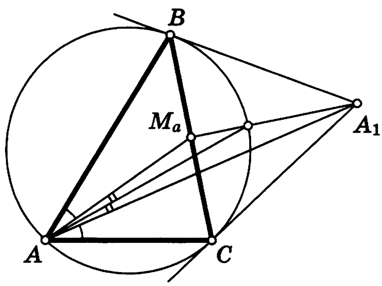

FIGURE 2.25

Indeed, the midpoint M_a of the side BC is the inverse of A_1 with respect to the circumcircle of the triangle ABC. Hence the foot of the bisector of the angle M_aAA_1 is the midpoint of the arc BC, i.e., this bisector coincides with the bisector of the angle BAC. But then the line AA_1 is symmetric to the median AM_a with respect to the bisector of the angle A.

Another elegant proof of this fact is based only on the existence of isogonal conjugation. We just identify the isogonal conjugate of A_1. This is the point symmetric to A with respect to M_a. It clearly lies on the median AM_a (we denote it A'; see Figure 2.26). It is not difficult to check that the reflections of the lines BA' and CA' in the bisectors of the corresponding angles are tangent to the circumcircle. Therefore A' transforms into A_1.

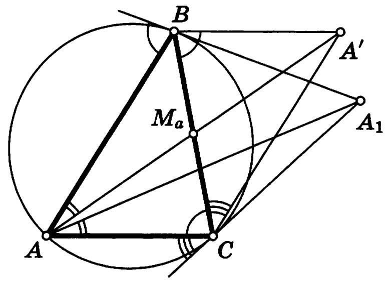

FIGURE 2.26

This theorem implies that a symmedian can be constructed with a straightedge, provided the circumcircle of the triangle is given. Projective transformations fixing the circumcircle of a triangle ABC take the symmedians into the symmedians and the Lemoine point into the Lemoine point. This is in a sense similar to the fact that affine transformations transform the centroid of a triangle into the centroid of the image triangle.

As a consequence, we have that the Lemoine point is the *Gergonne point* (see below) of the triangle $A_1B_1C_1$, where B_1 and C_1 are constructed similarly to A_1.

1b. *A symmedian cuts the side in the ratio equal to the ratio of the squares of the adjacent sides.*

Let L_a be the intersection of the symmedian of the angle A and the side BC, and M_a the midpoint of BC.

Since the areas of the triangles ABM_a and ACM_a are equal, the ratio of the distances from M_a to AB and AC is inversely proportional to the ratio of these sides. But since the line AL_a is the reflection of AM_a in the bisector of the angle A, the product of the ratios of the distances from M_a and L_a to AB and AC equals 1. Therefore the ratio of the distances from L_a to AB and AC equals the ratio of the lengths of these sides, and therefore the areas of the triangles ABL_a and ACL_a have the same ratio as the squares of AB and AC. On the other hand, this ratio equals the ratio of BL_a and CL_a since these triangles have a common height.

1c. *The sum of the squared distances from a point P to the vertices of a triangle attains its minimum when P is the intersection of the medians. The sum of the squared distances from P to the sides attains its minimum at the Lemoine point.*

It is not difficult to see that this statement can be deduced from the previous one.

2. Brocard points. It turns out that inside any triangle ABC there is a point Br_1 such that $\angle BABr_1 = \angle CBBr_1 = \angle ACBr_1$. If Br_2 is the isogonal conjugate of Br_1, then obviously $\angle ABBr_2 = \angle BCBr_2 = \angle CABr_2$ (Figure 2.27).

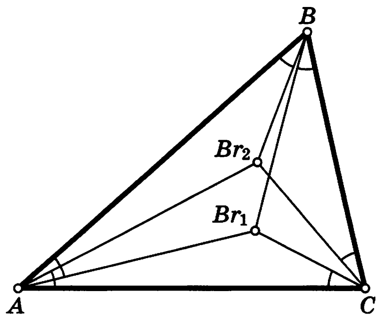

FIGURE 2.27

These two points are called the *first* and the *second Brocard points*, respectively. The ellipse tangent to the sides of the triangle and having foci at these points is called the *Brocard ellipse*. For an equilateral triangle these two points coincide with the center of the triangle and the ellipse coincides with the incircle.

We now prove the existence of Br_1. Using the sides of the triangle, construct the triangles BCA_1, B_1CA, BC_1A similar to it as shown in Figure 2.28.

Then the circumcircles of these triangles (call them ω_a, ω_b and ω_c) intersect in a point. Indeed, let Br_1 be the intersection of the circles ω_a and ω_b different from C. Then

$$\begin{aligned}\angle ABr_1B &= 360° - (\angle ABr_1C + \angle BBr_1C)\\ &= 360° - (180° - \angle AB_1C) - (180° - \angle BA_1C)\\ &= \angle BA_1C + \angle AB_1C = \angle C + \angle A\\ &= 180° - \angle B = 180° - \angle AC_1B.\end{aligned}$$

Therefore Br_1 lies on ω_c. We also have

$$\begin{aligned}\angle CBBr_1 &= 180° - \angle BBr_1C - \angle BCBr_1 = \angle BA_1C - \angle BCBr_1\\ &= \angle BCA - \angle BCBr_1 = \angle ACBr_1.\end{aligned}$$

Similarly, $\angle CBBr_1 = \angle BABr_1$.

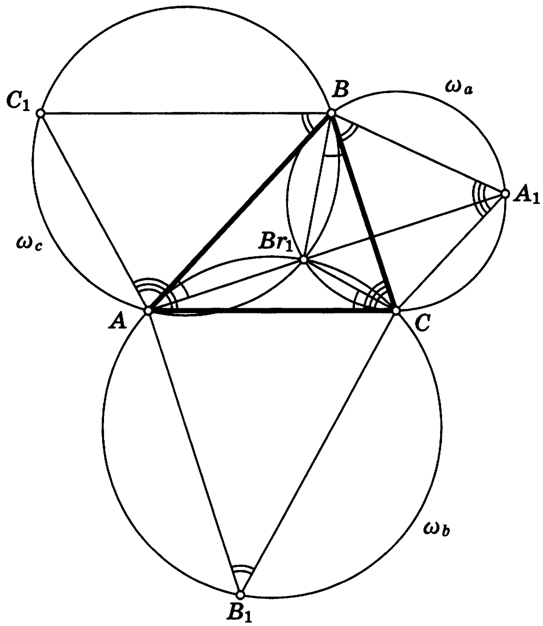

FIGURE 2.28

That such a point is unique can be easily seen from the construction. Indeed, suppose that there is another such point X. Then a similar argument shows that $\angle BXA = 180° - \angle B$, and therefore X lies on the circle ω_c. Similarly, it has to lie on the circles ω_a and ω_b, and therefore it coincides with Br_1.

The angle $BABr_1$ is called the *Brocard angle* of the triangle ABC.

Notice also that the points A, Br_1, A_1 lie on a line. Indeed,

$$\begin{aligned}\angle ABr_1A_1 &= \angle ABr_1C + \angle CBr_1A_1 = 180° - \angle AB_1C + \angle CBA_1 \\ &= 180° - \angle A + \angle A = 180°.\end{aligned}$$

The Brocard points have some interesting properties.

2a. *The pedal and the circumcevian triangles of the Brocard points are similar to the triangle ABC.*

Let A', B' and C' be the projections of Br_1 to the sides (Figure 2.29). The quadrilateral $B'AC'Br_1$ is inscribed, hence $\angle C'B'Br_1 = \angle C'ABr_1 = \angle ACBr_1$ (the last equality holds because Br_1 is the Brocard point). Similarly, $\angle A'B'Br_1 = \angle A'CBr_1$. Hence $\angle A'B'C' = \angle A'B'Br_1 + \angle C'B'Br_1 = \angle A'CBr_1 + \angle ACBr_1 = \angle C$. A similar argument shows that $\angle B'A'C' = \angle B$ and $\angle A'C'B' = \angle A$. Since the pedal triangle is similar to the circumcevian triangle, the circumcevian triangle of Br_1 is similar to the triangle ABC.

2b. *Let O be the center of the circumcircle of the triangle ABC. Then $OBr_1 = OBr_2$ and $\angle Br_1OBr_2$ equals twice the Brocard angle.*

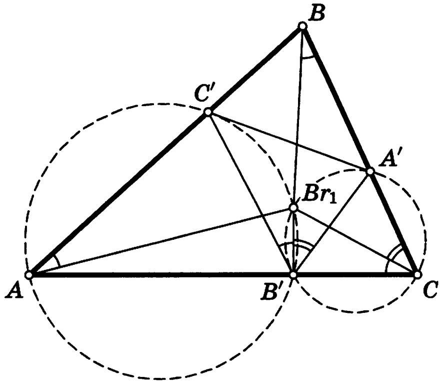

FIGURE 2.29

Let $A''B''C''$ be the circumcevian triangle of Br_1 with respect to the triangle ABC (Figure 2.30).

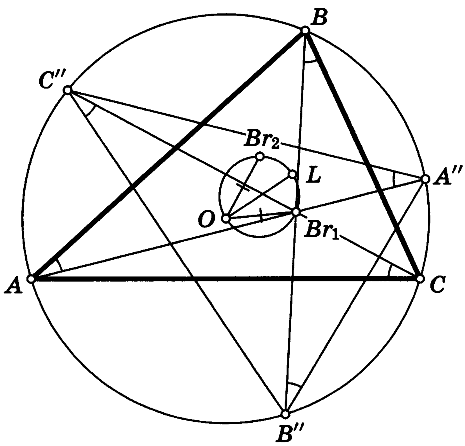

FIGURE 2.30

Notice that under the rotation about O through the double Brocard angle, A transforms into C''. The same is true for B and C. Therefore, under the rotation about O, the triangle ABC will transform into the triangle $C''A''B''$. Then Br_1 will be the second Brocard point of the triangle $A''B''C''$ ($\angle C''A''Br_1 = \angle C''A''A = \angle C''CA = \angle Br_1CA$).

Therefore, under the rotation about O through the double Brocard angle, Br_2 will transform into Br_1.

2c. *The Lemoine point lies on the circumcircle of the triangle ABr_1Br_2 and is antipodal to O.*

Consider a projective transformation putting the point Br_1 at the center of the circle ω circumscribed about the triangle ABC. The triangles ABC and $A''B''C''$ will clearly transform into the triangles symmetric with respect to the center of ω, and therefore their Lemoine points will also be symmetric. On the other hand, these points will be the images of the Lemoine points of the triangles ABC and $A''B''C''$, hence L, Br_1 and L'' lie on a line, where L'' is the Lemoine point of $A''B''C''$. Moreover, since the triangles ABC and $A''B''C''$ can be obtained from each other by rotation through twice the Brocard angle about O, the points L and L'' are equidistant from the ends of the chord passing through them. This, together with the preservation of the cross-ratio under projective transformations, implies that Br_1 is the midpoint of the segment LL'' and that the angle OBr_1L is right. Therefore Br_1 and Br_2 lie on the circle with diameter OL and are symmetric with respect to the line OL.

2d. *The Brocard ellipse is tangent to the sides of the triangle formed by the feet of the symmedians.*

Suppose triangles BCA_1 and C_2AB are similar to ABC and are positioned as in Figure 2.31.

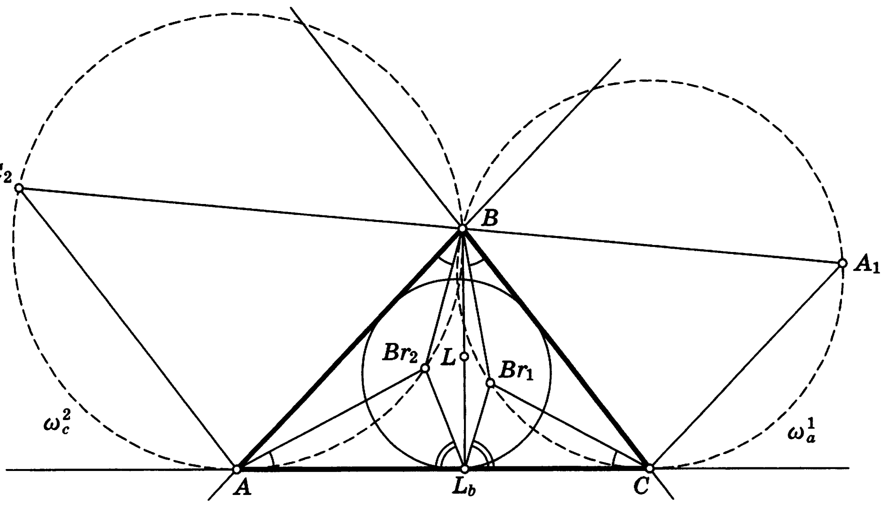

FIGURE 2.31

Let ω_a^1 and ω_c^2 be their circumcircles. Since Br_1 and Br_2 lie on ω_a^1 and ω_c^2, respectively, and the angles subtending the arcs CBr_1 and ABr_2 are equal, the ratio of the segments CBr_1 and ABr_2 equals the ratio of the radii of ω_a^1 and ω_c^2. On the other hand, the ratio of those radii equals the similarity coefficient of the triangles BCA_1 and C_2AB, and therefore equals

$$\frac{BC}{AC_2} = \frac{BC}{AB} \cdot \frac{AB}{AC_2} = \left(\frac{BC}{AB}\right)^2 = \frac{CL_b}{AL_b},$$

i.e., $\frac{CBr_1}{ABr_2} = \frac{CL_b}{AL_b}$. And since $\angle Br_1CA = \angle Br_2AC$ (the Brocard angles), the triangles CL_bBr_1 and AL_bBr_2 are similar. Therefore $\angle Br_1L_bC = \angle Br_2L_bA$. Hence the ellipse with foci at the Brocard points and sum

$Br_1L_b + Br_2L_b$ of the distances to the foci is tangent to the line AC at L_b. But such an ellipse is unique, and this is the Brocard ellipse.

That the Brocard ellipse is tangent to AB and BC at L_c and L_a is proved similarly.

3. The Steiner ellipse and the roots of the derivative. To each point on the plane with Cartesian coordinates (a, b) we associate the complex number $a + ib$.

Theorem 2.10. *Let p and q be the roots of the derivative of the polynomial $P(z) = (z - z_a)(z - z_b)(z - z_c)$. Then the points of the complex plane corresponding to p and q are isogonally conjugate with respect to the triangle ABC whose vertices correspond to the numbers z_a, z_b, z_c.*

Proof. We prove that $\angle BAP = \angle CAQ$, where the points P and Q correspond to p and q. Without loss of generality we may assume that $z_a = 0$, since subtracting z_a from z_a, z_b and z_c we will change the roots of the derivative of $P(z)$ by $-z_a$. Geometrically, this corresponds to the translation by the vector $-z_a$.

The polynomial $P(z)$ will then become $z^3 - (z_b + z_c)z^2 + z_c z_b z$ with derivative $3z^2 - 2(z_b + z_c)z + z_b z_c$. By Viète's theorem, the product of the roots of $P'(z)$ equals $\frac{1}{3}z_b z_c$. This means that the product of z_b and z_c has the same argument as the product of p and q, whence $\angle BAx + \angle CAx = \angle PAx + \angle QAx$. Therefore the angles $\angle BAP$ and $\angle CAQ$ are equal. A similar proof shows that $\angle ABP = \angle CBQ$ and $\angle ACP = \angle BCQ$. □

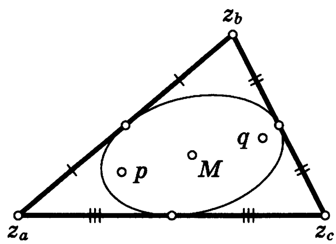

FIGURE 2.32

The ellipse with foci at those points and tangent to the sides of the triangle ABC is called the *inscribed Steiner ellipse* (Figure 2.32).

We prove that its center coincides with the centroid of the triangle. The centroid corresponds to $\frac{1}{3}(z_a + z_b + z_c)$, whereas the centroid of P and Q corresponds to $\frac{1}{2}(p+q)$. We have $P'(z) = 3z^2 - 2(z_a + z_b + z_c)z + z_a z_b z_c$, and therefore, by Viète's theorem, the sum of the roots of the derivative equals $\frac{2}{3}(z_a + z_b + z_c)$, i.e., their centroid is $\frac{1}{3}(z_a + z_b + z_c)$, which is what was to be proved.

Remarks. 1. Actually any polynomial of degree greater than one has the above property: *the centroid of the roots of the polynomial and the centroid of the roots of its derivative coincide.* The easiest way to prove this is to move the centroid to 0. Then the second coefficient of the polynomial, and therefore the second coefficient of its derivative, will vanish; hence the sum of the roots of the derivative (and therefore the centroid) will be zero, i.e., the centroid will coincide with the centroid of the roots of the polynomial.

2. Consider an affine transformation transforming an equilateral triangle into the triangle ABC. Then the incircle of the triangle will become an ellipse and the center of this ellipse will be the centroid of the triangle ABC. As we will show in Chapter 4, there is a unique conic tangent to the three given lines, with center at a given point. Therefore this will be the Steiner ellipse.

Since in an equilateral triangle the tangency points of the incircle are the midpoints of the sides, the Steiner ellipse will also be tangent to the sides of ABC at their midpoints.

Similar to the inscribed Steiner ellipse is the *circumscribed Steiner ellipse*; this is the ellipse that passes through the vertices of the triangle and has its center at the intersection point of the medians. It is the image of the circumcircle of an equilateral triangle under the affine transformation that sends it to the triangle ABC.

4. The points of Apollonius and Torricelli. Given a triangle ABC, the locus of points P such that $\frac{PA}{PB} = \frac{AC}{BC}$ is the Apollonius circle of the points A and B that contains the feet of the internal and external bisectors of the angle C as antipodal points. For points P_1, P_2, the intersections of this circle with the similar circle constructed using another pair of vertices, we have $P_iA \cdot BC = P_iB \cdot AC = P_iC \cdot AB$, hence the points P_i lie also on the third such circle. They are called the *Apollonius points* of the triangle ABC. Henceforth we shall denote these points Ap_1 (the first Apollonius point is usually taken to be the one inside the circumcircle) and Ap_2. Examining the angles we see that the Apollonius circles are orthogonal to the circumcircle of ABC. Hence the inversion with respect to the circumcircle preserves those circles and transforms the Apollonius points into each other. In particular, the line Ap_1Ap_2 passes through the center O of the circumcircle. Furthermore, the center of the Apollonius circle passing through C is the intersection of the line AB with the tangent to the circumcircle at C. Under the polar correspondence with respect to the circumcircle, these lines correspond to the intersection point of the tangents to it at A and B and the point C. Therefore the polars of the centers of the Apollonius circles are the symmedians, and the pole of the line containing those centers is the Lemoine point L. Thus L also lies on the line Ap_1Ap_2 and is the inverse of the midpoint of the segment Ap_1Ap_2 with respect to the circumcircle.

We now mention an important property of the Apollonius points which can be taken as their definition.

The pedal triangles of the Apollonius points are equilateral.

Proof. Let A', B' be the projections of Ap_i to BC and CA. Since the quadrilateral $CA'Ap_iB'$ is inscribed in the circle with diameter Ap_iC, we have $A'B' = Ap_iC \sin C = Ap_iC \cdot AB/2R$. This implies at once that all the sides of the pedal triangle are equal. □

It is now easy to see which points are isogonally conjugate to the Apollonius points. Indeed, the perpendiculars dropped from A and B to the corresponding sides of the pedal triangle intersect at a point T_i that is the isogonal conjugate of Ap_i. Since the pedal triangle is equilateral, the angle AT_iB equals either 60° or 120°. Thus, at T_i, all the sides of the triangle are seen at angles 60° or 120°. The points with this property are called the *Torricelli points* of the triangle ABC. They can be constructed as follows: let A', B' and C' be the vertices of the equilateral triangles based on the sides of the triangle ABC and pointing outside (inside). Then the lines AA', BB' and CC' intersect at the *first* (*second*) *Torricelli point* (Figure 2.33).

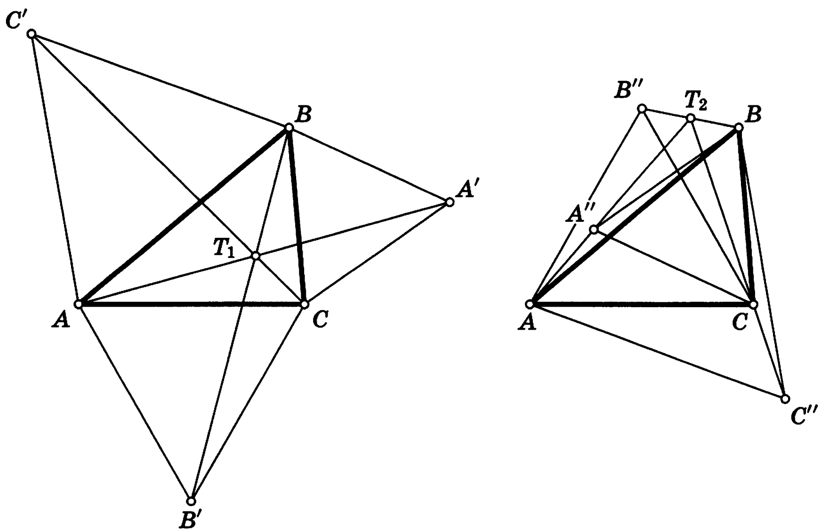

FIGURE 2.33

If the angles of the triangle are less than 120°, then the first Torricelli point lies inside the triangle and the sum of the distances from that point to the vertices is less than from any other point X in the plane. This can quickly be checked by turning the triangles AXC and AT_1C through 60° about A (Figure 2.34).

We now mention without proof three more properties of the Torricelli and Apollonius points.

1) *The lines* Ap_1T_1 *and* Ap_2T_2 *are parallel to the Euler line* OH;

2) *The line* T_1T_2 *passes through the Lemoine point, which implies, as we will show in* 3.3, *that the lines* Ap_1T_2 *and* Ap_2T_1 *intersect at the centroid of the triangle* ABC;

3) $\angle ApBrL = 60°$ (the indexing plays no role since Ap_1, Ap_2 and L lie on a line with respect to which Br_1 and Br_2 are symmetric); *in particular*

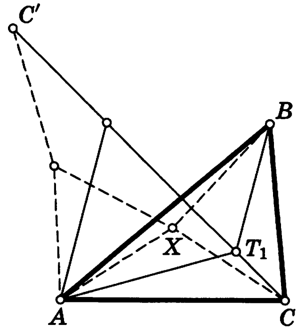

FIGURE 2.34

this means that the triangle OBr_1Br_2 has the same Apollonius points as the original triangle.

5. The Gergonne and Nagel points and the homothety centers of the circumcircle and the incircle.

Definition. Suppose the incircle of a triangle ABC is tangent to the sides at points G_a, G_b and G_c. Then the lines AG_a, BG_b and CG_c intersect in a single point G (this can be shown with the aid of Ceva's theorem or by using a projective transformation preserving the incircle and transforming the intersection of the lines AG_a and BG_b into the center), called the *Gergonne point*.

Draw the reflections of AG_a and BG_b in the bisectors AI and BI and find their intersection points A_1, B_1 with the incircle on the other side of A and B (Figure 2.35). We have

$$\angle G_cIA_1 = \angle G_cIG_a + \angle G_aIA_1 = 180^\circ - \angle B + 2\left(\angle B + \frac{1}{2}A - 90^\circ\right)$$
$$= \angle A + \angle B = \angle G_cIB_1.$$

Therefore $A_1B_1 \parallel AB$. Similarly, the reflection of CG_c in CI intersects the incircle at the point C_1 such that $C_1A_1 \parallel CA$ and $C_1B_1 \parallel CB$. Thus triangles ABC and $A_1B_1C_1$ are homothetic with respect to the isogonal conjugate of G. Under this homothety, the circumcircle of ABC transforms into the incircle, and therefore we have the following result.

The Gergonne point is isogonally conjugate to the inner center of homothety of the circumcircle and the incircle of the triangle.

Similarly, if we connect the vertices of the triangle and the tangency points of the opposite sides and the excircles, we obtain lines intersecting in a point, called the *Nagel point*. Repeating the above arguments, we have

The Nagel point is isogonally conjugate to the outer center of homothety of the circumcircle and the incircle of the triangle.

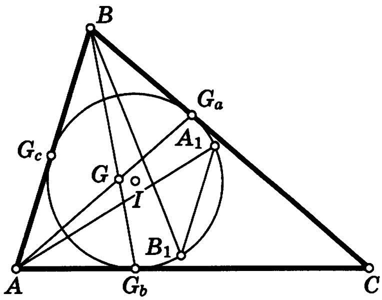

FIGURE 2.35

Besides isogonal conjugation with respect to a given triangle, one can also define the so-called isotomic conjugation, which is constructed as follows.

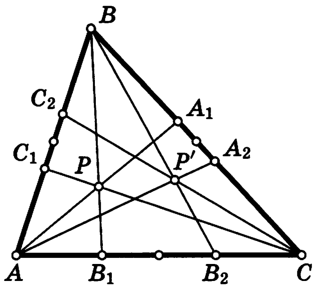

FIGURE 2.36

Definition. Suppose lines AP, BP, CP intersect the opposite sides of a triangle ABC at points A_1, B_1, C_1, and let A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 in the midpoints of the corresponding sides. Then the lines AA_2, BB_2, CC_2 intersect at a single point P', called the *isotomic conjugate* of P with respect to the triangle ABC (Figure 2.36).

As in the case of isogonal conjugation, a vertex of the triangle is isotomically conjugate to any point of the opposite side. In all other cases isotomic conjugation is bijective.

The fixed points of isotomic conjugation are the centroid of the triangle and the reflections of the vertices in the midpoints of the opposite sides. Notice also that isotomic conjugacy of points is preserved under affine transformations.

Among other properties of isotomic conjugation that are worth mentioning is the fact that the Gergonne point is isotomically conjugate to the Nagel point.

2.4. Radical axes and pencils of circles

Definition. Given a circle with center O and radius r and a point P, the quantity $OP^2 - r^2$ is called the *power of P* with respect to the circle.

The definition implies at once that the power is positive for outer points and negative for inner points.

Exercise 1. Find the locus of points of constant power with respect to a given circle.

Answer. Points of equal power with respect to a circle ω form a circle concentric with ω.

Lemma 2.7. *Suppose a line passing through P intersects the circle in X and Y. Then $PX \cdot PY$ does not depend on the line and equals the absolute value of the power of P with respect to the circle.*

Proof. Suppose we have two lines passing through P such that the first intersects the circle at points A and B and the second at points C and D. We prove that $PA \cdot PB = PC \cdot PD$. Clearly, the triangles PAC and PDB are similar and therefore

$$\frac{PA}{PC} = \frac{PD}{PB} \quad \Rightarrow \quad PA \cdot PB = PC \cdot PD.$$

It remains to show that this quantity equals the absolute value of the power of P with respect to the circle. Draw a line through the center of the circle. Then the product of distances from P to the intersection points equals $(OP + r) \cdot (OP - r)$ (where O is the center of the circle and r its radius). But this product obviously equals $OP^2 - r^2$. □

Exercise 2. Suppose the segments AB and CD intersect at a point P and $PA \cdot PB = PC \cdot PD$. Prove that the quadrilateral $ABCD$ is inscribed.

Solution. Since $PA \cdot PB = PC \cdot PD$, the triangles PBD and PCA are similar and therefore $\angle PBD = \angle PCA$. But this means that the quadrilateral $ABCD$ is inscribed.

Lemma 2.7 is needed, in particular, for the proof of the following important theorem.

Theorem 2.11. *The set of points whose powers with respect to two given nonconcentric circles are equal is a line. This line is called the* radical axis *of the circles.*

Proof. Suppose that the two circles intersect. Draw the line through the intersection points. We claim that this line is the radical axis. Suppose the circles intersect at points A and B. Take an arbitrary point P on the line AB. Then the absolute value of the power of P with respect to the two circles equals $PA \cdot PB$ and the signs obviously coincide.

There are no other points with this property. Draw the line through A and an arbitrary point X not lying on the line AB. Then, as is easy to see, either the distances from X to the other intersections with those circles are distinct (and therefore the absolute values of the powers are not equal) or the signs of the powers of points are distinct.

Now we prove the theorem in the case where the circles have no common points. To that end, we apply a useful trick. Consider two spheres intersecting our plane along the two circles and having common points. It is easy to see that such spheres exist. Indeed, take a point not in the plane and the spheres passing through that point and the two circles. For spheres, the power of a point can also be defined, and Lemma 2.7 is still true. A similar argument shows that the set of points whose powers with respect to the two spheres are equal is the plane passing through the intersection circle of the spheres (see Figure 2.37). That plane intersects our plane along a line which is obviously the radical axis of our circles. □

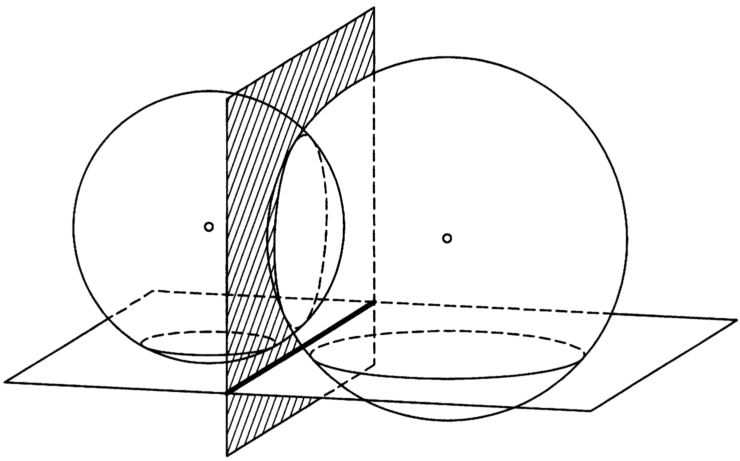

FIGURE 2.37

Suppose now that three circles are given. If their centers are not on a line, then there are two pairs of those circles whose radical axes intersect. The powers of their intersection point with respect to all three circles are equal and therefore the third radical axis also passes through that point, called the *radical center* of the three circles. If the centers of the circles are on a line, then the radical axes are either parallel or coincide. In the latter case the circles are said to be *coaxial*.

The set of all circles coaxial with two given circles is called a *pencil*. If the circles defining the pencil intersect at two points, then the pencil consists of all circles passing through these points. Such a pencil is said to be *hyperbolic*. If the two circles are tangent, then any circle of the pencil is tangent to their common tangent line at the same point. Such a pencil is said to be *parabolic*. Finally, two nonintersecting circles give rise to a pencil of the type shown in Figure 2.38. Such a pencil is said to be *elliptic*. Notice that two of the circles of an elliptic pencil degenerate into points, called the *limit points* of the pencil.

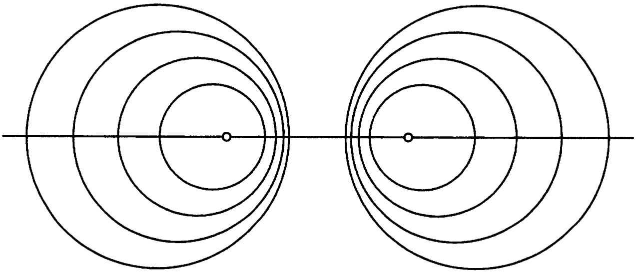

FIGURE 2.38

For any point on the radical axis which is outside the circles, the tangents to the circles of the pencil passing through that point are equal. Hence the circle centered at that point with radius the length of the tangent is perpendicular to all the circles of the pencil. All such circles form another pencil (Figure 2.39), and any two of them uniquely determine the original pencil. It now follows that the inversion in an arbitrary circle transforms pencils into pencils; moreover any pencil containing the circle of inversion transforms into itself. In particular, the limit points of an elliptic pencil transform into each other under inversion in any circle of that pencil. Notice also that the inversion centered at a limit point transforms the circles of the perpendicular pencil into lines. Therefore the original pencil is transformed into a pencil of concentric circles.

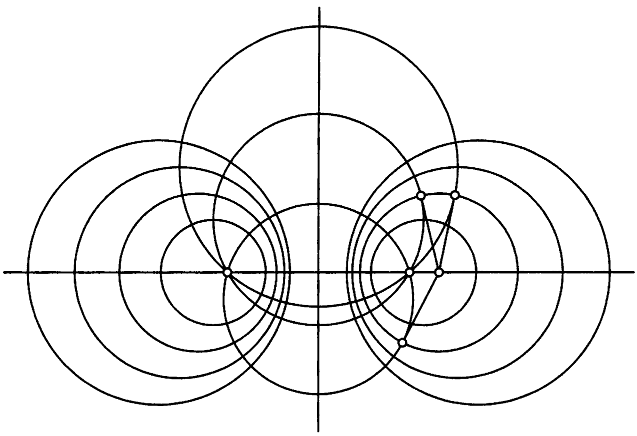

FIGURE 2.39

Exercise 3. Prove the last assertion by passing to three-space.

Solution. Consider the intersecting family of spheres which intersect our plane along the circles of the pencil. Those spheres also form a pencil.

Under inversion those spheres transform into spheres intersecting along a circle. Therefore they will intersect our plane along a pencil.

Pencils of circles have yet another very important property.

Theorem 2.12. *Given two circles ω_1 and ω_2, the locus where the ratio of powers with respect to those two circles is constant is a circle belonging to the pencil formed by ω_1 and ω_2.*

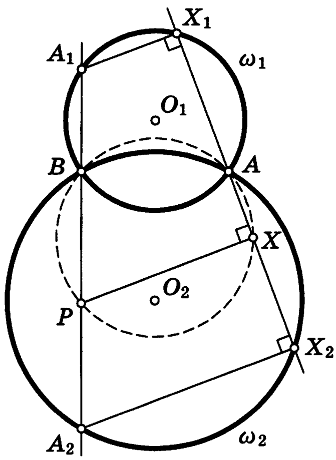

FIGURE 2.40

Proof. Suppose that ω_1 and ω_2 intersect at A and B (Figure 2.40). Let O_1 and O_2 be their centers and, respectively, r_1 and r_2 their radii. Let A_1 and A_2 be the reflections of A in O_1 and O_2. We shall show that the set of points X whose powers with respect to ω_1 and ω_2 equal k is a circle. Suppose the line XA intersects ω_1 and ω_2 at X_1 and X_2, respectively. Then k equals $\frac{XX_1}{XX_2}$ (with an appropriate sign). Since AA_1 and AA_2 are diameters of the corresponding circles, the angles AX_1A_1 and AX_2A_2 are right and therefore X_1 and X_2 are the projections of A_1 and A_2 to the line AX. Let P be the point on A_1A_2 such that $\frac{PA_1}{PA_2} = k$ (there are two points for which this ratio equals $|k|$; choose the one with the "appropriate" sign). Then, by Thales' theorem, X is the projection of P to the line AX and therefore it lies on the circle with diameter AP. Reversing the argument, it is easy to show that for any point on that circle the ratio of the powers with respect to ω_1 and ω_2 equals k.

To prove this assertion for nonintersecting circles, we switch again to three-space. Suppose we are given two intersecting spheres intersecting our plane along the two circles. Using the arguments as above we show that the locus where the ratio of the powers with respect to the two spheres equals k is a sphere from the pencil, i.e., a sphere containing the circle of intersection of the two spheres. The intersection of that sphere with our plane is a circle from the pencil determined by ω_1 and ω_2, which is the desired assertion. ☐

Using this theorem one can easily prove the theorem of Poncelet (for a pencil of circles) without algebraic arguments. In its general from, the theorem of Poncelet will be proved in 3.3.

Theorem 2.13 (Poncelet). *Suppose circles ω_i belong to the same pencil and A_0 is a point on ω_0. The tangent to ω_1 from A_0 intersects ω_0 again at A_1, the tangent to ω_2 from A_1 intersects ω_0 again at A_2, etc., the tangent to ω_{i+1} from A_i intersects ω_0 again at A_{i+1}. Suppose that for some n the points A_n and A_0 coincide. Then for any point B_0 on ω_0, the similarly constructed point B_n coincides with B_0* (Figure 2.41).

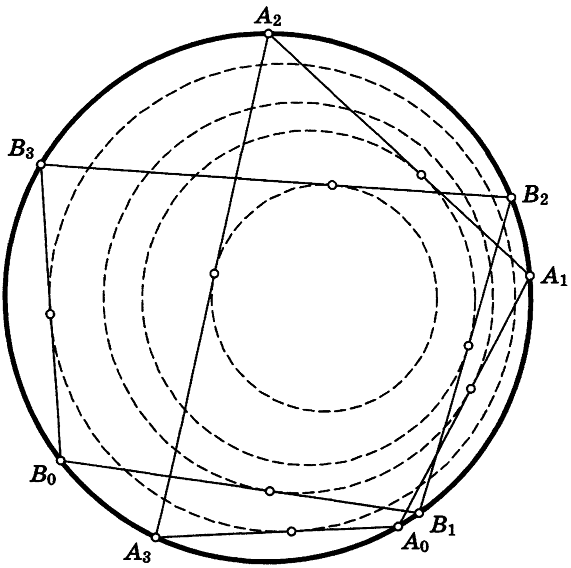

FIGURE 2.41

We note that it is not always possible to construct B_n. This is the case, for example, if B_0 lies inside ω_1. We assume that such B_n has been constructed.

Proof. It suffices to show that A_iB_i is tangent to some fixed circle from the pencil. Indeed, if A_0 coincides with A_n, then the tangent to the circle passing through A_0 coincides with the tangent passing through A_n (assuming the tangents run in the appropriate direction). Therefore their intersections with ω_0 must coincide, but these are B_0 and B_n.

Suppose the lines A_0A_1 and B_0B_1 are tangent to ω_1 at X and Y, respectively (Figure 2.42). Let Z be the intersection of these lines. The triangle XZY is isosceles (because ZX and ZY are simply the tangents to ω_1 passing through Z). Therefore the angles X and Y in thattriangle are equal.

Moreover the angles $B_1A_1A_0$ and $B_1B_0A_0$ are also equal. Thus the triangles XQA_1 and YPB_0 are similar, where P and Q are the intersections of the line XY with the segments A_0B_0 and A_1B_1. Hence the angles PQA_1 and QPB_0 are equal, and therefore there is a circle ω' tangent to A_0B_0 and A_1B_1 at P and Q, respectively.

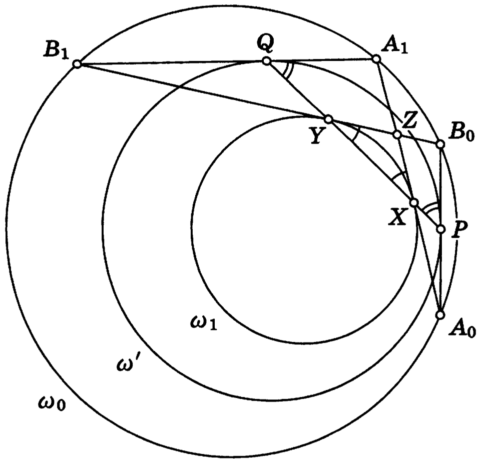

FIGURE 2.42

We want to show that it belongs to the pencil. It suffices to prove that ω_0 belongs to the pencil determined by ω_1 and ω'.

We show that the ratios of powers of A_0, A_1, B_0 and B_1 with respect to ω_1 and ω' are equal. Clearly, those ratios equal, respectively, $\frac{A_0X^2}{A_0P^2}$, $\frac{A_1X^2}{A_1Q^2}$, $\frac{B_0Y^2}{B_0P^2}$ and $\frac{B_1Y^2}{B_1Q^2}$.

By the similarity of the triangles A_0XP and B_1YQ, the ratios $\frac{A_0X^2}{A_0P^2}$ and $\frac{B_1Y^2}{B_1Q^2}$ are equal. Likewise, by the similarity of the triangles B_0YQ and A_1XQ, the ratios $\frac{A_1X^2}{A_1Q^2}$ and $\frac{B_0Y^2}{B_0P^2}$ are equal.

It remains to show that $\frac{A_0X}{A_0P} = \frac{B_0Y}{B_0P}$. But, by the sine theorem,

$$\frac{A_0X}{A_0P} = \frac{\sin\angle A_0PX}{\sin\angle A_0XP} = \frac{\sin\angle B_0PY}{\sin\angle B_0YP} = \frac{B_0Y}{B_0P}.$$

A similar argument shows that A_iB_i and $A_{i+1}B_{i+1}$ are tangent to the same circle from the pencil. It is easy to see that the segments A_iB_i can be tangent to only one such circle. Hence this circle is the same for all segments A_iB_i, and it must be ω'. □

The Poncelet theorem means, in particular, that if a polygon is inscribed in one circle and circumscribed about another, then it can be "rotated" between those circles. Moreover each diagonal of the polygon is tangent to a circle coaxial with the circumscribed and inscribed circles (Figure 2.43).

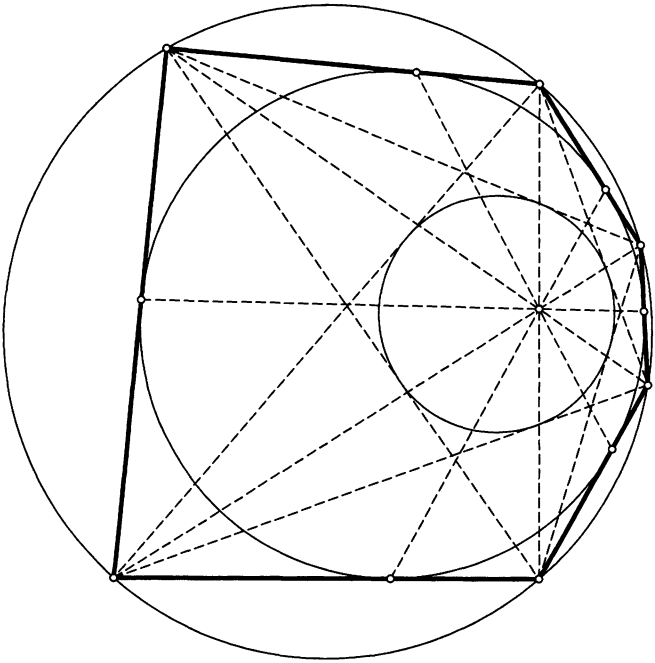

FIGURE 2.43

Besides the Poncelet theorem, the properties of radical axes also allow for a proof of Brianchon's theorem in its general form. First, we need to state it.

Theorem 2.14 (Brianchon). *Suppose lines l_i, $i = 1, \ldots, 6$, are tangent to the same conic, and let A_{ij} be the intersection of l_i and l_j. Then the lines $A_{12}A_{45}$, $A_{23}A_{56}$ and $A_{34}A_{61}$ intersect at a single point.*

Proof. Using a projective transformation we make the conic into a circle. We assume that we have a hexagon circumscribed about the circle, and the lines in question are its main diagonals. Our arguments can easily be adapted to the cases where the lines form other configurations.

Thus, given a circumscribed hexagon $ABCDEF$ (Figure 2.44), we have to show that AD, BE and CF intersect at a single point.

Consider the circles ω_1, ω_2 and ω_3, tangent to the pairs of lines AB and DE, BC and DE, CD and FA, respectively, and such that the tangency points are at a distance a from the corresponding tangency points of the sides of the hexagon and the circle (call them A_1, B_1, $\ldots$, F_1). Then the power of A with respect to ω_1 and ω_3 equals $-(AA_1 + a)^2$. Hence it lies on the radical axis of those two circles. The same is true for D. Thus AD is the radical axis of ω_1 and ω_3. A similar proof shows that BE is the radical

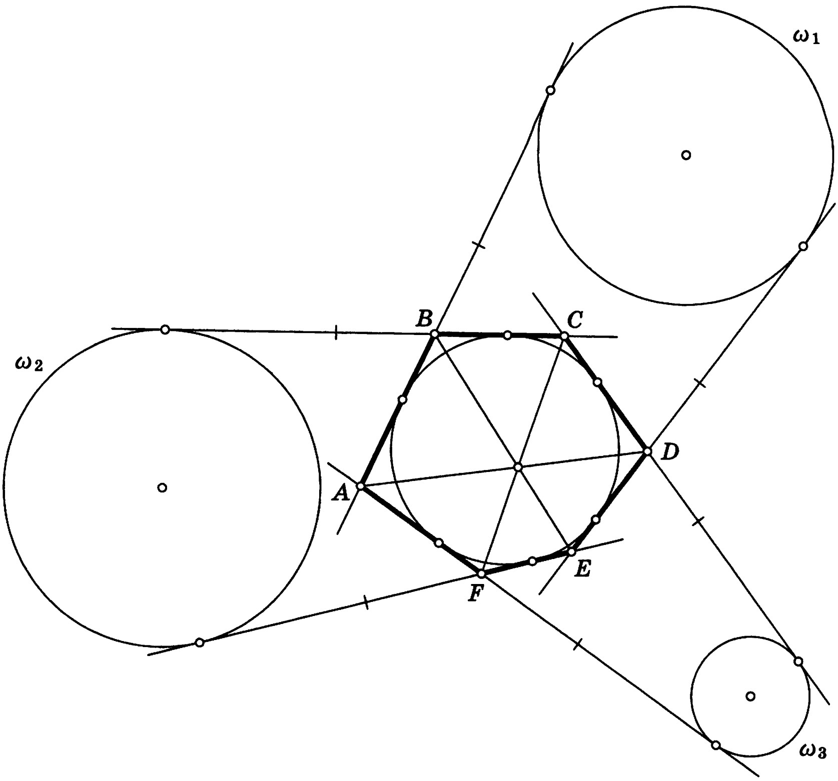

FIGURE 2.44

axis of ω_1 and ω_2, and CF is the radical axis of ω_2 and ω_3. Therefore AD, BE and CF intersect at a single point, namely, the radical center of ω_1, ω_2 and ω_3. □

Chapter 3

Projective Properties of Conics

3.1. The cross-ratio of four points on a curve. Parametrization. The converses of Pascal's and Brianchon's theorems

Projective equivalence of conics means that all the properties of the circle mentioned in the Introduction are also true for conics. In particular, for four points A, B, C, D on a conic, the cross-ratio of the lines XA, XB, XC, XD does not depend on the choice of a point X on the conic. This ratio is called the cross-ratio of A, B, C, D. Clearly, the cross-ratio is preserved under projective transformations.

We now choose and fix some point P of the conic and some line l that does not pass through P. Given a point X on the conic, consider the intersection point X' of PX and l (for the point P itself, we take the intersection of l with the tangent at P). Clearly, this correspondence is one-to-one and preserves the cross-ratio. Now any standard correspondence between the points of l and the real numbers gives rise to a parametrization of the conic. It is not difficult to see that under such a parametrization the coordinates of X are rational functions of the parameter.

Theorem 3.1 (The converse of Pascal's theorem). *For any six points X_i, $i = 1, \ldots, 6$, such that the intersections of the lines X_1X_2 and X_4X_5, X_2X_3 and X_5X_6, X_3X_4 and X_6X_1 are on a straight line, there is a conic passing through all the X_i.*

Proof. We use the fact that for any five points in general position, there is a unique conic containing them. Let α be such a conic for X_i, $i = 1, \ldots, 5$. Let A, B, C be the intersections of the lines X_1X_2 and X_4X_5, X_2X_3 and X_5X_6, X_3X_4 and X_6X_1, and let Y be the intersection point of α and BX_5 different from X_5. By Pascal's theorem, the intersection of X_3X_4 and X_1Y lies on AB, i.e., coincides with C. Hence Y coincides with X_6. □

Theorem 3.2 (The converse of Brianchon's theorem). *Given any six lines l_i, $i = 1, \dots, 6$, let A_{ij} be the intersection of the lines l_i and l_j. If the lines $A_{12}A_{45}$, $A_{23}A_{56}$ and $A_{34}A_{61}$ intersect at a single point, then there is a conic tangent to all the l_i.*

The proof of this theorem is similar to that of the previous one. However, here we first need to show that there is a unique conic tangent to five given lines. Using Brianchon's theorem, we can construct the tangency points of those lines with the conic. But only one conic can pass through five points. The construction of a tangency point is shown in Figure 3.1.

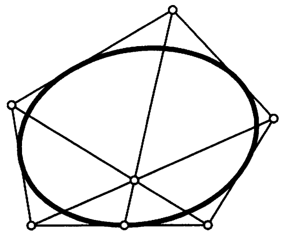

FIGURE 3.1

Using Brianchon's theorem we can get a new proof of Theorem 1.11. Indeed, suppose a parabola is inscribed in a triangle ABC. Through the orthocenter H of the triangle, draw a line l_1 tangent to the parabola and a line l_2 perpendicular to it. If we show that AB, BC, CA, l_1, l_2 and the line at infinity are tangent to a conic, then, since there is a unique conic tangent to five lines, it would coincide with the parabola. Therefore the tangents to the parabola passing through the orthocenter are perpendicular, and therefore the orthocenter lies on the directrix.

Let P be the intersection of AB with l_1, and X and Y the points of the line at infinity in the direction of l_2 and AC, respectively. Consider the "hexagon" $BPHXYC$. The main diagonals BX, PY and HC are obviously the heights of the triangle BPH, and therefore intersect at a single point. The converse of Brianchon's theorem implies that the hexagon $BPHXYC$ is circumscribed. It is now easy to see that its sides are the desired lines.

Pascal's theorem shows that using a straightedge we can construct any number of points on the conic passing through the five given points X_1, X_2, X_3, X_4, X_5. Indeed, let A be the intersection of X_1X_2 and X_4X_5, and l an arbitrary line passing through A. Let B and C be the intersections of l with X_2X_3 and X_3X_4. Then, by the converse of Pascal's theorem, the intersection point of the lines BX_5 and CX_1 lies on the conic. Similarly, by Brianchon's theorem, we can construct any number of tangents to the conic that is tangent to five given lines.

Problem 15. Prove that the diagonals of two quadrilaterals, one formed by the intersection points of two ellipses and the other formed by their common tangents, intersect at a single point (Figure 3.2).

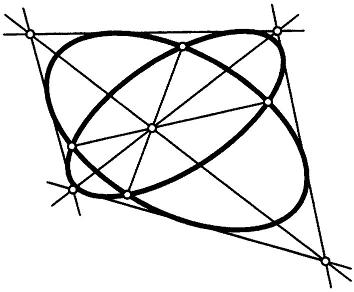

FIGURE 3.2

Problem 16. A hexagon $ABCDEF$ is inscribed in a conic. Prove that the lines AC, CE, EA, BD, DF and FB are tangent to some conic. Deduce from this the Poncelet theorem for triangles.

Problem 17. Suppose the tangent to a hyperbola at a point A intersects the asymptotes at points A_1 and A_2, and the tangent at a point B intersects the asymptotes at points B_1 and B_2. Prove that the lines A_1B_2, A_2B_1 and AB intersect at a single point.

Problem 18. Triangles ABC and $A'B'C'$ are centrally symmetric. Three parallel lines are drawn through A', B' and C'. Prove that their intersections with BC, CA and AB, respectively, lie on a straight line.

Problem 19. Prove that a conic circumscribed about a triangle ABC is an equilateral hyperbola if and only if it passes through the orthocenter of the triangle.

Problem 20. (The hexagon theorem.) Suppose a conic intersects the sides AB, BC and AC of a triangle ABC at points C_1 and C_2, A_1 and A_2, B_1 and B_2, respectively. Let A_3, B_3 and C_3 be the intersections of the pairs of tangents at the points A_1 and A_2, B_1 and B_2, C_1 and C_2 (Figure 3.3). Prove that the lines AA_3, BB_3 and CC_3 intersect at a single point.

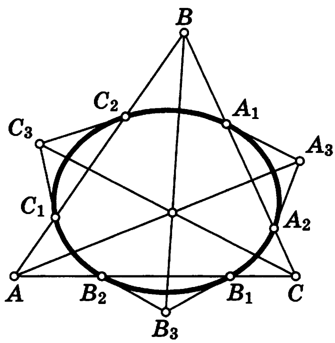

FIGURE 3.3

3.2. Polar correspondence. The duality principle

Suppose we are given a conic and a point A. Consider an arbitrary projective transformation sending the given conic into a circle. Let A' be the image of A under this transformation, a' the polar of A' with respect to the circle, and a the image of a' under the inverse transformation. Then a can be constructed as follows.

Consider two lines passing through A and intersecting the given conic at points X_1, X_2 and Y_1, Y_2. Let X be the intersection of the tangents to the conic at X_1 and X_2, and let Y be the intersection of the tangents at Y_1 and Y_2. Then the line XY coincides with a (Figure 3.4).

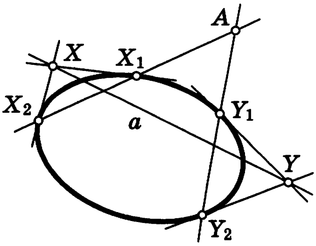

FIGURE 3.4

Indeed, applying this construction to the point A' and to the circle we have the line a'. Since projective transformations preserve the intersections and tangencies of lines and conics, the line a does not depend on the chosen projective transformation. There is another way to construct a: this is the line connecting the intersections of X_1Y_1 with X_2Y_2 and of X_1Y_2 with X_2Y_1 (Figure 3.5). In particular, if A is the center of an ellipse or a hyperbola, we get the line at infinity. Notice that the last construction also applies to degenerate curves of degree two; moreover, the constructed line passes through the common point O of the lines l_1 and l_2 comprising the curve, and the cross-ratio $(l_1l_2; OAa)$ of the lines equals 1 (Figure 3.6).

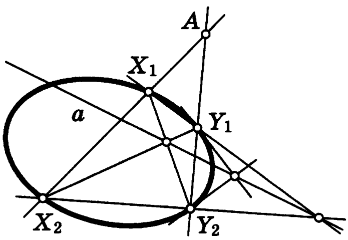

FIGURE 3.5

The defined correspondence between points and lines is called the *polar correspondence* with respect to a given conic; a is called the *polar* of A and A is the *pole* of a. Clearly, all the properties of the polar correspondence mentioned before are still true.

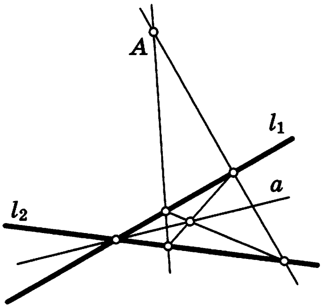

FIGURE 3.6

Notice that if a line p is the polar of a point P and an arbitrary line passing through P intersects p at Q and the conic at points A and B, then $(PQ; AB) = 1$. To see this, it suffices to consider the case when the conic is a circle and one of the points P and Q is on the line at infinity (Figure 3.7).

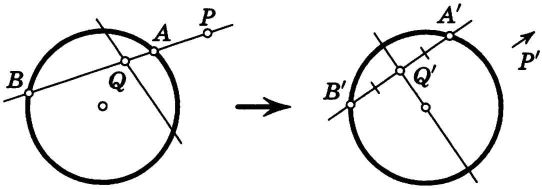

FIGURE 3.7

In particular, if the conic is an ellipse or a hyperbola, then the midpoints of all chords parallel to a fixed line lie on a line passing through the center of the conic (the directions of this line and the fixed line are said to be *conjugate* with respect to the conic); if the conic is a parabola, then the midpoints lie on a line parallel to its axis (see Problem 10). The dual assertion is also true: if tangents a and b to a conic and two arbitrary lines p and q meet at one point, then the pole of p lies on q if and only if $(ab; pq) = 1$.

The duality principle also remains true. Hence, for example, the converse of Brianchon's theorem is a consequence of the converse of Pascal's theorem. Furthermore, for any five lines in general position, there is a unique conic tangent to them.

Definition. The *dual curve* of a smooth curve is the set of duals to all the tangents of this curve.

An example of a curve and its dual curve is shown in Figure 3.8.

The following theorem provides an important property of the duality operation.

Theorem 3.3. *Let $R(\gamma)$ be the dual curve of a curve γ. Then $R(R(\gamma)) = \gamma$.*

Proof. Suppose a point X moves along γ toward A. Then, clearly, the intersections of the tangents at X and A (call them x and a, respectively)

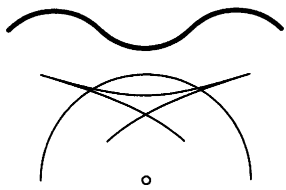

FIGURE 3.8

tend to A. Let Y be the intersection of x and a (Figure 3.9). What happens with the duals of x and a on the curve $R(\gamma)$? Clearly, $R(x)$ tends to $R(a)$ and therefore the segment $R(a)R(x)$ tends to the tangent to $R(\gamma)$ at $R(a)$. But $R(a)R(x)$ is nothing but $R(Y)$, and therefore $R(Y)$ tends to the tangent to $R(\gamma)$ at $R(a)$. Then the dual of $R(Y)$ tends to the dual of the tangent at $R(a)$. But this is the point Y which, under the motion, tends to A. It follows that the dual of the tangent at $R(a)$ is A. But this means that the duals of the tangents to $R(\gamma)$ form the curve γ. □

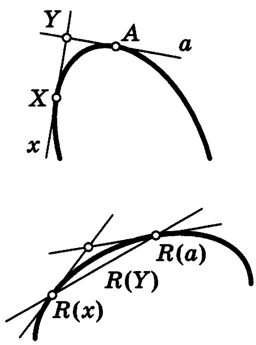

FIGURE 3.9

The polar correspondence provides yet another way of constructing conics. Consider the circle α with center A and radius r and another circle ω with center O. The polars of all the points of α with respect to ω envelop some curve, called the *polar curve* of α. The polar curve can also be constructed as the set of the poles of all the tangents to α.

Theorem 3.4. *The polar curve of a circle with respect to another circle is a conic.*

Proof. Let ω be a circle with center O and ω_1 a circle with center O_1 (for convenience, assume that O lies inside ω_1; see Figure 3.10). Suppose that the inversion with respect to ω transforms ω_1 into a circle ω_2 centered at O_2.

Let $p(X)$ be the line passing through X and perpendicular to OX. It is the polar of the inverse of X with respect to ω. As X moves along ω_2, its corresponding line sweeps the set of the polars of the points on ω_1.

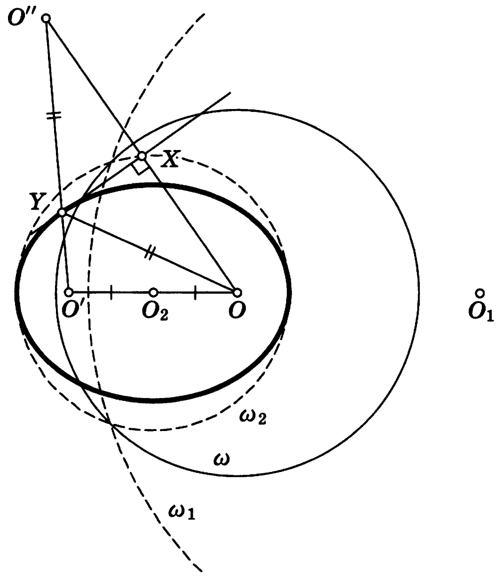

FIGURE 3.10

Thus we need to show that the set of all such lines is tangent to a conic. Consider the reflection O' of O in O_2, and the reflection O'' of O in X. It is easy to see that the length of the segment $O'O''$ equals the diameter of ω_2. Suppose $O'O''$ intersects $p(X)$ at a point Y. Then, since $p(X)$ is the midpoint perpendicular to OO'', the segments YO and YO'' are of equal length. Moreover, the angles formed by the line $p(X)$ with YO and YO' are equal. Therefore $p(X)$ is tangent at Y to the ellipse with foci O and O' and major semi-axis equal to the diameter of ω_2. Moreover, it is easy to see that, as X moves along ω_2, the point Y sweeps the entire ellipse.

Thus we have constructed a conic which is the polar curve of our circle.

If O is outside ω_1, then identical arguments show that the polar curve is a hyperbola, and in the case when O lies on ω_1, the polar curve is a parabola. □

Because of the projective equivalence of conics, the theorem just proved can be generalized.

Theorem 3.5. *The polar of a conic with respect to another conic is also a conic.*

The only thing that is not clear is why any two conics can be made into circles by a projective transformation. In general, this is not true (although this is true over complex numbers). But this is possible if the conics intersect at no more than two points. This can easily be achieved by scaling (with center at the center of the conic) the conic with respect to which we perform a polar transformation so that it would intersect our conic at most at two points. The dual curve will then scale in the opposite way. Now any two conics intersecting at most at two points can be made into two circles using a projective transformation.

In fact, this theorem can be proved by using only the theorems of Pascal and Brianchon. Choose and fix five points X_1, X_2, X_3, X_4 and X_5 on the conic α whose polar curve we consider. Then the polars of those five points are tangent to some other conic, which we denote α_1. Suppose a point X is moving along α. Then the corollary of Pascal's theorem can be applied to the points $X_1, \dots, X_5$ and X. Hence the corollary of Brianchon's theorem applies to the polars of those points. But the converse of Brianchon's theorem implies that all these six lines are tangent to a conic. It could only be the conic α_1, since five of the lines (the polars of X_i, $i = 1, \dots, 5$) can be tangent to only one conic. Thus the polars of all points on α are tangent to α_1. Reversing the argument, one can easily show that the conic α_1 is traversed completely.

Finally, we mention yet another approach to defining conics and polar correspondences. Suppose we have a one-to-one correspondence between points and lines in the projective plane which has the duality property, i.e., if a point A belongs to the image of a point B, then B belongs to the image of A. Then the set of points belonging to their own images is a conic (possibly, imaginary) and the polar correspondence with respect to that conic coincides with the given correspondence.

We now show that a focus and the corresponding directrix of a conic are polar to each other. In fact, this has already been established for the parabola. We prove this for the remaining conics.

Theorem 3.6. *A focus and the corresponding directrix of a conic are polar to each other* (Figure 3.11).

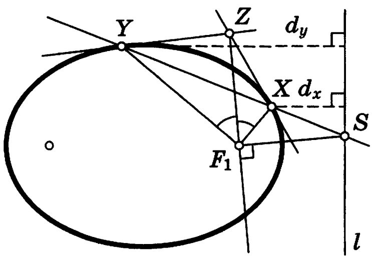

FIGURE 3.11

Proof. Consider a focus F_1 and its polar l. We prove that for any two points X and Y on the conic, the ratios of the distances to F_1 and l are equal. Let S be the intersection of the lines XY and l. Let Z denote the intersection of the tangents to the conic at X and Y. By the properties of the polar transformation, F_1Z is the polar of S. By the corollary of Theorem 1.2, the angle SF_1Z is right. Moreover, Theorem 1.4 implies that F_1Z is the bisector of the angle XF_1Y. Therefore F_1S is the bisector of the exterior angle XF_1Y. Therefore $\frac{F_1X}{SX} = \frac{F_1Y}{SY}$. Since the ratio of SX and SY equals the ratio of distances from X and Y to any line containing S (and, of course, different from XY), we have $\frac{F_1X}{d_x} = \frac{F_1Y}{d_y}$, where d_x and d_y are the distances from X and Y to l. □

This assertion can also be proved using Dandelin's construction, which we applied to prove that a conic is a projection of a circle (Figure 3.12). In three-dimensional (and, more generally, in n-dimensional) space, one also has duality transformations. They are constructed the same way as in two dimensions. Moreover, points become planes and vice versa, whereas lines are sent to lines.

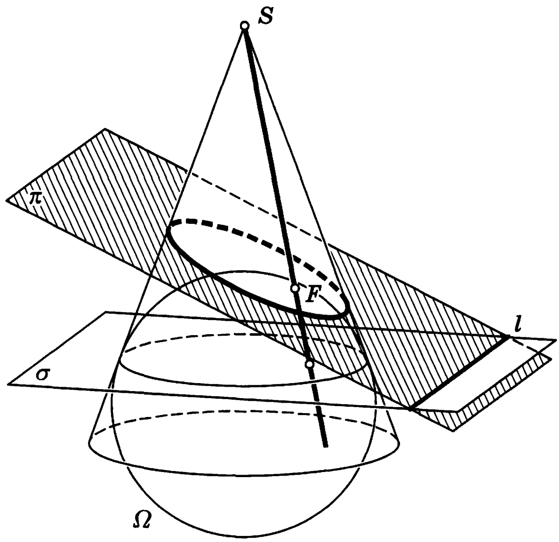

FIGURE 3.12

The polar plane of the point S with respect to the sphere Ω is the plane σ, and the polar plane of the point F is the plane π. Hence the polar of the line SF is the line l. It is clear that the pole of l with respect to the intersection circle of Ω and σ is the intersection of the line SF and the plane σ. Therefore F is the pole of l with respect to the intersection ellipse of our cone and the plane π (consider the projection of σ to π from the point S).

An interesting example of assertions interchanged by a polar transformation is Theorem 1.11 and Problem 19.

Indeed, let H be the orthocenter of a triangle ABC. The polar transformation with respect to a circle ω with center H maps the triangle ABC into a homothetic (with respect to H) triangle $A'B'C'$. If a parabola is tangent to the sides of the triangle ABC, then the conic dual to it with respect to ω will pass through A', B' and C' and also through H, because it is the pole of the line at infinity. By Problem 19, this conic is an equilateral hyperbola. The intersection points of this hyperbola and the line at infinity give rise to perpendicular directions, and therefore their polars will be perpendicular. On the other hand, the polars of the intersection points of this hyperbola with the line at infinity are the tangents to our parabola passing through H (Figure 3.13).

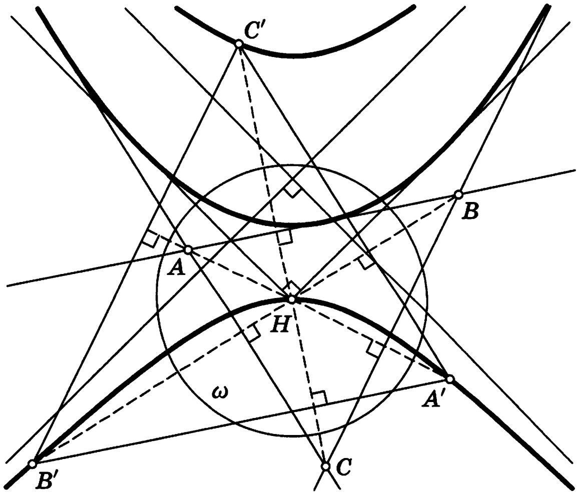

FIGURE 3.13

Similar arguments allow one to deduce the assertion of Problem 19 from Theorem 1.11. Therefore these assertions are dual to each other.

Now we mention a few more results related to the Ceva triangles of triangles inscribed in a conic.

Theorem 3.7. *A triangle ABC is self-polar (i.e., its sides are the polars of the corresponding vertices) with respect to a conic if and only if it is the Ceva triangle of a point on the conic with respect to a triangle inscribed in the conic.*

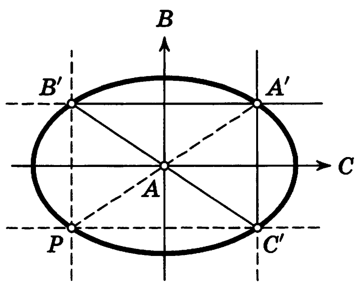

FIGURE 3.14

Proof. We begin by moving the vertices B and C of the triangle to points at infinity with perpendicular directions. Then our conic will, obviously, become a conic centered at A (since the center of the conic is the pole of the line at infinity). Consider the rectangle inscribed in the conic with sides parallel to the directions given by B and C (such a rectangle exists because the triangle ABC is self-polar). Its vertices can be viewed as a triangle and a point on the conic for which the triangle ABC is self-polar (Figure 3.14). □

Theorem 3.8. *Suppose we are given a triangle ABC and a point Z. For an arbitrary line passing through Z, let A' and B' be its intersection points with BC and AC. Then the locus of the intersections of the lines AA' and BB' is a conic passing through A, B and C and tangent to the lines AZ and BZ.*

Proof. Apply a projective transformation making the triangle ABC into a right isosceles triangle ($AC = BC$) and sending the point Z to infinity in the direction perpendicular to AB. Then the triangles $AA'P$ and $B'BP$, where P is the intersection of AB and $A'B'$, are equal, and therefore the lines AA' and BB' are perpendicular; i.e., their point of intersection lies on the circumcircle of the triangle ABC. Moreover, the lines BZ and AZ are tangent to that circle at A and B. □

Now let Y be the intersection of the tangents to the conic at A and C. Consider the intersections A' and C' of the lines passing through Y with BC and AB. Then the intersection of the lines AA' and CC' lies on the conic. This means that there is family of triangles $A'B'C'$, with vertices on the corresponding sides of the triangle ABC, with the following properties.

1. For each triangle in the family, the lines AA', BB' and CC' intersect at a single point. The set of all such points is a conic passing through A, B, C.

2. All lines $A'B'$ pass through the pole of the line AB with respect to the circumscribed conic. Similarly, all lines $A'C'$ pass through the pole of the line AC, and all lines $B'C'$ pass through the pole of the line BC.

Projective properties of conics may be useful for proving results seemingly unrelated to conics. As an example, we have the following.

Theorem 3.9. *Suppose we are given a triangle ABC and points P and Q, and suppose that the lines AP, BP and CP intersect the respective sides of the triangle at points A_1, B_1, C_1 and that the lines AQ, BQ, CQ intersect the respective sides at points A_2, B_2, C_2. Let C_3, C_4 be the intersections of the lines CC_1 and A_2B_2, CC_2 and A_1B_1, respectively; the points A_3, A_4, B_3, B_4 are defined similarly. Then the lines A_1A_4, A_2A_3, B_1B_4, B_2B_3, C_1C_4, C_2C_3 intersect at a single point* (Figure 3.15).

Proof. The points A, B, C, P, Q determine a conic. Without loss of generality we may assume that the conic is a circle and PQ is its diameter. Since A_1, B_1, C_1 are the intersections of the opposite sides and the diagonals of the inscribed quadrilateral $ABCP$, we have that A_1B_1 is the polar of C_1. Since PQ is a diameter, the feet of the heights of the triangle PQC_1

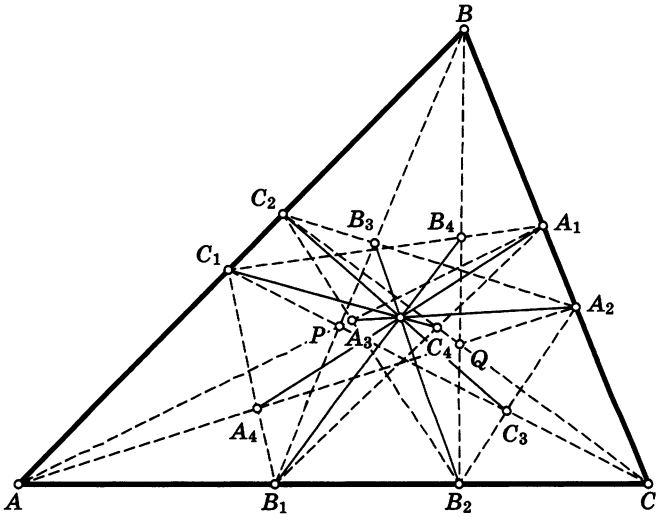

FIGURE 3.15

dropped from P and Q lie on the circle. Examining the quadrilateral formed by those feet and the points P and Q, we see that the orthocenter of the triangle PQC_1 lies on the polar of C_1, i.e., on A_1B_1. Since it also lies on the line QC, it coincides with C_4. Thus the line C_1C_4 is perpendicular to the diameter PQ, i.e., it passes through the pole of the diameter. Similarly, the remaining five lines also pass through the pole of PQ, hence the pole is the point mentioned in the theorem. □

Using duality we can prove the following nice theorem.

Theorem 3.10 (Frégier). *Suppose we are given a conic and a point P on it. Then all the chords seen from P at a right angle pass through a single point* (Figure 3.16).

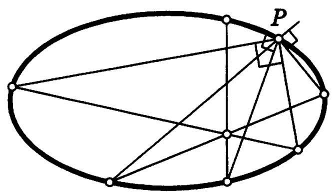

FIGURE 3.16

Proof. Apply the polar correspondence with respect to a circle with center P. Since the given conic passes through P, its transform will be a parabola. The perpendicular lines passing through P will transform into the two points at infinity corresponding to the perpendicular directions, and their second intersections with the conic will become perpendicular tangents

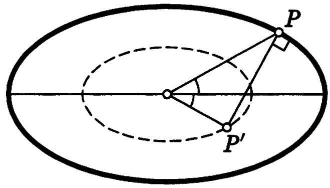

FIGURE 3.17

to the parabola. Since the intersection of those tangents lies on the directrix, the corresponding chord passes through its pole P'. □

Clearly, P' is the intersection of the diameter symmetric to the diameter passing through P and the normal to the conic at P (Figure 3.17).

It is not difficult to see that P' cuts the diameter in the ratio equal to that of the squares of the axes of the conic, and therefore when P moves along the conic, P' sweeps a conic which is homothetic to the original conic with respect to its center (if the original conic is a parabola, then the new conic is obtained from it by parallel translation (Figure 3.18)).

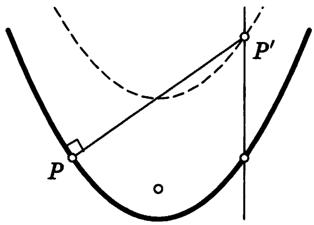

FIGURE 3.18

Similar arguments prove the following generalization of the Frégier theorem.

The chords of a conic seen from a fixed point P on it at an angle ϕ or $180° - \phi$ are tangent to some conic (Figure 3.19).

The duality with respect to a circle with center P transforms the desired envelope into a conic (a hyperbola) from which the parabola is seen at an angle ϕ or $180° - \phi$.

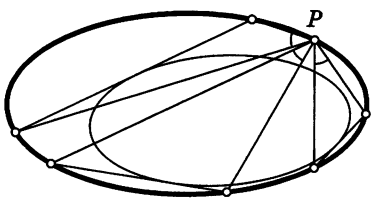

FIGURE 3.19

Problem 21. Let C be the center of a conic which is the polar of a circle α with respect to a circle ω. Prove that the polar of C with respect to ω coincides with the polar of the center of ω with respect to α.

Problem 22. 1. Prove that the directions conjugate with respect to an equilateral hyperbola are symmetric with respect to its asymptotes.

2. Prove that the angle between concentric equilateral hyperbolas is equal to twice the angle between their asymptotes.

Problem 23. What kind of curve is enveloped by the sides of the rhombi inscribed in a fixed ellipse?

3.3. Pencils of curves. Poncelet's theorem

Definition. Suppose we are given two conics with equations

(1) $$f(x,y)=0 \quad \text{and} \quad g(x,y)=0.$$

Then the *pencil of conics* is the set of curves with equations

(2) $$af(x,y)+bg(x,y)=0,$$

where a and b are arbitrary numbers.

Clearly, the pencil is determined by any two conics in it. Moreover, if the two conics defining the pencil intersect at a single point, then all the conics in the pencil pass through that point. If the two conics are tangent to each other, then all the conics in the pencil are tangent to each other at that point.

The pencils of circles defined in 2.4 are special cases of pencils of conics. Indeed, if the line containing the centers of the circles is viewed as the x-axis and the radial axis is viewed as the y-axis, then the equations of the circles become

$$x^2+y^2+ax+c=0,$$

where c is the power of the origin with respect to the circles of the pencil and a is an arbitrary number. Clearly, this is a special case of equation (2).

The Fundamental Theorem of Algebra implies that any two curves of orders m and n intersect at mn points (such points may be imaginary or may coincide). In particular, when $m=n=2$, we have that any two conics intersect at four points. Then any conic in the corresponding pencil will also pass through those points.

The converse is also true: for any conic passing through four intersection points of the conics with equations $f(x,y)=0$ and $g(x,y)=0$, there are numbers a and b such that the equation of the conic can be written as $af(x,y)+bg(x,y)=0$. This assertion is called the *theorem on pencils of conics*.

It is not difficult to show that a hyperbola is equilateral if and only if $a_{11}+a_{22}=0$ (see equation (1) in 1.2). Moreover, it is convenient to regard a degenerate curve consisting of two perpendicular lines as an equilateral hyperbola. It now follows from the theorem on pencils of conics that if two equilateral hyperbolas intersect at four points, then any conic in the pencil determined by those points will also be an equilateral hyperbola. Thus we have yet another proof of the fact that a conic circumscribed about a triangle is an equilateral hyperbola if and only if it passes through the orthocenter.

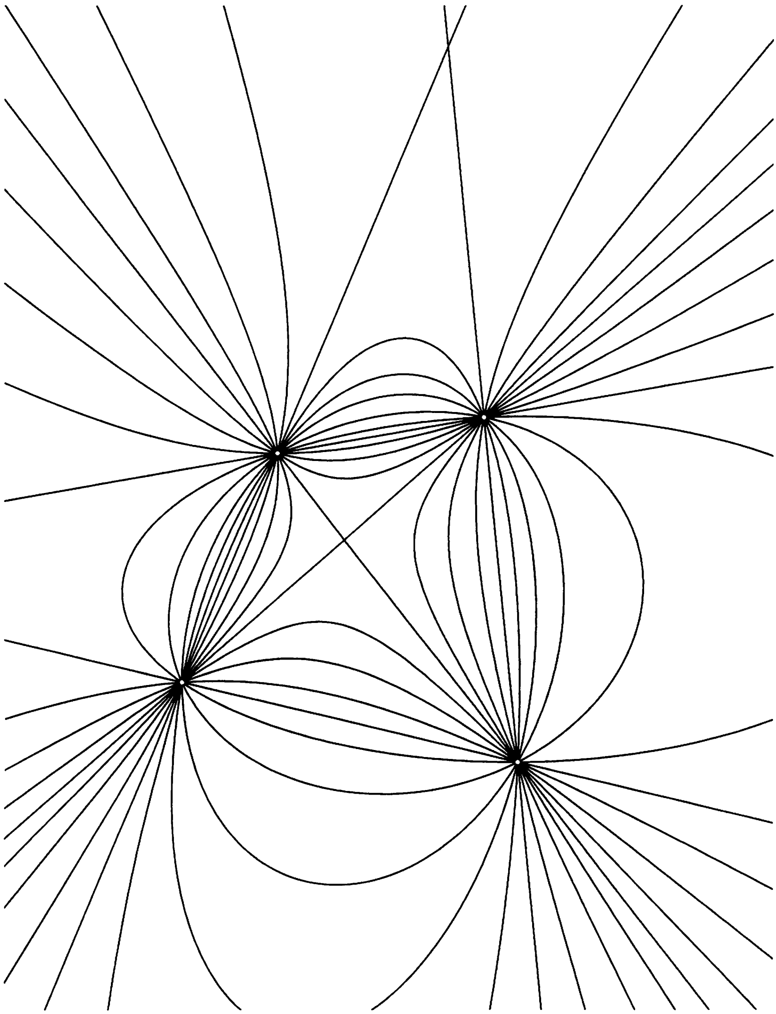

FIGURE 3.20

The theorem on pencils of conics allows one to define the pencil as the set of conics passing through four given points A, B, C, D in general position. Moreover, for any point X different from A, B, C, D, there is exactly one conic of the pencil passing through X.

Some of the points defining the pencil can be imaginary. For example, any circle intersects the line at infinity in two fixed complex points so that a hyperbolic pencil of circles is defined by those points and two common points on the circles, whereas an elliptic pencil is defined by four complex points two of which are finite and two are at infinity. It can also happen that some of the points defining the pencil coincide. If two points coincide, then all the conics of the pencil are tangent to each other at the double point as in a parabolic pencil of circles. If three of the four points coincide, then the tangency at that point is of order two. If all four points coincide, then the order of tangency is three. For example, the set of concentric circles is a pencil formed by two pairs of coinciding points.

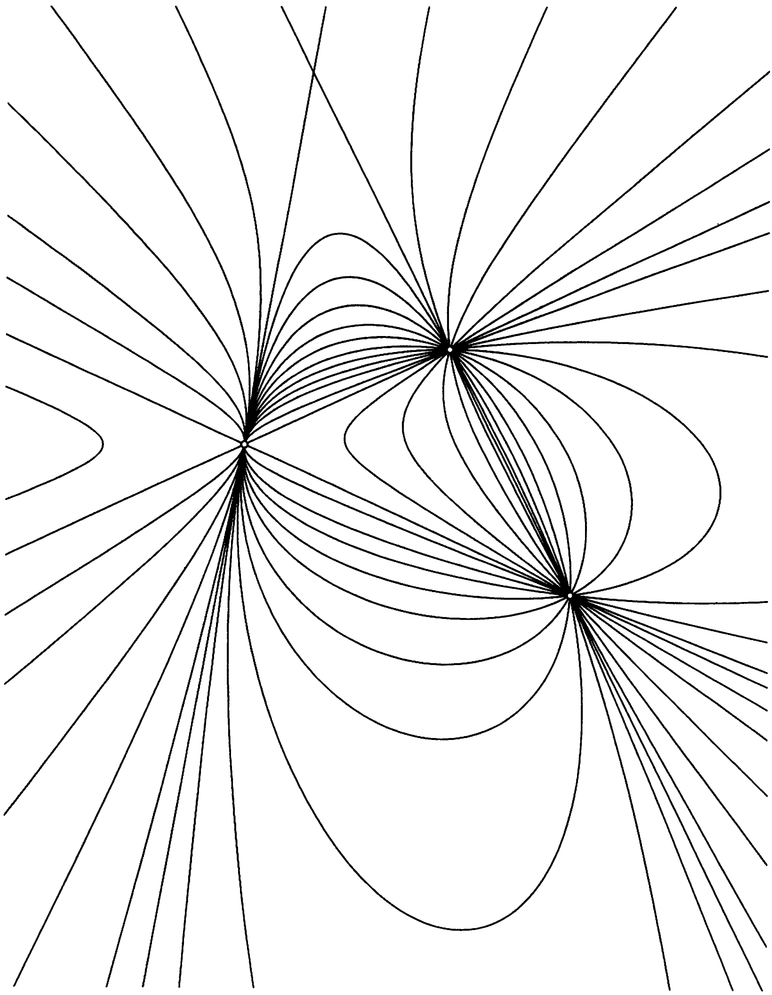

FIGURE 3.21

If all points defining the pencil are distinct, then there are three degenerate curves in that pencil: $AB \cup CD$, $AC \cup BD$ and $AD \cup BC$.

Now we describe the various types of pencils in more detail.

1. Pencils passing through four distinct points (Figure 3.20). Elliptic and hyperbolic pencils of circles belong to this type.

2. Pencils passing through four points two of which coincide, i.e., pencils tangent to a given line at a fixed point (Figure 3.21). Parabolic pencils of circles belong to this type.

3. A pencil defined by two coinciding pairs of points. It consists of conics tangent to two given lines at two given points (Figure 3.22). Pencils of concentric circles or parabolas with equation $y = ax^2$ belong to this type.

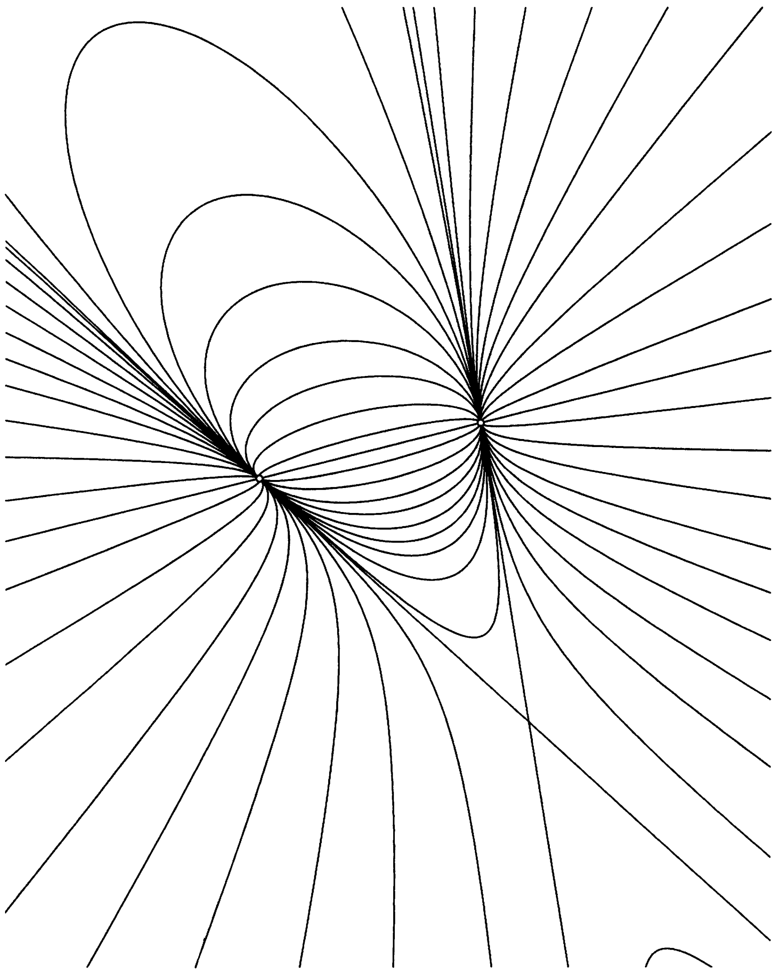

FIGURE 3.22

4. A pencil defined by four points three of which coincide (Figure 3.23). The conics of such a pencil are tangent to a circle.

5. A hyperosculating pencil whose four defining points coincide (Figure 3.24). An example of such a pencil is given by the parabolas with equation $y = x^2 + a$.

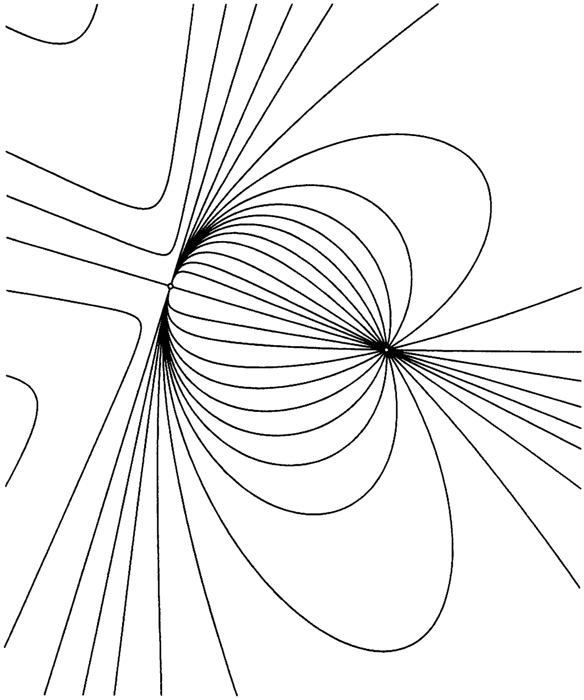

FIGURE 3.23

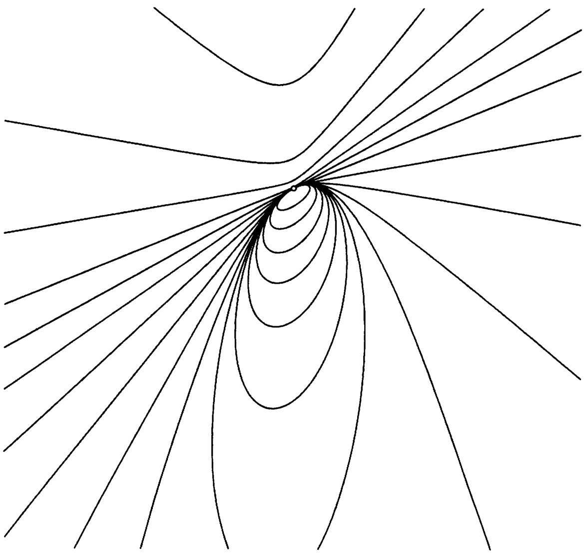

FIGURE 3.24

In addition to pencils defined by four points one can consider *dual pencils*, i.e., sets of conics tangent to four given lines. Dual pencils are then classified according to the number of coinciding defining lines. If two lines coincide, then all the conics of the pencil are tangent to them, and therefore to each other, at a point.

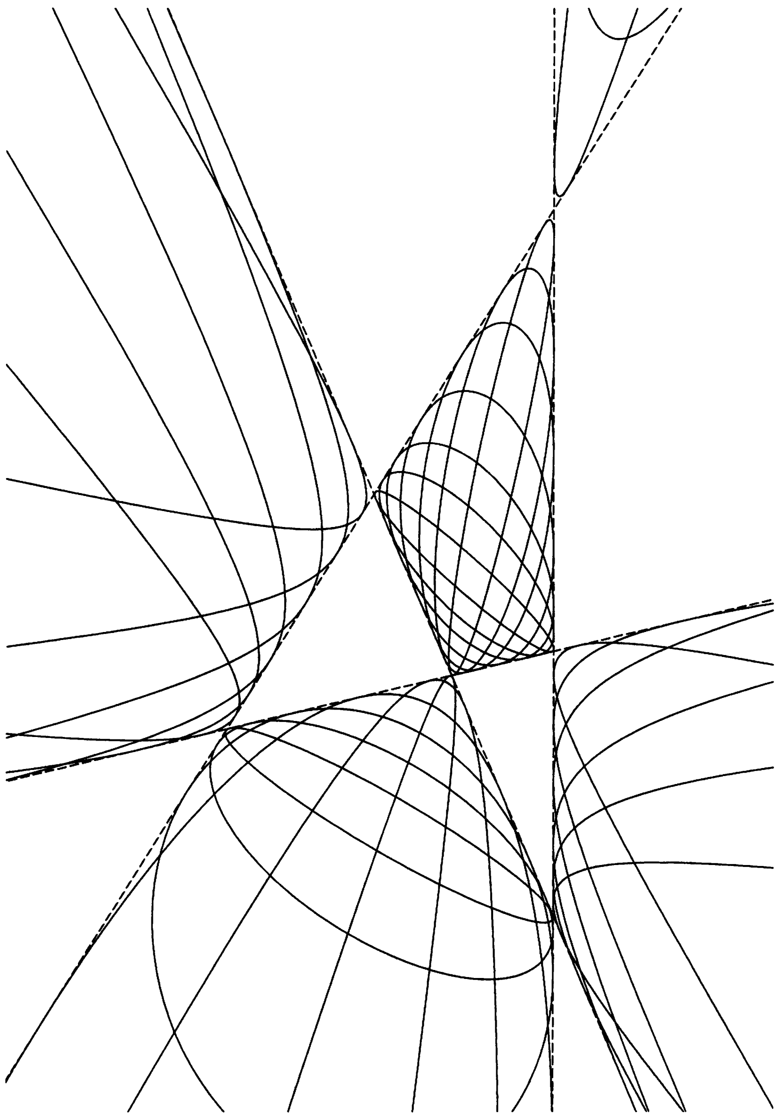

FIGURE 3.25

If three of the lines coincide, then the conics of the pencil osculate. If all four lines coincide, then the conics hyperosculate. Notice that double tangent and hyperosculating pencils are self-dual, i.e., are transformed into themselves under the polar correspondence with respect to any conic in the pencil. The duality principle allows one to state, for each assertion concerning usual pencils, the corresponding assertion concerning dual pencils, and vice versa.

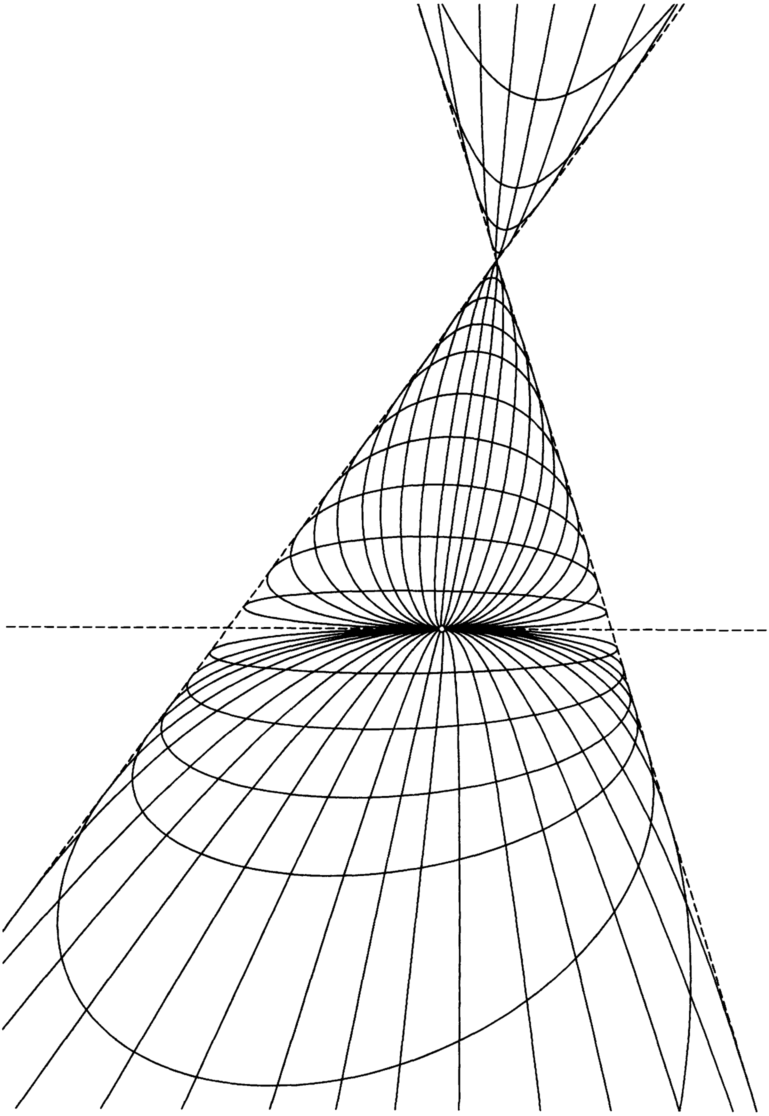

FIGURE 3.26

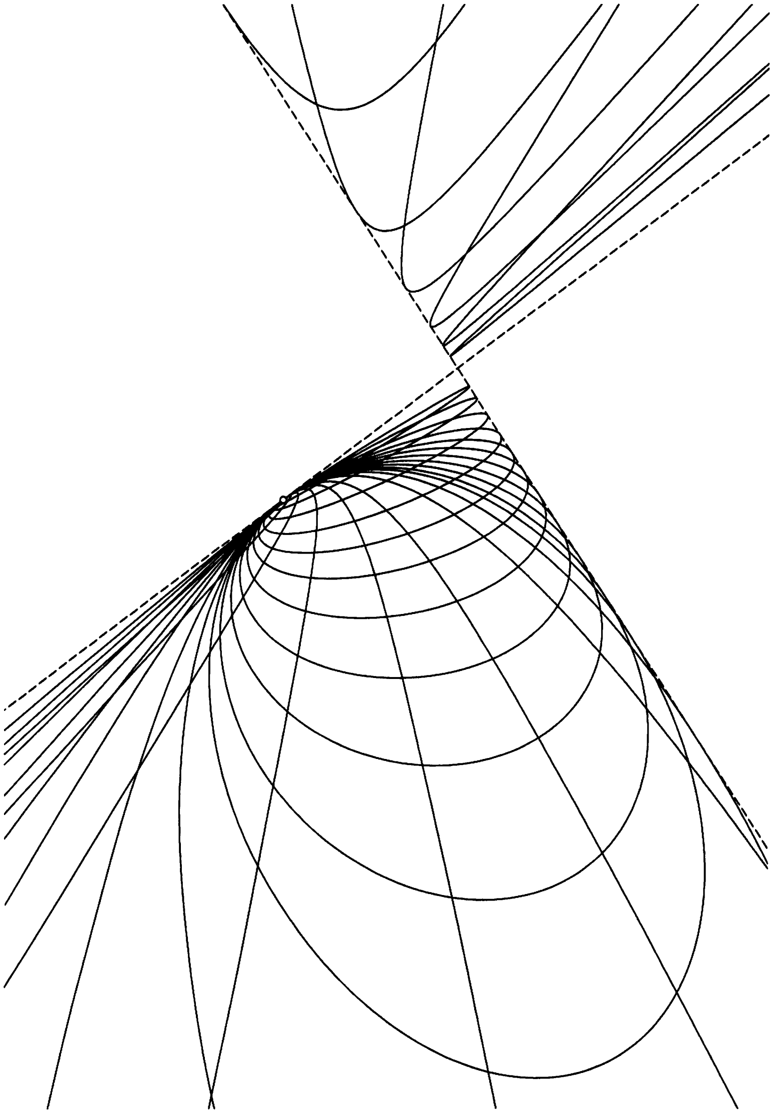

FIGURE 3.27

Using the theorem on pencils of conics one can prove the following nice result.

Theorem 3.11 (The four conics theorem). *Suppose we are given three conics α_1, α_2, α_3, and let P_1, Q_1, P_1', Q_1' be the intersections of α_2 and α_3; P_2, Q_2, P_2', Q_2' the intersections of α_1 and α_3; and P_3, Q_3, P_3', Q_3' the intersections of α_2 and α_1. If the points P_1, Q_1, P_2, Q_2, P_3, Q_3 lie on one of the conics, then the lines $P_1'Q_1'$, $P_2'Q_2'$, $P_3'Q_3'$ intersect at a single point* (Figure 3.28).

Proof. We first need the following auxiliary result.

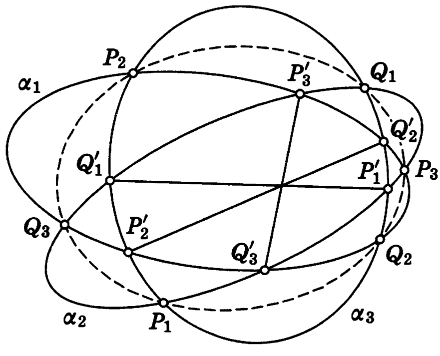

FIGURE 3.28

Theorem 3.12 (The three conics theorem). *Suppose three conics have two common points. Then their common chords passing through the remaining intersections of each pair meet at a single point* (Figure 3.29).

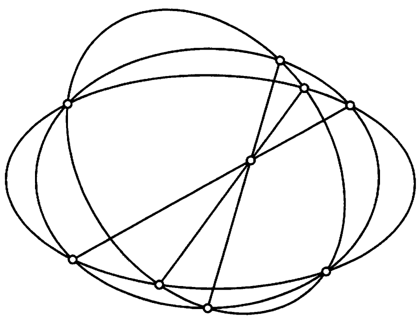

FIGURE 3.29

To prove this claim it suffices to transform the common points of the conics into the intersection points of the line at infinity and circles. Then all three conics will transform into circles and the desired assertion will follow from the existence of the radical center.

Now let α_0 be the conic passing through the points P_i, Q_i, and let $F_i(x,y) = 0$ be an equation of α_i. Since the degenerate conic consisting of the lines P_2P_3 and Q_2Q_3 belongs to the same pencil as α_0 and α_1, we may assume that its equation is of the form $F_0 = F_1$. Similarly, the conics consisting of the lines P_1Q_1, P_3Q_3 and P_1Q_1, P_2Q_2, will be given, respectively, by the equations $F_0 = F_2$ and $F_0 = F_3$. Therefore for each point on the line P_3Q_3 we have $F_1 = F_2$. Since this also holds for the points P_3' and Q_3', this is an equation of the degenerate conic consisting of the lines P_3Q_3 and $P_3'Q_3'$. Accordingly, the conics consisting of the lines P_2Q_2 and $P_2'Q_2'$, is given by the equation $F_1 = F_3$, and the conic consisting of the lines P_1Q_1 and $P_1'Q_1'$ is given by the equation $F_2 = F_3$. Thus these three conics belong to the same pencil. Three of the points determining this pencil can be found using the three conics theorem applied to the triples α_0, α_1, α_2; α_0, α_1, α_3; and α_0, α_2, α_3: they are the intersections of the triples of lines P_1Q_1, P_2Q_2, $P_3'Q_3'$; P_1Q_1, P_3Q_3, $P_2'Q_2'$; and P_2Q_2, P_3Q_3, $P_1'Q_1'$. Hence the fourth point belongs to all the lines $P_i'Q_i'$. Similar arguments prove the converse: if each of the four triples of lines P_1Q_1, P_2Q_2 and $P_3'Q_3'$; P_1Q_1, P_3Q_3 and $P_2'Q_2'$; P_2Q_2, P_3Q_3 and $P_1'Q_1'$; and $P_1'Q_1'$, $P_2'Q_2'$ and $P_3'Q_3'$ intersects at a single point, then the points P_1, Q_1, P_2, Q_2, P_3, Q_3 (as well as P_1, Q_1, P_2', Q_2', P_3', Q_3' and the two similar sextuples) lie on a conic. □

Using the duality principle we have the following results.

Theorem 3.13 (Dual to the three conics theorem). *Suppose three conics are tangent to two given lines. Then the intersections of the common tangents to each pair of conics which are different from the given lines lie on a straight line* (Figure 3.30).

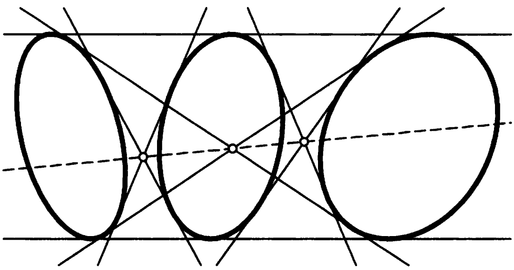

FIGURE 3.30

Theorem 3.14 (Dual to the four conics theorem). *If two of the common tangents to each pair of three given conics are tangent to the same conic, then the intersections of the other two tangents to each pair lie on a straight line* (Figure 3.31).

We now mention some important properties of pencils.

Theorem 3.15. *Let A, B, C, D be four distinct points, and X, Y, Z the intersections of the lines AB and CD, AC and BD, AD and BC. Let P be a point different from X, Y and Z. Then the polars of P with respect to all the conics of the pencil determined by the points A, B, C and D pass through a single point.*

Here is an interesting special case. If the points A, B, C and D form an orthocentric quadruple (i.e., each point is the orthocenter of the triangle formed by the remaining points), then the obtained point is isogonally conjugate to P with respect to the triangle XYZ.

Indeed, the polar of P with respect to the degenerate curve formed by the lines AB and CD is the line symmetric to XP with respect to AB. Since AB and CD are the bisectors of the angle YXZ, this line passes through

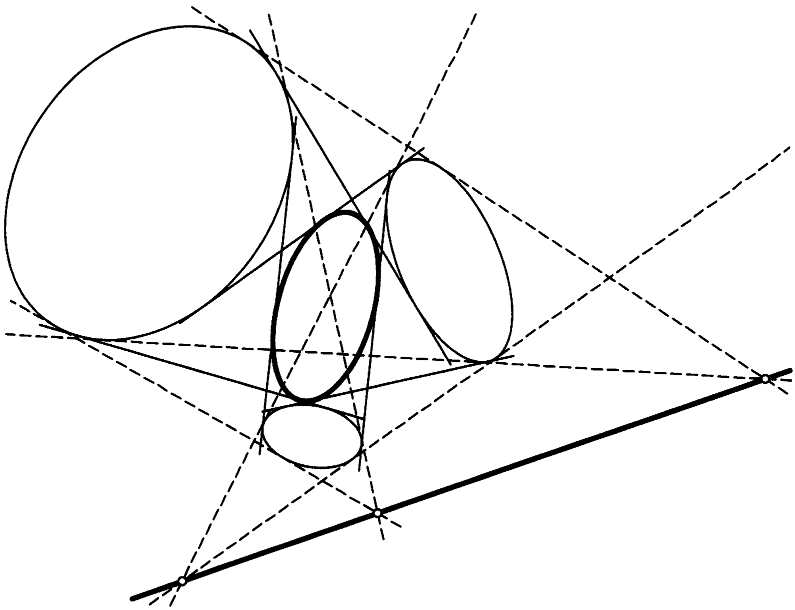

FIGURE 3.31

the isogonal conjugate P' of P. Similarly, P' lies on the polar of P with respect to another degenerate curve, and therefore with respect to any curve of the pencil.

In the same way, one can show that if one of the points A, B, C and D is the centroid of the triangle formed by the other three points, then the above transformation is the isotomic conjugation with respect to XYZ.

Summarizing the above discussion we conclude that *isotomic and isogonal conjugations are projectively equivalent.*

Instead of proving Theorem 3.15 we prove the dual theorem.

Theorem 3.16. *Suppose we are given four lines l_i, $i = 1, \ldots, 4$, and let X_{ij} be the intersection of the lines l_i and l_j. Then the locus of the poles of any line different from $X_{12}X_{34}$, $X_{13}X_{24}$, $X_{14}X_{23}$ with respect to the conics of the pencil given by the lines l_i is a straight line* (Figure 3.32).

Proof. Using a projective transformation, we move the line in question to infinity. It follows from the assumptions that in this case the lines l_i form a quadrilateral $ABCD$ which is not a parallelogram. We prove that the centers of the conics inscribed in it lie on the so-called *Gauss line* passing through the midpoints of the diagonals of the quadrilateral.

Notice that the Gauss line is the locus of points P such that $S_{PAB} + S_{PCD} = S_{PBC} + S_{PDA}$ (the area could be positive or negative depending on the orientation of the corresponding triangle). Indeed, the area of each of the four triangles is a linear function of the coordinates of P; hence the

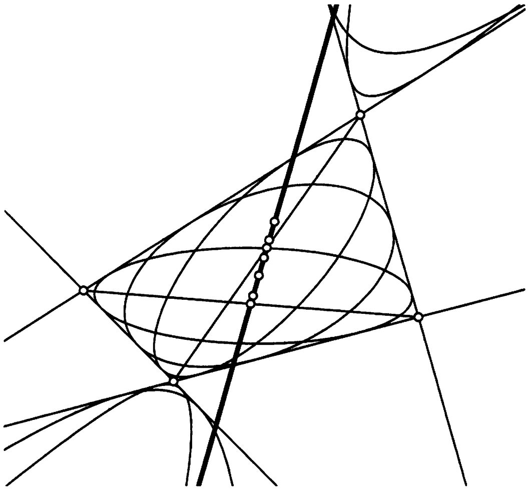

FIGURE 3.32

set of points satisfying the above condition is a straight line. Clearly, the midpoints of the diagonals belong to that line.

Suppose now that a conic with foci F_1 and F_2 is inscribed in the quadrilateral $ABCD$. Since its center is the midpoint of the segment F_1F_2, the assertion of the theorem is equivalent to the assertion that $S_{F_1AB}+S_{F_1CD}+S_{F_2AB}+S_{F_2CD}=S_{F_1BC}+S_{F_1DA}+S_{F_2BC}+S_{F_2DA}$.

Let F_1' be the reflection of F_1 in AB. Then

$$S_{F_1AB}+S_{F_2AB}=S_{F_1'AF_2B}=\frac{1}{2}AF_1'\cdot AF_2\sin\angle F_1'AF_2+BF_1'\cdot BF_2\sin\angle F_1'BF_2.$$

But the points F_1 and F_2 are isogonally conjugate with respect to $ABCD$, and therefore $\angle F_1'AF_2=\angle F_1AB+\angle F_2AB=\angle A$, $\angle F_1'BF_2=\angle B$ and

$$S_{F_1'AF_2B}=\frac{1}{2}(AF_1\cdot AF_2\sin\angle A+BF_1\cdot BF_2\sin\angle B).$$

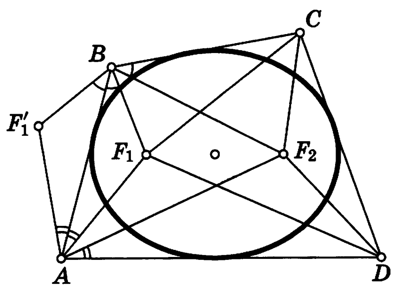

FIGURE 3.33

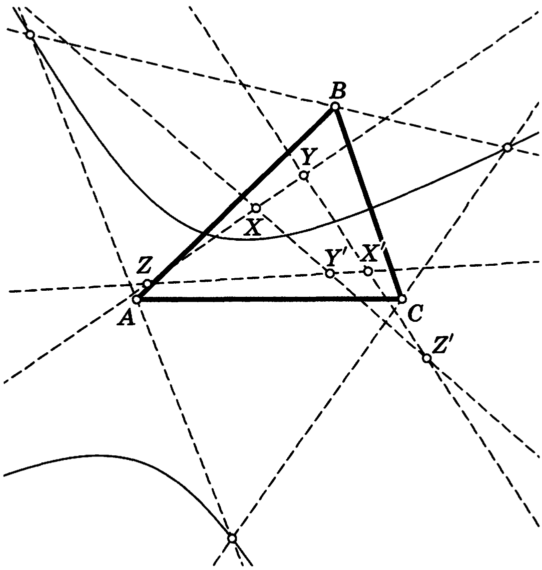

FIGURE 3.34

This, together with similar equalities, implies that the left- and the right-hand sides of the desired equality are equal to

$$\frac{1}{2}(AF_1 \cdot AF_2 \sin \angle A + BF_1 \cdot BF_2 \sin \angle B + CF_1 \cdot CF_2 \sin \angle C + DF_1 \cdot DF_2 \sin \angle D).$$

□

As a special case of Theorem 3.16 we have *Monge's theorem* asserting that if a circle is inscribed in a quadrilateral, then its center lies on the Gauss line.

Recall that the foci of such conics are isogonally conjugate in the polygon. The pedal circles of those foci with respect to the polygon exist and coincide. Their centers are the midpoints of the segments connecting the foci. Hence the centers of all such circles lie on the Gauss line.

Theorem 3.15 has the following nice consequence.

Corollary. *Suppose we are given a triangle ABC and two pairs of isogonally (isotomically) conjugate points X, X' and Y, Y'. Then the intersections of XY with $X'Y'$ and of XY' with $X'Y$ are also isogonally (isotomically) conjugate* (Figure 3.34).

Proof. Consider the pencil of conics giving rise to the conjugation in question. In that pencil, choose a conic with respect to which the polar of X coincides with the line $X'Y'$. Then the polar of Y' passes through X, i.e., it coincides with XY, and the pole of the line XY' is the intersection point of the lines XY and $X'Y'$. Therefore the conjugate of that point lies on XY'. Similarly, it lies on $X'Y$. □

Theorem 3.17. *The poles of a fixed line with respect to all the conics of the pencil defined by points A, B, C and D form a conic.*

Proof. Transform the given line into the line at infinity. Then its poles will be the centers of the conics of the pencil. It follows from the converse to Pascal's theorem that the midpoints K, L, M and N of the sides of the quadrilateral $ABCD$ belong to the set of the centers. Hence it suffices to show that for the center O of any conic of the pencil, the cross-ratio of the lines OK, OL, OM and ON is the same. This cross-ratio equals the ratio of the poles of those lines, which are the intersection points of the line at infinity with the sides of the quadrilateral $ABCD$ and are independent of the choice of the conic. □

Notice that the center of a degenerate curve of degree two is the intersection of the lines making that curve. Therefore the conic mentioned in Theorem 3.17 always passes through the intersections of the lines AB and CD, AC and BD, AD and BC.

Similar to Theorem 3.15, Theorem 3.17 also has an important special case.

Corollary. *Suppose we are given a triangle ABC and a line l not passing through its vertices. Then the isogonal (isotomic) conjugate of l is a conic passing through A, B and C.*

This corollary yields another proof of the assertion of Problem 19. The isogonal conjugates of the conics passing through the vertices of the triangle are lines. The infinite points of those conics transform into points on the circumcircle, and the points in the perpendicular direction transform into antipodal points (as is easy to check). Hence those lines pass through the center of the circumcircle of the triangle. The isogonal conjugate of that center is the orthocenter of the triangle; i.e., the conic must contain the orthocenter of the triangle. The converse is proved similarly.

Theorem 3.15 has another nice proof under the assumption that the pencil in question consists of circles.

Notice that the radical axes of P and the circles of the pencil $\mathcal{W}$ intersect at a single point, which we denote Q. This is clear, since on the radical axis of $\mathcal{W}$, the powers of points with respect to all circles are equal. Therefore the desired P is a point on the radical axis whose power with respect to some circle in the pencil equals the square of the distance to P. The polar of P with respect to any circle is parallel to the radical axis of P and the circle and is twice as far from P (Figure 3.35). Thus all the polars of P with respect to the circles of $\mathcal{W}$ pass through the reflection of P in Q.

Problem 24. Prove that an equilateral hyperbola is self-dual with respect to the circle that is tangent to the hyperbola at its vertices.

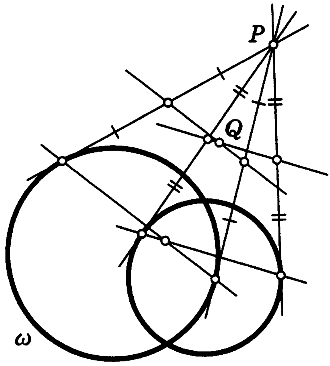

FIGURE 3.35

Problem 25. Inside a convex quadrilateral, a point T is given which is equidistant from the opposite sides. Prove that T lies on the line connecting the midpoints of the diagonals of the quadrilateral if and only if the quadrilateral is either inscribed or circumscribed or a trapezoid.

Problem 26. Inside an angle with vertex O, points A and B are given. A billiard ball can move from A to B by reflecting either in one side at point X or in the other side at point Y. Let C and Z be the midpoints of the segments AB and XY.

1. Prove that if the angle O is right, then C, Z and O lie on a straight line.
2. Prove that if the angle O is different from 90°, then the line CZ passes through O if and only if the polygonal lines AXB and AYB are of the same length.

Problem 27. Two points of a conic lie on two circles one of which intersects the conic at points X_1 and Y_1, and the other at points X_2 and Y_2. Prove that the lines X_1Y_1 and X_2Y_2 are parallel.

Problem 28. Prove that for any quadrilateral, the midpoints of its sides and of the diagonals, as well as the intersections of the diagonals and of the opposite sides, lie on a conic. What kind of conic is it if the vertices of the quadrilateral form an orthocentric quadruple?

Problem 29. Prove that the centers of the conics circumscribed about a quadrilateral $ABCD$ form an equilateral hyperbola if and only if the quadrilateral $ABCD$ is inscribed.

Problem 30. Suppose we are given three circles, each lying outside the other two. Prove that the common inner tangents to each pair of circles form a hexagon whose main diagonals intersect at a single point.

Poncelet's theorem.

Consider now the pencil determined by points A, B, C, D and a line l not passing through these points. If one of the conics of the pencil intersects l at P, then it intersects l at another point P' (which may coincide with P). The transformation $P \to P'$ will be called the *involution* of l defined by the pencil. Applying Pascal's theorem to the hexagon $ABCDPP'$, we obtain a method of constructing P', shown in Figure 3.36.

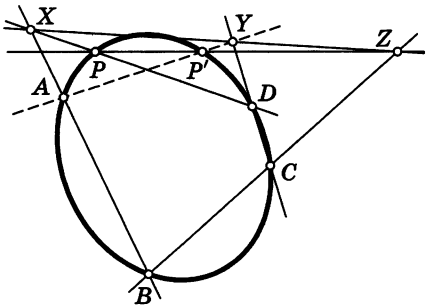

FIGURE 3.36

Since this construction can be represented as the composition of central projections, the involution preserves cross-ratios. In particular, this implies that l is tangent to at most two conics of the pencil. Moreover, the involution is uniquely determined by two pairs of the corresponding points.

Notice also that if not all four points defining the pencil are real, then the property of involution can be established without using the complex plane; all we need is a projective transformation making the pencil into circles. Let P be the intersection of the line and the radical axis of the pencil. Then the involution of the line will be just the inversion with center P.

With the aid of involution, we prove Poncelet's theorem in the general case, i.e., for a pencil of conics.

Theorem 3.18 (Poncelet). *Suppose conics $\alpha_0, \alpha_1, \dots, \alpha_n$ belong to a pencil $\mathcal{F}$. From an arbitrary point A_0 on α_0 draw a tangent to α_1 and find its second intersection point A_1 with α_0. From A_1 draw a tangent to α_2 and find its second intersection point A_2 with α_0, etc. If for some point A_0 the point A_n coincides with A_0, then the same is true for any other point of the conic α_0.*

Proof. We use induction on n. First we establish a result which is of independent interest.

Lemma 3.8. 1. *Suppose points A, B and C lie on a conic α_0, the line AB is tangent to the conic α_1 at K, and the line AC is tangent to the conic α_2 at L. Then there is a point D on α_0 such that α_1 is tangent to the line CD and α_2 is tangent to the line BD at the points of their intersections with*

KL. Moreover, there is a conic of the pencil $\mathcal{F}$ tangent to the lines AD and BC at their intersections with KL.

2. *Suppose that points A, B, C and D lie on a conic α_0 and that the conic α_1 is tangent to AB at a point K and to CD at a point M. Then there is a conic of the pencil $\mathcal{F}$ tangent to the lines AC and BD at their intersection points with KM* (Figure 3.37).

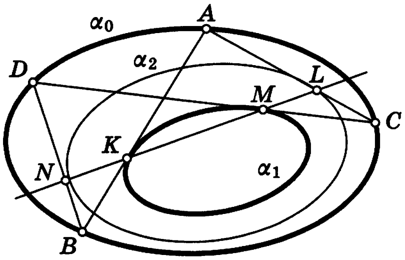

FIGURE 3.37

Proof. 1. Let M be the second intersection point of α_1 and KL, and let $\mathcal{G}$ be the pencil containing α_1 and the degenerate curve consisting of the lines CM and AB. On the line AC, the pencils $\mathcal{F}$ and $\mathcal{G}$ give rise to the same involution defined by the points A and C and the intersections of AC with α_1 (which are not necessarily real). Therefore the point L is a double involution point defined by the pencil $\mathcal{G}$, i.e., $\mathcal{G}$ contains the double line KL. Therefore all conics of $\mathcal{G}$, including α_1, are tangent to CM at M. Let D be the second intersection point of CM and α_0. Applying the foregoing argument to the points B, C and D we have that α_2 is tangent to the line BD at its intersection point with KL. Next we determine the intersection point of the lines AD and KL and take the conic of $\mathcal{F}$ passing through it. The same argument shows that that conic is tangent to the line AD.

The fact that it is also tangent to BC and assertion 2 of the theorem are proved similarly. □

Corollary. *Suppose the line AB is tangent to the conic α_1 at X and the line AC is tangent to the conic α_2 at Y. Then there are exactly two conics of the pencil $\mathcal{F}$ tangent to BC at points Z_1 and Z_2; moreover X, Y and Z_1 lie on a straight line and the lines AZ_2, BY and CX intersect at a single point.*

Now we can prove Poncelet's theorem for $n = 3$. Suppose the lines A_0A_1, A_1A_2, A_2A_0 are tangent to the conics α_1, α_2, α_3 at points X_1, X_2, X_3 that do not lie on a straight line, and the line B_0B_1 is tangent to the conic α_1 at a point Y_1. By Lemma 3.8, there is a conic α' in $\mathcal{F}$ tangent to the lines A_0B_0 and A_1B_1 at their intersection points Z_0 and Z_1 with X_1Y_1, and a point B_2 on α_0 such that α_2 is tangent to B_1B_2 and α' is tangent to B_2A_2 at their intersection points Y_2 and Z_2 with Z_1X_2. Moreover, there is a conic α'' tangent to the lines A_2A_0 and B_2B_0 at their intersection points K and Y_3 with Z_2Z_0. Applying Desargues' theorem to the triangles $A_0A_1A_2$ and $Z_0Z_1Z_2$, we see that the points X_1, X_2, K are not on a straight line. Therefore $K = X_3$ and $\alpha'' = \alpha_3$.

Now suppose that n is arbitrary. Use the points A_0, A_1 and A_2 to construct a conic Q' tangent to the lines A_0A_2 and B_0B_2. Since the sides of the polygon $A_0A_2 \dots A_{n-1}$ are tangent to the conics $\alpha', \alpha_3, \dots, \alpha_n$, we can make the induction step from $n-1$ to n. □

Conics with the common focus and directrix.

Consider a family $\mathcal{Q}$ of conics with a fixed focus F and the corresponding directrix l.

By Theorem 3.15, the polar transformations with respect to those conics act the same way on the lines passing through F. More precisely, a line a transforms into the intersection of the perpendicular to a at F with the line l.

Using a projective transformation, make one of our conics into a circle such that F will transform into its center F'. Then the directrix will transform into the polar of the center, i.e., the line at infinity. What happens with the other conics under such a transformation?

They will transform into the pencil of circles with center F'!

Indeed, they will transform into conics such that the polar of the line at infinity is F'. But for any conic, the pole of the line at infinity is its center, and therefore the center F' is the center of all such conics. Moreover, the pole of any line passing through F' must be the point on the line at infinity in the direction perpendicular to that line (because this is true for a circle, and the dual transformation on the lines passing through F' is the same for all conics). Obviously, this is possible only if all the conics are circles.

Notice that concentric circles transform into each other under the dual polar transformation with respect to one of those circles. This property is also preserved under the dual transformation! Thus we have proved the following result.

Theorem 3.19. *The polar transformation with respect to one of the conics of the family $\mathcal{Q}$ (which is a family with fixed focus and directrix) leaves the family $\mathcal{Q}$ unchanged; i.e., the conics of this family transform into conics of the same family.*

The fact that this pencil is projectively equivalent to a pencil of concentric circles allows one to describe the action of this transformation on the conics.

Theorem 3.20. *Let R be the polar transformation with respect to some conic of $\mathcal{Q}$. Suppose a point X on the conic α (from $\mathcal{Q}$) transforms into the line $R(X)$ tangent to $R(\alpha)$ at Y. Then the points X, Y and F lie on a straight line* (Figure 3.38).

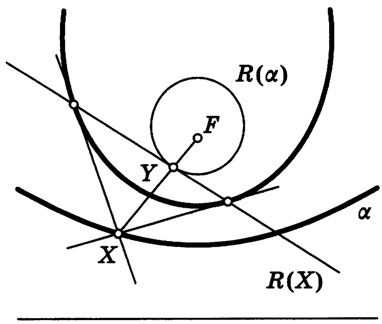

FIGURE 3.38

Under such a transformation, the eccentricity changes as follows.

Theorem 3.21. *Suppose conics α_1 and α_2 are dual with respect to a conic α (assuming that all of them are from $\mathcal{Q}$). Then $\epsilon_1\epsilon_2 = \epsilon^2$, where ϵ, ϵ_1 and ϵ_2 are the eccentricities of the conics α, α_1 and α_2, respectively.*

Proof. Let F_l be the projection of F to the line l. Let X, Y and Z be the points where the segment FF_l intersects the conics α, α_1 and α_2, respectively. We need to show that

$$\frac{FY}{F_lY}\cdot\frac{FZ}{F_lZ}=\frac{FX^2}{F_lX^2}.$$

After the right-hand side is divided by the left-hand side and the expression obtained is rearranged, we see that the equality

$$\frac{FY\cdot F_lX}{FX\cdot F_lY}\cdot\frac{FZ\cdot F_lX}{FX\cdot F_lZ}=1$$

is to be proved. Note that the left-hand side equals $(XY;F_lF)\cdot(XZ;F_lF)$, and therefore it does not change under projective transformations. It remains to show that this equality holds when the conics are concentric circles.

Thus we perform a projective transformation making α, α_1 and α_2 into three concentric circles α', α_1' and α_2' with center F' (which is the image of F). The points X, Y, Z and F_l transform into points X', Y', Z' and F_l' lying on a straight line (which also contains F'), and F_l' transforms into a point at infinity.

By the definition of the polar transformation with respect to a circle, $F'Y'\cdot F'Z' = F'X'^2$. Hence

$$\frac{F'Y'\cdot F_l'X'}{F'X'\cdot F_l'Y'}\cdot\frac{F'Z'\cdot F_l'X'}{F'X'\cdot F_l'Z'}=\frac{F'Y'\cdot\infty}{F'X'\cdot\infty}\cdot\frac{F'Z'\cdot\infty}{F'X'\cdot\infty}$$
$$=\frac{F'Y'}{F'X'}\cdot\frac{F'Z'}{F'X'}=1. \qquad \square$$

Notice also that the conics mentioned in the generalized Frégier theorem can be obtained from $\mathcal{Q}$ by the dual transformation. Therefore they form a pencil of the third kind and, using a projective transformation, we can make them into concentric circles.

Problem 31. Prove that the foci different from F of two conics from the pencil $\mathcal{Q}$ which are dual with respect to a parabola are symmetric with respect to l.

Chapter 4

Euclidean Properties of Curves of Second Degree

4.1. Special properties of equilateral hyperbolas

Recall that a hyperbola is said to be equilateral if its asymptotes are perpendicular. In the previous chapter we gave several proofs that the conic circumscribed about a triangle is an equilateral hyperbola if and only if it passes through the orthocenter of the triangle. In this section we establish some other interesting properties.

Theorem 4.1. *The centers of all equilateral hyperbolas passing through the vertices of a triangle ABC lie on the Euler circle of the triangle.*

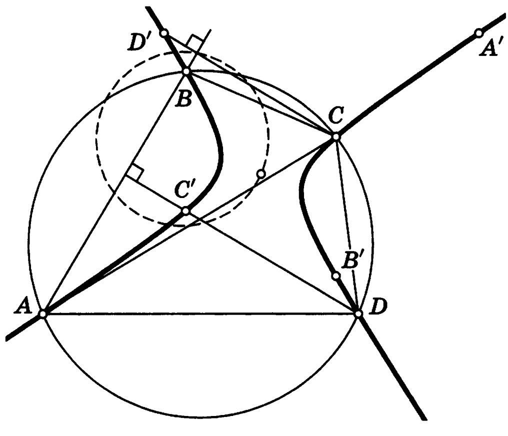

FIGURE 4.1

Proof. Let D be the fourth (besides A, B and C) intersection point of the hyperbola and the circumcircle of the triangle ABC, and A', B', C' and D' the orthocenters of the triangles BCD, CDA, DAB and ABC, respectively (Figure 4.1).

Since $CD' = 2R\,|\cos\angle BCA| = 2R\,|\cos\angle BDA| = DC'$, we have that $CDC'D'$ is a parallelogram, i.e., $C'D' \parallel CD$ and $C'D' = CD$.

Therefore the quadrilaterals $ABCD$ and $A'B'C'D'$ are centrally symmetric. The center of symmetry is the center of the hyperbola on which, by the main property of equilateral hyperbolas, all the eight points lie. Moreover, it coincides with the midpoint of the segment DD' and therefore lies on the Euler circle of the triangle ABC (as well as on the Euler circles of the triangles BCD, CDA and DAB). □

Notice that Theorem 4.1 implies the following fact. If lines a and b rotate about points A and B, respectively, with velocities that are equal in the absolute value but have different directions, then their intersection point sweeps an equilateral hyperbola and the points A and B are symmetric with respect to the center of the hyperbola. Indeed, if A and B are symmetric with respect to the center and X and Y are arbitrary points of the hyperbola, then its center lies on the Euler circle of the triangle AXY. Let M be the midpoint of the segment XY. Then $\angle A = \angle M = \angle O = \angle B$ (Figure 4.2). The latter is true because the homothety with center A and coefficient 2 transforms the Euler circle of the triangle AXY into the circumcircle of the triangle BXY.

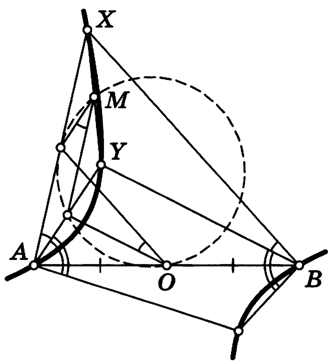

FIGURE 4.2

Now consider the triangle ABC and the point P. The circles symmetric to the circumcircles ABP, BCP, CAP with respect to AB, BC, CA, intersect at a single point. This will be the point P' symmetric to P with respect to the center of the hyperbola $ABCP$. Indeed, the previous assertion implies that the circles ABP and ABP' have equal radii; i.e., they are symmetric with respect to AB.

Theorem 4.2. *Suppose we are given a triangle ABC and a point P different from its orthocenter. Then the centers of the incircle and the excircles of the Ceva triangle of P with respect to the triangle ABC lie on an equilateral hyperbola passing through A, B, C and P.*

Proof. This property is a special case of the following fact.

Lemma 4.9. *Suppose we are given two triangles $A_1B_1C_1$ and $A_2B_2C_2$, and let A', B' and C' be the intersections of B_1C_1 and B_2C_2, C_1A_1 and C_2A_2, A_1B_1 and A_2B_2, respectively. If the triangle $A'B'C'$ is in perspective with the triangle $A_1B_1C_1$ as well as with $A_2B_2C_2$ (from the centers of perspective D_1 and D_2), then the points $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2$ lie on a conic* (Figure 4.3).

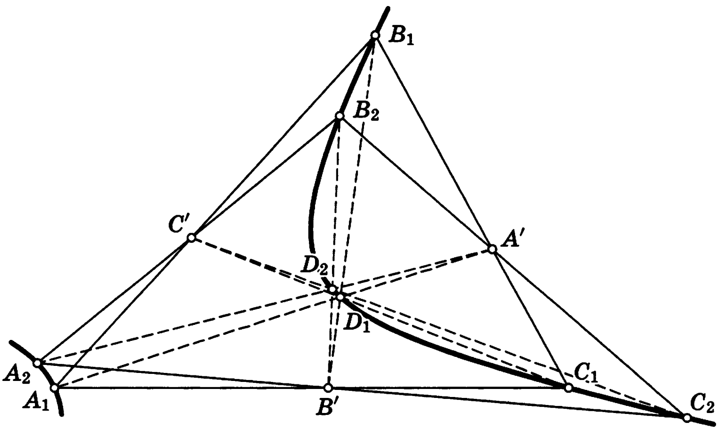

FIGURE 4.3

Proof. Using an appropriate projective transformation, we can make the quadrilateral $A_1B_1C_1D_1$ a square. Since the points A' and C' will go to infinity in perpendicular directions, the quadrilateral $A_2B_2C_2D_2$ will become a rectangle whose sides are parallel to the sides of the square. Moreover, the image of the point B' will be the center of both the square and the rectangle. Clearly, the conic passing through the vertices of the square and one of the vertices of the rectangle also passes through the other three vertices. □

Suppose now that $A'B'C'$ is the Ceva triangle of the point P, I' is the center of the circle inscribed in it, and I'_a, I'_b, I'_c are the centers of the excircles. Then the triangles ABC and $I'_aI'_bI'_c$ satisfy the hypotheses of the lemma. Therefore the points A, B, C, P, I'_a, I'_b, I'_c, I' lie on a conic. Since I is the orthocenter of the triangle $I'_aI'_bI'_c$, this conic is an equilateral hyperbola. □

Theorem 4.3. *Suppose points A, B, C, D lie on an equilateral hyperbola. Then the Ceva circle of D with respect to the triangle ABC passes through the center of the hyperbola* (Figure 4.4).

Proof. By Theorem 4.2, the centers I'_a, I'_b, I'_c of the excircles of the Ceva triangle lie on the hyperbola. Since the Ceva circle is the nine-point circle of the triangle $I'_aI'_bI'_c$, it passes through the center of the hyperbola. □

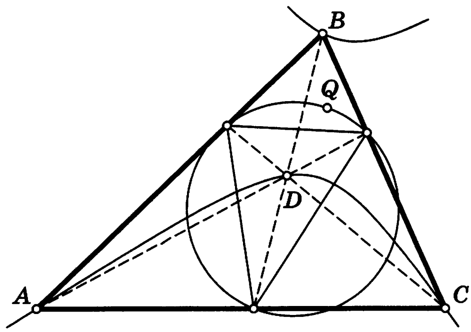

FIGURE 4.4

Theorem 4.4. *Suppose points A, B, C and D lie on an equilateral hyperbola. Then the pedal circle of D with respect to the triangle ABC passes through the center of the hyperbola.*

Proof. Let $A'B'C'$ be the pedal triangle of D, and B_1, C_1 the midpoints of the segments BD and CD (Figure 4.5).

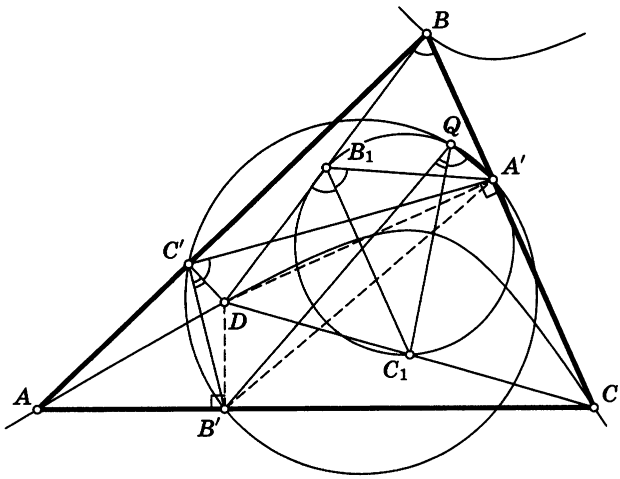

FIGURE 4.5

To prove that the circumcircle of the triangle $A'B'C'$ also passes through the center Q of the hyperbola, it suffices to show that the angles $A'C'B'$ and $A'QB'$ are equal.

Notice that $\angle DC'A' = \angle DBA'$ since the quadrilateral $C'BA'D$ is inscribed. The segment B_1C_1 is a midline of the triangle DBC and therefore $\angle DB_1C_1 = \angle DBA'$. Since the points D and A' are symmetric with respect to B_1C_1, the angles DB_1C_1 and $A'B_1C_1$ are equal. The point Q, as the center of an equilateral hyperbola, lies on the Euler circle of the triangle BCD. Therefore $\angle A'B_1C_1 = \angle A'QC_1$. Thus

$$\angle DC'A' = \angle DBA' = \angle DB_1C_1 = \angle A'B_1C_1 = \angle A'QC_1.$$

Similarly $\angle DC'B' = B'QC_1$. Therefore

$$\angle A'C'B' = \angle A'C'D + \angle DC'B' = \angle A'QC_1 + \angle C_1QB' = \angle A'QB'. \quad \square$$

Theorem 4.5 (Emelyanov and Emelyanova). *Let A_1, B_1 and C_1 be the feet of the bisectors of a triangle ABC, A_2, B_2, C_2 the feet of its heights, C^*, B^*, A^* the intersections of the lines A_1B_1 and A_2B_2, C_1A_1 and C_2A_2, B_1C_1 and B_2C_2 (henceforth such points will be called the poles), and A' and B' the intersections of an arbitrary line passing through C^* with BC and AC, respectively. Then:*

1. *The lines $A'B^*$, $B'A^*$ and AB intersect at a single point (call it C').*
2. *The lines AA', BB' and CC' intersect at a single point*
3. *The circumcircle of the triangle $A'B'C'$ passes through the Feuerbach point of the triangle ABC* (Figure 4.6).

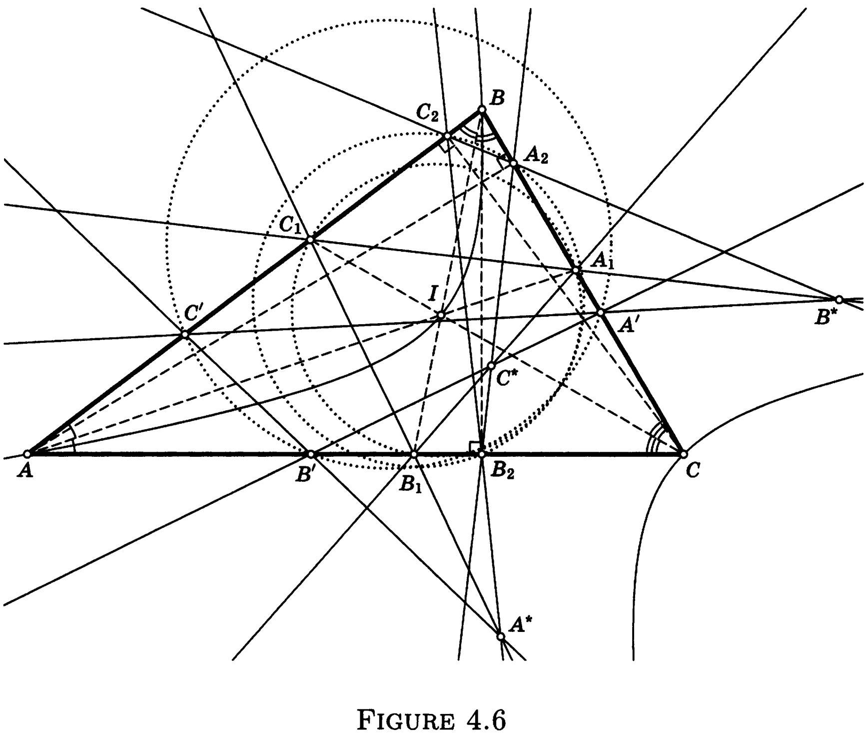

FIGURE 4.6

Proof. Parts 1 and 2 follow from Theorem 3.9. We now prove part 3. Since the set of the centers of perspective of the triangles from the Feuerbach family lies on an equilateral hyperbola (it is called the *Feuerbach hyperbola*), their Ceva circles pass through the center of the hyperbola. Their pedal circles also pass through that center. But the pedal circles of the points I and H have a single common point—the Feuerbach point. Therefore it is the center of the hyperbola. $\square$

This theorem admits a generalization: If a family of Ceva triangles satisfying the conditions of parts 1 and 2 of Theorem 3.9 contains the orthotriangle, then their Ceva circles have a common point.

Since the center of the hyperbola lies both on the Ceva and the pedal circles of its points and the pedal circles of any two isogonally conjugate points coincide, the pedal circle of the points isogonally conjugate to the points of the Feuerbach hyperbola pass through the Feuerbach point. But the isogonal transform of a conic passing through the vertices of a triangle is a line. In our case, this line passes through the center O of the circumcircle of the triangle ABC, which is isogonally conjugate to the orthocenter, and through the self-conjugate point I. Thus we have proved:

If a point lies on the line OI, then its pedal circle passes through the Feuerbach point.

This result also admits a generalization: Suppose l is a line passing through the point O. Then the pedal circles of all points of l have a common point.

Consider again the line l passing through O. For each of its points P, define another point P' such that the points isogonally conjugate to P and P' are symmetric with respect to the center of the equilateral hyperbola passing through those points and the vertices of the triangle. The properties of the isogonal conjugation established in 3.3 show that the transformation $P \to P'$ preserves cross-ratios. Since that transformation fixes the intersection points of the line l with the circumcircle and interchanges O and the point at infinity, it coincides with the transformation generated on l by the inversion with respect to the circumcircle. Thus we have proved the following.

Theorem 4.6. *Two points are inverses of each other with respect to the circumcircle of a given triangle if and only if their isogonal conjugates are symmetric with respect to the center of the corresponding equilateral hyperbola.*

In conclusion, we mention two more interesting facts. Clearly, the line OI contains the centers of homothety of the incircle and the circumcircle of the triangle. The Gergonne and Nagel points isogonally conjugate to them lie on the Feuerbach hyperbola, i.e., we have the following result.

Let AA_1, BB_1, CC_1 be the bisectors of a triangle ABC, AA_2, BB_2 and CC_2 its heights, A_3, B_3 and C_3 the tangency points of the sides BC, CA, AB and the incircle, and A_4, B_4 and C_4 the tangency points of the sides and the corresponding excircles. Then the lines A_1B_1, A_2B_2, A_3B_3, A_4B_4 intersect at a single point.

Now consider the line passing through O and the Lemoine point L. That line contains two Apollonius points whose pedal triangles are equilateral. The isogonal transform of that line is the *Kiepert hyperbola* passing through the centroid M and the Torricelli points T_1 and T_2. Moreover, T_1 and T_2, obviously, have the following property: the circles symmetric to the circles

ABT_1, BCT_1 and CAT_1 pass through T_2. Therefore the midpoint of the segment T_1T_2 is the center of the Kiepert hyperbola and therefore lies on the nine-point circle.

Moreover, since the lines T_1T_1' and T_2T_2' are parallel to the Euler line and the lines T_1T_2 and $T_1'T_2'$ pass through L, we conclude that: a) the lines T_1T_2' and $T_1'T_2$ pass through M, and b) the centers of the two equilateral pedal triangles lie on the line LM.

Problem 32. Given a quadrilateral $ABCD$, find the locus of points P such that the radii of the circumcircles of the triangles ABP, BCP, CDP and DAP are equal.

Problem 33. Let P be an arbitrary point on an equilateral hyperbola. Let Q be the point symmetric to P with respect to the center of the hyperbola. The circle with center P and radius PQ intersects the hyperbola at three more points A, B and C. Prove that the triangle ABC is equilateral.

Problem 34. Let P be the center of an equilateral hyperbola passing through the vertices of an inscribed quadrilateral $ABCD$. Prove that P lies on the line connecting the center of the circumcircle and the centroid of the quadrilateral $ABCD$.

Problem 35. Prove that points A, B, C, A', B', C' lie on a conic if and only if there is a conic with respect to which both triangles ABC and $A'B'C'$ are self-polar.

Problem 36. A triangle ABC is self-polar with respect to a conic with center O. Prove that this conic is homothetic to the conic passing through the midpoints of the segments AB, BC, CA, OA, OB, OC.

4.2. Inscribed conics

Consider a conic inscribed in a triangle ABC. Let A', B', C' be its tangency points on the sides BC, CA, AB. Using a projective transformation that makes the conic into a circle, we have that the lines AA', BB', CC' intersect at a single point. That point is called the *perspector* of the conic. Since there is a unique projective transformation fixing the vertices of the triangle and transforming the given point P into the Gergonne point (the perspector of the incircle), there is a unique conic with perspector P.

The next result describes a connection between the perspector and the center of the conic.

Theorem 4.7. *Let P be the perspector of the conic, Q its center, and M the centroid of the triangle. Then M lies on the segment $P'Q$, where P' is the isotomic conjugate of P and $P'M = 2MQ$* (Figure 4.7).

Proof. First, we note that if the conic is inscribed in a triangle ABC, then the pole of the median CM_c lies on the line c_m passing through C and parallel to AB. Indeed, the cross-ratio of the lines c_m, CM_c, CA, CB equals 1.

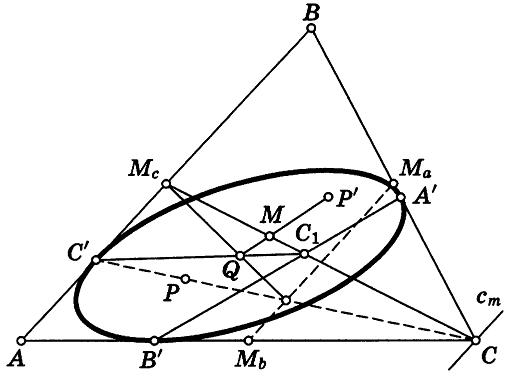

FIGURE 4.7

Suppose now that a conic with center Q is tangent to the sides of the triangle at points A', B', C', and let C_1 be the intersection of the lines $C'Q$ and $A'B'$. Since Q is the center of the conic, the pole of the line $C'Q$ is the point at infinity of the line AB, and the pole of the line $A'B'$ is C. Thus the polar of C_1 is the line c_m, and therefore C_1 lies on the median CM_c.

By Theorem 3.9, the intersection of the lines CC' and M_cQ lies on the midline M_aM_b, i.e., Q is isotomically conjugate, with respect to the triangle $M_aM_bM_c$, to the image of the perspector P under the homothety with center M and coefficient $-\frac{1}{2}$. This immediately implies the assertion of the theorem. □

Theorem 4.7 implies that for each point, there is a unique inscribed conic with center at that point. In particular, if the center of the conic coincides with M, then M is also the perspector and the conic is the inscribed Steiner ellipse, i.e., the preimage of the incircle under an affine transformation making the triangle equilateral. Notice that the Steiner ellipse has the largest area of all ellipses inscribed in a given triangle. This follows from the facts that an equilateral triangle has the smallest area among all triangles circumscribed about a circle and that affine transformations preserve ratios of areas.

Theorem 4.8. *The center of an inscribed conic with perspector P is the pole of the line PM with respect to the conic passing through A, B, C, M and P* (Figure 4.8).

Proof. This follows from the theorem just proved and Theorem 3.9. □

Finally, consider the pencil of conics tangent to four given lines. Let U, U' and V, V' be the foci of two conics from that pencil. Then the points of each pair of foci are isogonally conjugate with respect to the triangle formed by any three of the given lines. As was shown in 3.3, this implies that the intersections of UV with $U'V'$ and of $U'V$ with UV' are also isogonally conjugate with respect to all of these four triangles and therefore are the

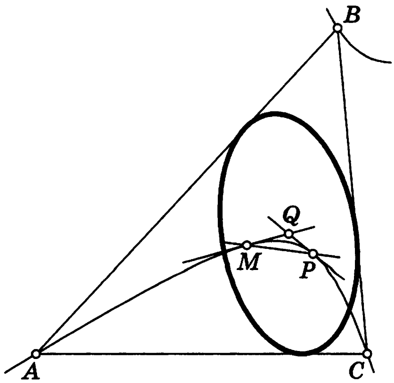

FIGURE 4.8

foci of some conic from the pencil. Since the projections of the focus to the given lines lie on a circle, it is not difficult to see that the locus of the foci is a cubic, i.e., a curve of degree three. The correspondence between the foci of each conic of the pencil gives rise to an involution on that cubic. We have shown that for any two pairs of the corresponding points U, U' and V, V' on the cubic, the intersection of the lines UV and $U'V'$ also lies on the cubic. As a limit case, we have that the tangents to the cubic at the corresponding points U and U' intersect at the cubic and their intersection point corresponds to the third intersection point of the cubic with the line UU' (Figure 4.9).

Suppose a parabola is inscribed in a triangle. By Theorem 4.7, we have that the point isotomically conjugate to the perspector of the parabola is a point at infinity. If the triangle is equilateral, then the isotomic conjugation coincides with the isogonal conjugation and the image of the line at infinity is the circumcircle of the triangle. In the general case, we have the preimage of that circle under an affine transformation making the triangle equilateral, i.e., the *circumscribed Steiner ellipse*, whose tangents at the vertices of the triangle are parallel to the opposite sides. Notice that this ellipse has the smallest area among all ellipses circumscribed about the given triangle. Thus we have proved the following.

Theorem 4.9. *The locus of the perspectors of the parabolas inscribed in a given triangle is the circumscribed Steiner ellipse.*

Problem 37. A parabola is tangent to the sides of a triangle at points A', B', and C'. Prove that the intersection of the line passing through C' and parallel to the axis of the parabola with the line $A'B'$ lies on the median CM_c.

Problem 38. Prove that the circumscribed and the inscribed Steiner ellipses are homothetic. Find the center and the coefficient of the homothety.

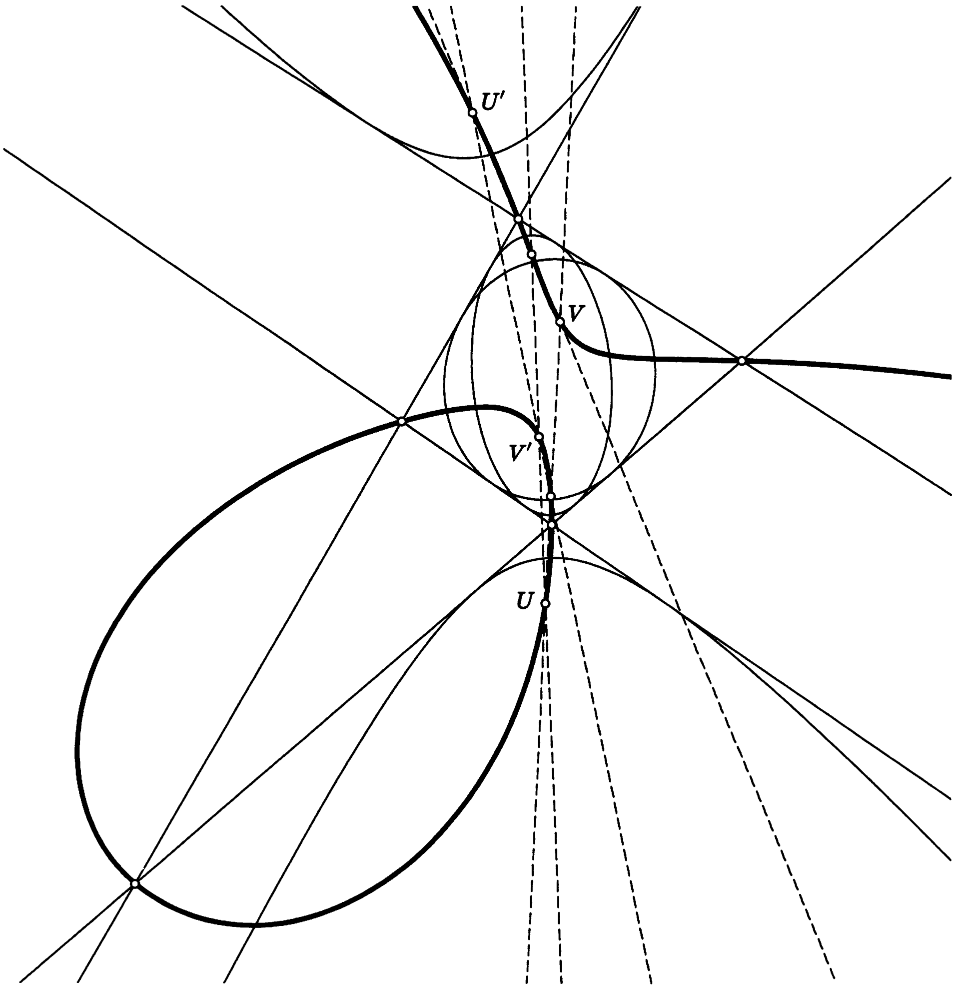

FIGURE 4.9

Problem 39. Prove that the locus of the centers of conics passing through the vertices of a triangle and its centroid is the inscribed Steiner ellipse.

Problem 40. Suppose points P and P' are isogonally conjugate with respect to a triangle ABC, and let A', B', C' be the intersections of the sides of the triangle and the lines connecting P with the centers of the corresponding excircles. Prove that the lines AA', BB', CC' and PP' intersect at a single point.

Problem 41. Suppose we are given a triangle and the center of a conic inscribed in it. Determine if the conic is an ellipse or a hyperbola.

The parabola tangent to four lines.

Since for any five lines (in general position) there is a unique conic to which these lines are tangent, for any four lines in general position (such a configuration is called a *complete quadrilateral*) no two of which are parallel, there is a unique parabola to which these lines are tangent. The fifth line in this case is the line at infinity.

Using Theorems 1.10 and 1.11 we see that for the four triangles formed by those lines, we have the following two theorems.

Theorem 4.10 (Miquel). *Suppose we are given a complete quadrilateral. Then the circumcircles of the four triangles formed by its lines intersect at a single point.*

This point is called the *Miquel point* of the complete quadrilateral (Figure 4.10).

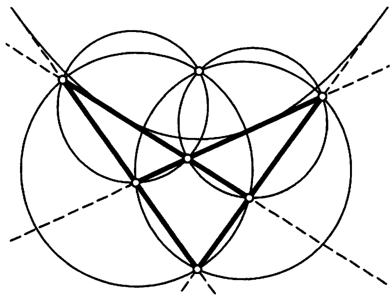

FIGURE 4.10

Proof. Consider the parabola tangent to the sides of our quadrilateral. By Theorem 1.10, the circumcircles of the corresponding triangles pass through the focus of the parabola. Therefore the focus is the desired point. □

Theorem 4.11. *Suppose we are given a complete quadrilateral. Then the orthocenters of the four triangles formed by its lines lie on a straight line. That line is perpendicular to the Gauss line of the quadrilateral.*

That line is called the *Aubert line* of the complete quadrilateral (Figure 4.11).

Proof. As in the proof of the preceding theorem, consider the parabola tangent to the sides of the quadrilateral. By Theorem 1.11, the orthocenters of the corresponding triangles lie on the directrix of that parabola.

We now prove the second part of the theorem. Using the corollary to Lemma 1.2 and Problem 10, one easily shows that the projection of the midpoint of a diagonal of the quadrilateral to the directrix is the centroid of the projections of the tangency points of the quadrilateral and the parabola; i.e., those three points have the same projection to the directrix. Hence they all lie on a straight line (this yields another proof of the existence of the

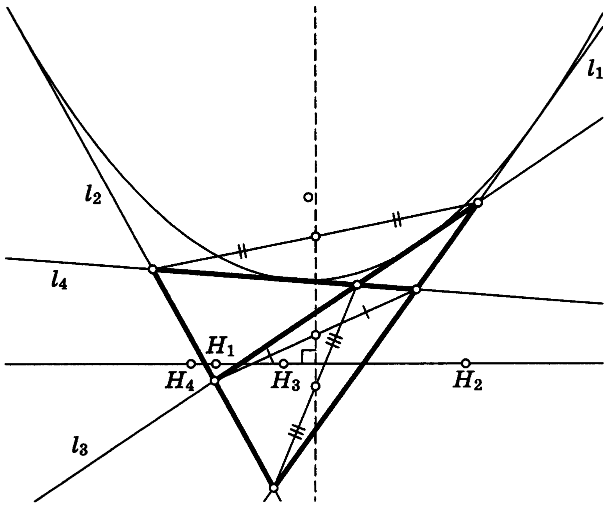

FIGURE 4.11

Gauss line), which is actually perpendicular to the directrix (and parallel to the axis of the parabola). □

The above two theorems can easily be proved without using parabolas. The former is quickly proved by computing the angles; the latter, by using radical axes. However, the next theorem does not seem to allow for a short and simple proof without using the inscribed parabola.

Theorem 4.12 (Emelyanov). *The Euler circle of the triangle formed by the diagonals of a complete quadrilateral passes through the Miquel point of that quadrilateral* (Figure 4.12).

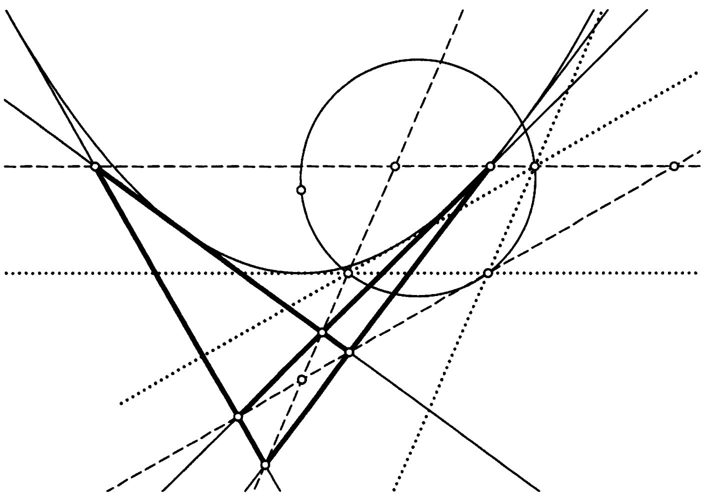

FIGURE 4.12

The proof requires a lemma dual to Lemma 4.9.

Lemma 4.10 (Dual to Lemma 4.9). *If two complete quadrilaterals have the same diagonals, then there is a conic tangent to all the sides of those quadrilaterals (obviously, such a conic is unique).*

Proof of Theorem 4.12. Consider the midlines of the triangle formed by the diagonals of the quadrilateral and the line at infinity. Those four lines form a quadrilateral whose diagonals are nothing but the sides of the triangle. Therefore, by Lemma 4.10, there is a conic tangent to the sides of the quadrilateral, the line at infinity, and the midlines of the triangle. Since it is tangent to the line at infinity, it must be a parabola and the Miquel point of the quadrilateral is the focus of that parabola. The Euler circle is the circumcircle of the midline triangle whose sides, as we have shown, are tangent to the parabola, and therefore the Euler circle passes through the focus of the parabola, i.e., through the Miquel point of the quadrilateral.

Similar arguments show that the *center of the circumcircle of the triangle formed by the diagonals of a complete quadrilateral lies on the Aubert line.*

Indeed, the center of the circumcircle is the orthocenter of the midline triangle whose sides, as we have seen, are tangent to the parabola that is tangent to the sides of the quadrilateral. Therefore its orthocenter lies on the directrix of the parabola, which, as shown above, coincides with the Aubert line of the quadrilateral. □

Notice that for each point P on the Euler circle of a triangle ABC we can construct a parabola that has focus at that point and whose directrix is the line passing through the reflections of P in the midlines of the triangle ABC. Consider any line tangent to that parabola. Suppose it intersects the sides AB and AC of the triangle at points C_1 and B_1. Suppose also that the lines CC_1 and BB_1 meet at a point Q and that AQ intersects BC at a point A_1. Then, using Lemma 4.10, one easily shows that A_1B_1, A_1C_1 and the trilinear polar of Q are tangent to the parabola, and therefore P is the Miquel point of the quadrilateral formed by the sides of the Ceva triangle of Q and its trilinear polar with respect to the triangle ABC. Hence the circumcircle of the triangle $A_1B_1C_1$ passes through P.

Thus the set of all tangents to our parabola gives rise to the set of all points Q such that P is the Miquel point of the sides of the Ceva triangle of Q and its trilinear polar. This set (in fact, it is a curve of degree four shown in Figure 4.13), together with the equilateral hyperbola circumscribed about ABC and centered at P, yields the set of all points whose Ceva circles pass through P.

The above construction gives a simple proof of yet another rather involved theorem:

Theorem 4.13 (Droz–Farny). *Suppose a line l_1 passing through the orthocenter H of a triangle ABC intersects its sides at points A_1, B_1 and C_1. Another line l_2, perpendicular to l_1 and also passing through H, intersects*

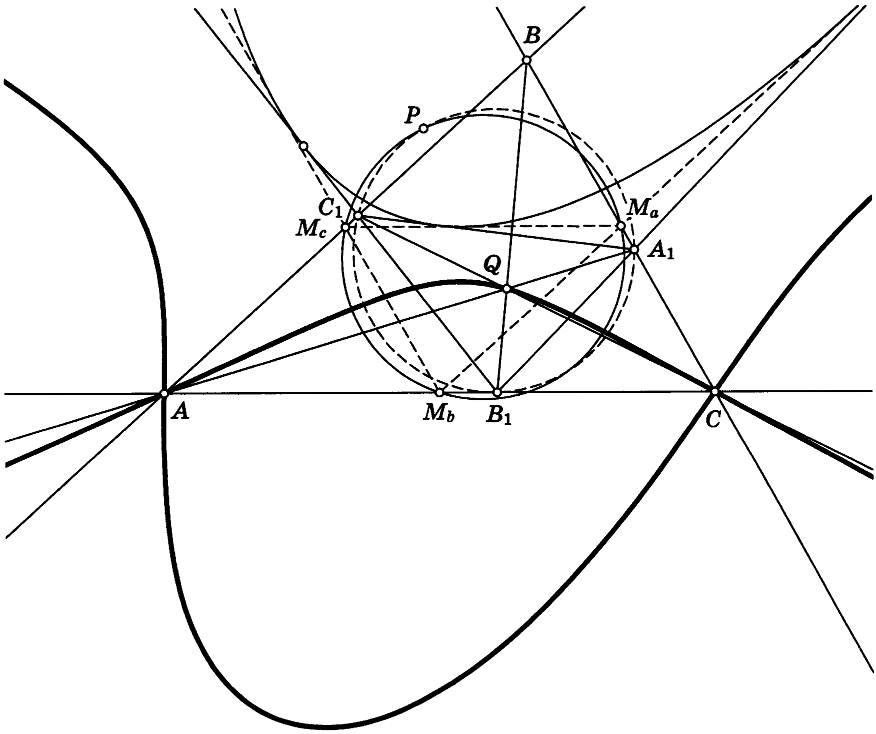

FIGURE 4.13

the sides of the triangle at points A_2, B_2 and C_2. Then the midpoints of the segments A_1A_2, B_1B_2 and C_1C_2 lie on a straight line.

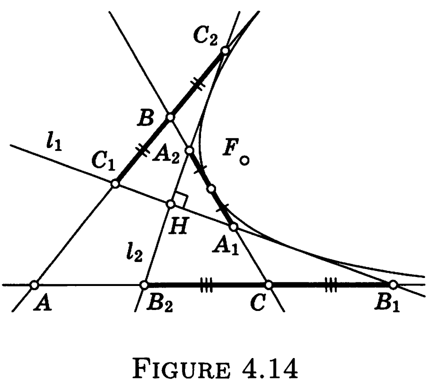

FIGURE 4.14

Proof. Consider the parabola tangent to the sides of the triangle and the line l_1. By Theorems 1.11 and 1.7, the tangent to that parabola passing through H and different from l_1 is perpendicular to l_1, and therefore it coincides with l_2. By Theorem 1.10, the circumcircles of the triangles A_1A_2H, B_1B_2H and C_1C_2H pass through the focus F of the parabola.

Hence the centers of those circles lie on the midpoint perpendicular to FH. Since all these triangles are right, the centers of their circumcircles are the midpoints of the hypotenuses, i.e., of the segments A_1A_2, B_1B_2 and C_1C_2 (Figure 4.14).[1] □

When a point P moves along the circumcircle of the triangle, the midpoint perpendiculars to PH are tangent to the conic with foci at H and O inscribed in the triangle (this follows from the construction described in Theorem 3.4). Thus we have also proved that all those lines envelop the conic with foci at O and H inscribed in the triangle.

We remark that the above theorem easily follows from Problem 18 at the end of 3.1. For the centrally symmetric triangle we take the triangle whose vertices are the reflections of the center of the circumcircle in the sides of the triangle. Each line obtained this way gives rise to two perpendicular lines passing through the orthocenter and generating the line from the Droz–Farny theorem. The existence of those two lines can easily be proved by computing the angles.

4.3. Normals to conics. Joachimstahl's circle

Definition. Suppose we are given a conic and a point P on it. The *normal* to the conic at P is the line passing through P and perpendicular to the tangent to the conic at P.

Given an arbitrary point not on the conic, one has four (possibly complex) normals to the conic passing through that point. It turns out that the feet of those normals have the following property.

Theorem 4.14. *Let P_1, P_2, P_3, P_4 be four points of a conic α with center O. Suppose the normals at those points pass through a single point Q. Then P_1, P_2, P_3, P_4, O, Q lie on an equilateral hyperbola whose asymptotes are parallel to the axes of the conic.*

Proof. Given an arbitrary circle ω with center Q, consider the locus of the centers of the conics from the pencil generated by the circle and the conic α. By Theorem 3.17, it is a conic and, since the pencil contains a circle, it is an equilateral hyperbola (see Problem 29). We denote it γ (Figure 4.15). The infinite points of the hyperbola γ are the fixed points of the involution defined by the pencil on the line at infinity, i.e., the points of the axes of α. Any point X on γ is the center of a conic from the pencil. Its polar with respect to the conic is the line at infinity, and therefore, by Theorem 3.15, the polars of X with respect to all conics of the pencil are parallel. Since the line QX is perpendicular to the polar of X with respect to the circle ω, it is also perpendicular to the polar of X with respect to the conic α. Hence

[1] The last part of the proof is a special case of Problem 7 at the end of Chapter 1.

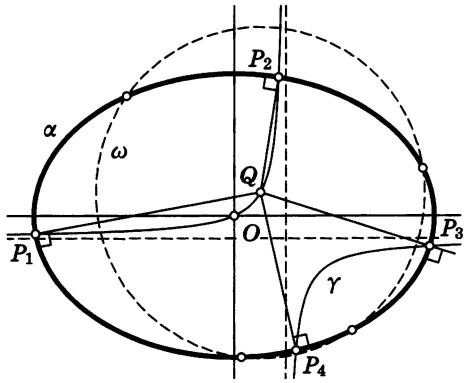

FIGURE 4.15

if P belongs to the intersection of α and γ, then the tangent to α at P is perpendicular to QP, and therefore QP is the normal to α.

Moreover, it is easy to recover the tangent to γ at O. The direction conjugate to the direction of the tangent with respect to α must be perpendicular to QO. Therefore the hyperbola γ does not depend on the radius of the circle ω because the latter can be defined as the conic passing through Q, two points of the line at infinity (through which the axes of α pass), and the point O, and which is also tangent at O to the corresponding line. It now follows that there are no other points P such that PQ is normal to α. Therefore any such point can be obtained by the foregoing construction if ω is the circle with center Q and radius QP. □

The constructed hyperbola is called the *Apollonius hyperbola* of α with respect to Q. The midpoints of the sides of the quadrilateral whose vertices are the intersection points of ω and α, lie on γ and form a parallelogram whose center coincides with that of γ and with the centroid of the quadrilateral. Hence the centroid of the intersection is the center of γ and does not depend on the radius of the circle.

Theorem 4.15. *The points P_1, P_2, P_3 and the point symmetric to P_4 with respect to the center of the conic lie on a circle.*

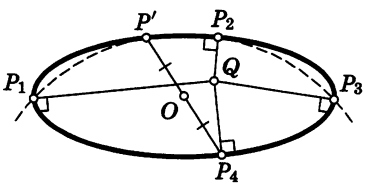

FIGURE 4.16

Proof. Consider the case where the given conic is an ellipse. Let P' be the point symmetric to P_4 with respect to O (Figure 4.16). The assertion of the theorem is equivalent to the statement that the conic of the centers of the pencil determined by P_1, P_2, P_3, P' is an equilateral hyperbola. Consider the compression to the minor axis of the ellipse transforming it into a circle. Since affine transformations take centers of conics into centers of conics, the locus in question also transforms into a conic and, in fact, into an equilateral hyperbola, since the new pencil contains a circle (the image of the original ellipse). This hyperbola contains the images of O (i.e., O itself) and of the midpoints of the segments $P'P_1$, $P'P_2$, $P'P_3$. The homothety with respect to the image of P' and with coefficient 2 transforms those points into the points P_4, P_1, P_2 and P_3. But since there is only one equilateral hyperbola passing through the images of P_4, P_1, P_2 and P_3, it must be the corresponding Apollonius hyperbola (under the compression it is taken into an equilateral hyperbola, because its asymptotes run in the directions of the axes of the ellipse); i.e., the conic of the centers of the pencil determined by P_1, P_2, P_3, P' is homothetic to the Apollonius hyperbola passing through P_1, P_2, P_3, P_4, and is therefore an equilateral hyperbola. □

The circle mentioned in the theorem is called *Joachimstahl's circle*.

Using complex affine transformations, one can also prove Theorem 4.15 for hyperbolas. In the case where the conic is a parabola, one of the four feet of the normals passing through the given point is at infinity. Accordingly, Theorem 4.15 is stated as follows:

The feet of the three normals to a parabola passing through a given point and the vertex of the parabola lie on a circle (Figure 4.17).

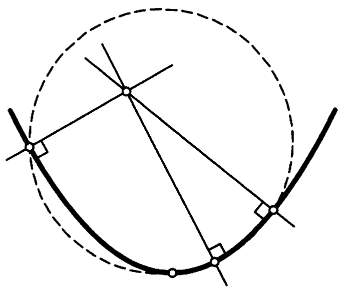

FIGURE 4.17

4.4. The Poncelet theorem for confocal ellipses

Suppose we are given a circle with center at one of the foci of confocal conics. Applying first the polar correspondence and then the inversion with respect to the circle, we transform the confocal conics into concentric circles. Hence the polar circles also form a pencil. Therefore the confocal conics form the dual pencil.

We consider two confocal ellipses such that there is a polygon inscribed in the larger ellipse and circumscribed about the smaller ellipse (Figure 4.18). By the Poncelet theorem, there are infinitely many such polygons (if there is at least one). It turns out that they have some interesting properties.

In 1.4 we established the following fact. Suppose a string is put on an ellipse α which is then pulled tight using a pencil. If the pencil is rotated about the ellipse, it will traverse another ellipse confocal with α.

As an important consequence of this fact we have the following theorem.

Theorem 4.16. *If a convex n-gon is inscribed in a given ellipse α and has the longest perimeter among all such n-gons, then it is circumscribed about an ellipse α_n confocal with α.*

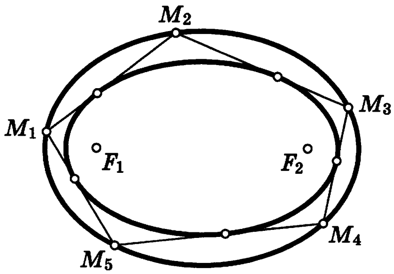

FIGURE 4.18

Proof. Let $M_1M_2\dots M_n$ be a polygon with the longest perimeter. We shall prove that for each $i = 1, 2, \dots, n$, the bisector of the exterior angle $M_{i-1}M_iM_{i+1}$ is tangent to α at M_i.

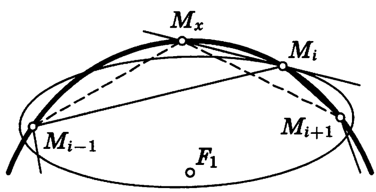

FIGURE 4.19

Suppose this is not the case. Let α' be the ellipse with foci M_{i-1} and M_{i+1} that passes through M_i (Figure 4.19). Then the tangent to α', which is also the bisector of the exterior angle $M_{i-1}M_iM_{i+1}$, intersects α at some other point M_x. But the sum $M_{i-1}M_x + M_{i+1}M_x$ is larger than $M_{i-1}M_i + M_{i+1}M_i$, since M_x is outside the ellipse α'. Thus we have a convex n-gon $M_1M_2\dots M_{i-1}M_xM_{i+1}\dots M_n$ whose perimeter is longer than the perimeter of $M_1M_2\dots M_n$. But this contradicts our choice of the polygon.

Now we show that an ellipse can be inscribed in the n-gon $M_1M_2\dots M_n$. Let F_1 and F_2 be the foci of the ellipse α. Consider the ellipse K_n that has the same foci and is tangent to the line M_1M_2. The angle between the second tangent to α_n passing through M_2 and M_2F_1 equals $\angle F_1M_2M_1$. But such a line must be the line M_2M_3 since the angles $M_3M_2M_1$ and $F_1M_2F_2$ have equal exterior bisectors (the tangent to α at M_2).

Similarly, examining the vertex M_3, we see that α_n is tangent to M_3M_4, etc.

By the Poncelet theorem, the polygon $M_1M_2\dots M_n$ can be "rotated" between α and α_n. We show that the perimeter of the polygon does not change under this rotation.

Indeed, that perimeter can easily be computed. Let Q_i be the tangency points of α_n and the sides M_iM_{i+1}. By Theorem 1.6, for an arbitrary point M on α, the quantity $MX + MY + \smile XY$, where MX and MY are the tangents to α_n, does not depend on M. Hence $C = Q_{i-1}M_i + M_iQ_i + \smile Q_iQ_{i-1}$ does not depend on M_i. Computing the sum of the lengths of such loops for each M_i, we have nC. Each side of the polygon will be counted once, whereas the arcs of the ellipse are counted $n-1$ times (the shortest of the two arcs Q_iQ_{i+1} belongs to all loops except for the loop corresponding to M_{i+1}). Thus the perimeter of the n-gon $M_1M_2\dots M_n$ is equal to the difference between nC and the perimeter of the ellipse α_n counted $n-1$ times. Therefore it does not change under the rotation. □

A convex Poncelet n-gon inscribed in one of the two confocal ellipses and circumscribed about the other has yet another extremal property, which in a sense is "dual" to the one established in Theorem 4.16.

Theorem 4.17. *A convex n-gon circumscribed about a given ellipse α has the shortest perimeter among all such n-gons if and only if all of its vertices lie on an ellipse confocal with α.*

Proof. Fix the tangency points M_{i-1}, M_{i+1} of the sides of the polygon and the ellipse. Let T be the intersection of the tangents at those points (Figure 4.20). For the sake of definiteness, suppose that the arc $M_{i-1}M_{i+1}$ is less than half of the ellipse. We want to find a point M_i on it such that the length of the polygonal line $M_{i-1}XYM_{i+1}$, where X and Y are the intersection points of the tangent passing through M_i with TM_{i-1} and TM_{i+1}, is the least possible. Let X' and Y' be the tangency points of the incircle of the triangle TXY and its sides TY and TX. Then $M_{i-1}XYM_{i+1} = M_{i-1}Y' + X'M_{i+1}$, i.e., the desired minimum is attained when the incircle is of largest possible radius. Therefore the incircle and the ellipse are tangent to the line XY at the same point but on the opposite sides. Thus, by Theorem 3.16, the center O of the ellipse lies on the line IT', where I is the center of the incircle of the triangle TXY and T' is the midpoint of XY.

Let F_1 and F_2 be the foci of the given ellipse. Since it is inscribed in the triangle TXY, the lines F_1X and F_2X form equal angles with the

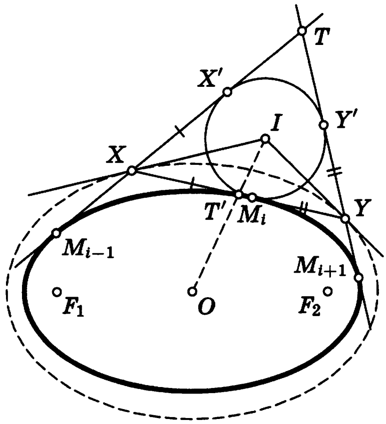

FIGURE 4.20

line IX, and the lines F_1Y and F_2Y form equal angles with the line IY. Moreover, the line connecting the midpoints of the segments F_1F_2 and XY passes through I. We now invoke Problem 26 (in 3.3). Since the angle XIY is obtuse, the lengths of the polygonal lines F_1XF_2 and F_1YF_2 are equal, i.e., X and Y lie on an ellipse confocal with K. This, obviously, implies the assertion of the theorem.

The case where the arc $M_{i-1}M_{i+1}$ is longer than half of the ellipse is argued similarly except that the incircle of the triangle TXY must be replaced by an excircle. Finally, if M_{i-1} and M_{i+1} are antipodal points, then the desired assertion is established by passing to limit or by deducing it from the fact that the sum of the sides of a parallelogram circumscribed about an ellipse cannot be less than the sum of its axes. This last assertion, in turn, can be proved by a simple calculation. □

Problem 42. Suppose that an ellipse and a circle are externally tangent to each other and that their common tangents are parallel. Prove that the distance between their centers equals the sum of the semiaxes of the ellipse.

Chapter 5

Solutions to the Problems

1. This equation defines a curve of order two because it is equivalent to the equation $xy = 1$. Since

$$xy = \frac{1}{4}((x+y)^2 - (x-y)^2),$$

in coordinates $\xi = x + y$ and $\zeta = x - y$ this hyperbola has the equation

$$\frac{\xi^2}{4} - \frac{\zeta^2}{4} = 1.$$

Let us find its foci. Suppose a point X moves along the hyperbola toward infinity. Then the lines F_1X and F_2X will tend to becoming parallel (where F_1 and F_2 are the foci of the hyperbola), and therefore the quantity $|F_1X - F_2X|$ equals the length of the projection of the segment F_1F_2 to the axis Ox. On the other hand, it also equals the real axis of the hyperbola, i.e., $2\sqrt{2}$. Since the angle between F_1F_2 and Ox equals $45°$, the quantity F_1F_2 equals $2\sqrt{2} \cdot \sqrt{2} = 4$. Hence $OF_1 = OF_2 = 2$ and therefore F_1 has coordinates $(\sqrt{2}, \sqrt{2})$ and F_2 has coordinates $(-\sqrt{2}, -\sqrt{2})$.

2. Connect the tangency points and the vertices of the polygon with F. Color the obtained angles containing black sides, red and containing white sides, blue. By Theorem 1.4, the angles at the same vertex of the polygon are equal and have different colors. Hence the sum of the red angles equals the sum of the blue angles, i.e., $180°$.

3. The assumption implies that the lines symmetric to the diagonal AC with respect to the bisectors of the angles A and C intersect at a single point P of the diagonal BD. Applying the sine theorem to triangles ABP, ADP, ABL and ADL, where L is the intersection of the diagonals, we have $\frac{BL}{DL}\frac{BP}{DP} = \frac{AB^2}{AD^2}$. Similarly, $\frac{BL}{DL}\frac{BP}{DP} = \frac{CB^2}{CD^2}$. Therefore $\frac{AB}{AD} = \frac{CB}{CD}$, whence the desired assertion.

4. The eccentricity of all equilateral hyperbolas obviously equals $\sqrt{2}$ (check this!). Hence the directrices of the hyperbolas in question are at a distance $FP/\sqrt{2}$ away from P. Therefore they are tangent to the circle ω with center P and radius $FP/\sqrt{2}$. It is easy to see that they envelop the entire circle (because each tangent to the circle gives rise to an equilateral hyperbola with focus P and directrix the tangent). On the other hand, the directrices are taken into asymptotes under a rotational homothety centered at F (with rotation through $\pm 45°$ and the coefficient of homothety $1/\sqrt{2}$). Hence all asymptotes of such equilateral hyperbolas will be tangent to one of the two circles obtained from ω by the above homothety (see Figure 5.1).

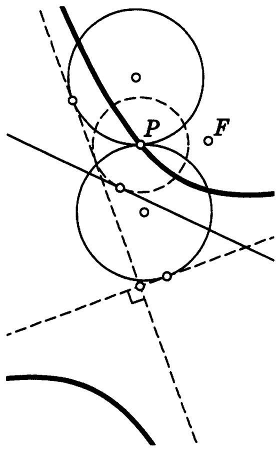

FIGURE 5.1

5. Let X' denote the projection of X to the directrix of the parabola. Notice that $FY \parallel XX'$ and $XY \parallel X'F$ (both lines are perpendicular to the tangent to the parabola at X). Thus $XYFX'$ is a parallelogram, and therefore the length of YZ equals $X'F'$, where F' is the projection of F to XX'. But the length of the segment $X'F'$ is constant and equals the distance from the focus to the directrix of the parabola (see Figure 5.2).

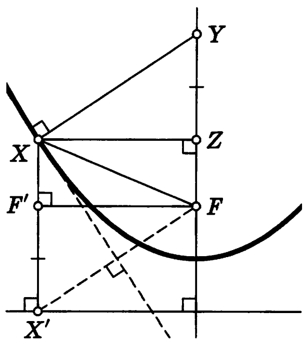

FIGURE 5.2

6. Let X and Y denote the positions of the travelers and A the intersection of the roads. The intersection point of the midpoint perpendiculars to AX and AY moves along some line l (since its projections to the roads move with constant speed). At all times, the circumcircle of the triangle AXY

passes through the reflection A' of A in l. Consider the parabola tangent to the sides of the triangle AXY with focus A'. Its directrix is the line passing through the reflections of A' in AX and AY. But those points are fixed. Hence the directrix and the focus of the parabola are fixed and therefore the parabola itself is fixed and tangent to XY.

7. Suppose the line tangent to the parabola intersects AP and AQ at points X and Y. Let M be the midpoint of XY. Then, by Theorem 1.10, the circumcircle of the triangle AXY passes through the focus F of the parabola. Notice that the angles of the triangle XFY do not depend on the position of the tangent. Therefore the angle XMF and the ratio $\frac{FX}{FM}$ are also constant. Thus X transforms into M under the rotational homothety with center F, rotation angle XMF, and coefficient $\frac{FX}{FM}$. Hence M moves along the line which is the image of AP under this homothety.

8. Apply a projective transformation taking P to the centroid of a triangle ABC. Then the points A', B' and C' will move to infinity and will therefore lie on a line.

Similarly, one can show that for any line (not passing through the vertices of the triangle) there is a unique trilinear pole, i.e., a point for which that line is the trilinear polar.

9. Apply an affine transformation such that the line becomes parallel to one of the axes of the hyperbola. Then the desired equality would follow from symmetry. But since the segments in question lie on a line, their images under the affine transformation will also be equal.

10. Apply an affine transformation taking the lines in question to lines parallel to the directrix of the parabola. Then the line connecting the midpoints of the segments AB and CD will, obviously, become the axis of the parabola. But the affine transformation preserves the lines parallel to the axis of the parabola (as a set). These are the lines passing through the tangency point of the parabola and the line at infinity.

11. If C lies inside the circle, then use a projective transformation to take it to the center. It is easy to see that the images of the points D and E are symmetric with respect to the center, i.e., C, and therefore C, D, and E lie on a straight line.

If C is outside the circle, then use a projective transformation that takes it to a point at infinity.

12. Use an affine transformation taking the ellipse to a circle. Then all the circles in question become ellipses. Those ellipses are similar (i.e., the ratio of the minor and major semiaxes is the same) and positioned the same way (the corresponding axes are parallel). We need to show that they are of equal "size". The conjugate diameters of the original ellipse become perpendicular diameters of our circle. It remains to invoke the result of 1.4. If the ellipses were of different "sizes", then the radii of the circles from the points of which they are seen at a right angle would also be different. But all those radii equal the radius of our big circle (Figure 5.3).

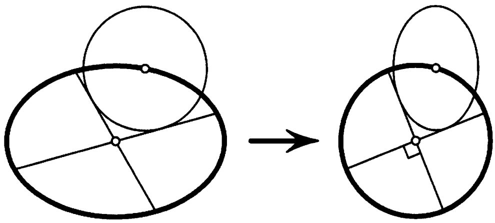

FIGURE 5.3

13. Let O be the center of the circumcircle of the triangle ABC. By the sine theorem, the radius OA of that circle and the angle BOC do not change. Therefore the length of the segment OP does not change (see Figure 5.4). Moreover, the direction of the bisector of the exterior angle AOP does not change because the bisector of the angle AOB is always perpendicular to AB and the angle BOP does not change. Let P' be the reflection of P in that bisector. Then P' lies on the line OA and the length of the segment AP' is constant. Hence the projections of AP and AP' to a line parallel to the exterior bisector of the angle AOP are equal and the projection of AP to a line parallel to the interior bisector of the angle AOP equals the projection of AP' to the same line multiplied by $\frac{AO-OP}{AO+OP}$. Therefore P moves along an ellipse obtained by squeezing the circle with center A and radius AP' by a factor $\frac{AO-OP}{AO+OP}$ toward the line parallel to the exterior bisector of the angle AOP.

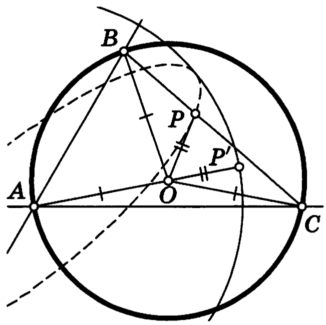

FIGURE 5.4

14. Assume first that the conic intersecting the triangle at the points in question is a circle. Then, for example, $BA_1 \cdot BA_2 = BC_1 \cdot BC_2$. Hence this ratio equals 1. It now immediately follows that if the conic is an ellipse, then this quantity is also preserved, since an affine transformation taking that ellipse into a circle does not change the ratio ("we multiply and divide by two collinear segments"). This argument does not work for a hyperbola, since there is no affine transformation taking a circle to a hyperbola. But this can be done by a projective transformation.

We shall show that under a projective transformation the above quantity does not change.

Each projective transformation is a central projection. Let P be the center of that projection. Then

$$\begin{aligned}\frac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} &= \frac{S_{\triangle PBA_1} \cdot S_{\triangle PBA_2}}{S_{\triangle PCA_1} \cdot S_{\triangle PCA_2}} \\ &= \frac{(PB \cdot PA_1 \cdot \sin\angle BPA_1) \cdot (PB \cdot PA_2 \cdot \sin\angle BPA_2)}{(PC \cdot PA_1 \cdot \sin\angle CPA_1) \cdot (PC \cdot PA_2 \cdot \sin\angle CPA_2)} \\ &= \frac{PB^2}{PC^2} \cdot \frac{\sin\angle BPA_1 \cdot \sin\angle BPA_2}{\sin\angle CPA_1 \cdot \sin\angle CPA_2}.\end{aligned}$$

Multiplying the similar equalities for the remaining two sides we have:

$$\begin{aligned}&\frac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} \cdot \frac{CB_1 \cdot CB_2}{AB_1 \cdot AB_2} \cdot \frac{AC_1 \cdot AC_2}{BC_1 \cdot BC_2} \\ &\quad = \Big(\frac{PB^2}{PC^2} \cdot \frac{\sin\angle BPA_1 \cdot \sin\angle BPA_2}{\sin\angle CPA_1 \cdot \sin\angle CPA_2}\Big) \cdot \Big(\frac{PC^2}{PA^2} \cdot \frac{\sin\angle CPB_1 \cdot \sin\angle CPB_2}{\sin\angle APB_1 \cdot \sin\angle APB_2}\Big) \\ &\quad\quad \cdot \Big(\frac{PA^2}{PB^2} \cdot \frac{\sin\angle APC_1 \cdot \sin\angle APC_2}{\sin\angle BPC_1 \cdot \sin\angle BPC_2}\Big) \\ &\quad = \Big(\frac{\sin\angle BPA_1 \cdot \sin\angle BPA_2}{\sin\angle CPA_1 \cdot \sin\angle CPA_2}\Big) \cdot \Big(\frac{\sin\angle CPB_1 \cdot \sin\angle CPB_2}{\sin\angle APB_1 \cdot \sin\angle APB_2}\Big) \\ &\quad\quad \times \Big(\frac{\sin\angle APC_1 \cdot \sin\angle APC_2}{\sin\angle BPC_1 \cdot \sin\angle BPC_2}\Big).\end{aligned}$$

But it is not difficult to see that this quantity does not depend on the plane of projection. Thus the equality also holds for a hyperbola.

It remains to show that if the points satisfy this condition, then they lie on a conic. Consider the conic passing through five of the six points, say, all except C_2. That conic must intersect AB at some other point C_2' besides C_1. The chosen five points and C_2' satisfy the given condition. It is easy to see that this condition uniquely determines C_2', and therefore the latter must coincide with C_2.

15. Apply a projective transformation taking the intersections of the ellipses to the vertices of a square. Then the center O of the square is also the common center of the ellipses and therefore their common tangents are also pairwise centrally symmetric with respect to O.

16. Let A_1 be the intersection of the diagonals AC and FB. Define points $B_1, \ldots, F_1$ similarly (Figure 5.5).

By Pascal's theorem, the intersection P of the lines AD and FC lies on the line F_1C_1. Let Q be the intersection of the lines AD_1 and B_1F. Then, by Pappus' theorem, C_1, P and Q lie on a straight line. Using Pappus' theorem for the points A, A_1, B_1 and F, E_1, D_1, we see that the same line contains the intersection point of the lines A_1D_1 and B_1E_1. Thus the lines A_1D_1, B_1E_1 and C_1F_1 intersect at a single point. Now, by the converse of Brianchon's theorem, the hexagon $A_1B_1C_1D_1E_1F_1$ is circumscribed about

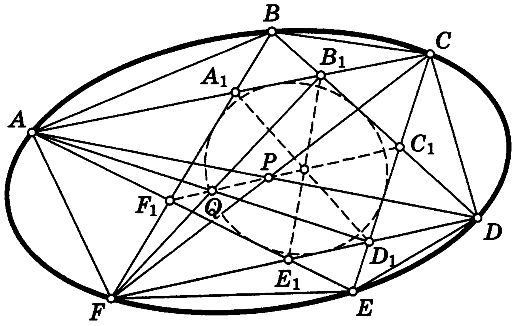

FIGURE 5.5

the conic. Notice that our proof does now use any specific order of the points.

Based on this we can now prove the Poncelet theorem for triangles. Suppose a triangle ABC is inscribed in a conic α_1 and circumscribed about a conic α_2. Take an arbitrary point D on α_1 which is outside α_2 and draw tangents to α_2. Suppose they intersect α_1 again at points E and F. We need to show that EF is tangent to α_2. As we have shown, there is a conic tangent to the lines AB, BC, CA, DE, EF and FD. On the other hand, there is a unique conic tangent to the five lines AB, BC, CA, DE, DF, and this is α_2. Therefore EF is also tangent to α_2, which is the desired claim.

17. *Hint.* The quadrilateral formed by the given lines and the asymptotes is circumscribed about the hyperbola.

18. By the converse of Brianchon's theorem, there is a conic tangent to the sides of both triangles. Suppose an arbitrary line tangent to that conic intersects AC in P and BC in Q. It suffices to show that the lines PB' and QA' are parallel. But this follows from Brianchon's theorem applied to the hexagon $A'B'XQPY$, circumscribed about the same conic, where X and Y are the infinite points of the lines BC and AC.

19. Let X and Y be two points on the line at infinity whose corresponding directions are perpendicular. Draw the lines: through A and B, parallel to the direction of X, and through C and H, parallel to the direction of Y. Let UV be the diagonal formed by those lines of the rectangle and B' the foot of the triangle's height dropped from B (Figure 5.6).

Since the quadrilaterals $BB'CV$ and $AUB'H$ are inscribed in the circles with diameters BC and AH, we have $\angle AB'U = \angle AHU$ and $\angle VB'C = \angle VBC$. But $\angle AHU = \angle VBC$ as the angles with perpendicular sides, and therefore U, B', V lie on a straight line. By the converse of Pascal's theorem, the hexagon $AXBHYC$ is inscribed in a conic, i.e., the equilateral hyperbola $ABCXY$ passes through H.

Conversely, suppose the conic passes through A, B, C and H. Since A, B, C, H are not the vertices of a convex quadrilateral, that conic must be a hyperbola. If X is one of the points of its intersection with the line at infinity,

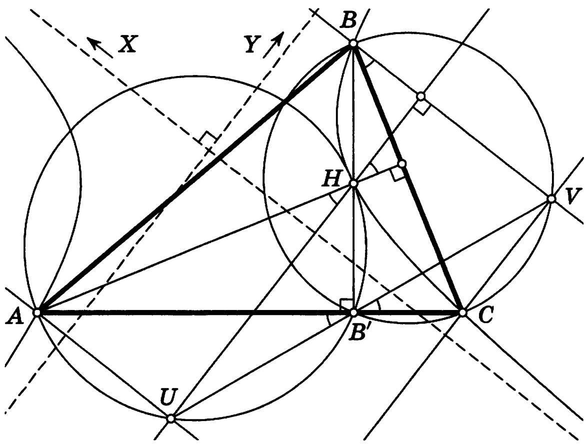

FIGURE 5.6

and Y is the point at infinity corresponding to the perpendicular direction, then Y also belongs to the conic. Therefore the conic is an equilateral hyperbola.

20. *Hint.* One can use *Sondat's theorem*, asserting that if two triangles are in perspective (i.e., the lines connecting their corresponding vertices intersect at a single point) and are orthologic (i.e., the perpendiculars dropped from the vertices of each triangle to the sides of the other intersect at a single point), then the centers of perspective and orthology lie on a straight line. As a consequence of this theorem, one easily deduces that orthologic triangles with coinciding orthology centers are in perspective. The assertion of the problem is a special case of the latter. Notice also that if the conic is a circle and its interior points are viewed as points of the Klein model of Lobachevsky geometry, then the assertion of the problem becomes an analog of the theorem that the heights of a triangle intersect at a single point.

The assertion of the problem admits an alternative formulation: two triangles are in perspective if and only if they are polar to each other (i.e., there is a conic such that the polar correspondence with respect to that conic takes the vertices of one triangle to the sides of the other and vice versa; that conic may be imaginary). For the proof it suffices to transform the axis of perspective of the triangles to the line at infinity, so that they become homothetic, and then use an affine transformation (possibly, imaginary) to take the center of homothety to the common orthocenter.

21. The center of a conic is the polar of the line at infinity with respect to that conic. Hence their preimages with respect to ω are a polar and a pole with respect to α. But the preimage of the line at infinity is the center of ω.

22. 1. Suppose a line intersects the hyperbola at points A and B and its asymptotes at points X and Y. Since $AX = BY$, the midpoints of the segments AB and XY coincide and the line connecting the midpoint of AB

with the center O of the hyperbola is a median of the right triangle OXY. Hence it forms the same angles with the asymptotes of the hyperbola as the line AB.

2. Let P be an intersection point of two equilateral hyperbolas with center O. Since the tangents to the hyperbolas at that point are symmetric to OP with respect to lines parallel to the asymptotes, the angle between them is twice as large as the angle between the asymptotes.

23. Apply the polar transformation with respect to a circle centered at the center O of the ellipse. Then the sides of a rhombus will transform into points from which the image of the ellipse (which is also an ellipse) is seen at a right angle. By Theorem 1.5, this is a circle also centered at O. Therefore its image, i.e., the envelope of all the rhombi, is a circle with center O.

24. The circle and the hyperbola form a double tangent pencil. Under the polar correspondence with respect to the circle the hyperbola transforms into a conic from that pencil. The points at infinity of that conic are the images of the asymptotes of the hyperbola, which, obviously, coincide with the points at infinity of the hyperbola. Therefore the hyperbola is transformed into itself.

We shall now show that the circle is self-polar, too, with respect to the hyperbola. Choose two perpendicular lines in space and consider the two cones of revolution obtained by rotating the lines about the bisectors of the angles between them. Those lines are common rulings of the cones along which the cones are tangent to each other. Given a plane perpendicular to the axis of one of the cones, the cross-section of that cone is a circle and the cross-section of the other is an equilateral hyperbola tangent to the circle at its vertices (Figure 5.7). Therefore there is a projective transformation interchanging the circle and the hyperbola. Moreover, it preserves self-polarity.

Notice also that if a secant plane is perpendicular to the common ruling of the cones, then the cross-sections are equal parabolas tangent to each other at their vertices. Hence such parabolas are also self-polar with respect to each other.

25. Since T lies on the Gauss line, it is the center of a conic inscribed in the quadrilateral. Suppose the quadrilateral is not circumscribed; then that conic is not a circle, i.e., the distance from its center to the tangent takes any value at most four times. By assumption, the opposite sides of the quadrilateral are equidistant from the center. Hence either at least one pair consists of parallel sides and the quadrilateral is a trapezoid, or the sides in each pair are symmetric with respect to one of the axes of the conic. Those axes cannot coincide, because otherwise a circle could be inscribed in the quadrilateral. Hence the bisectors of the angles between the opposite sides of the quadrilateral are perpendicular, which is equivalent to its being inscribed. The converse for an inscribed quadrilateral is proved similarly, and for a circumscribed quadrilateral and a trapezoid it is obvious.

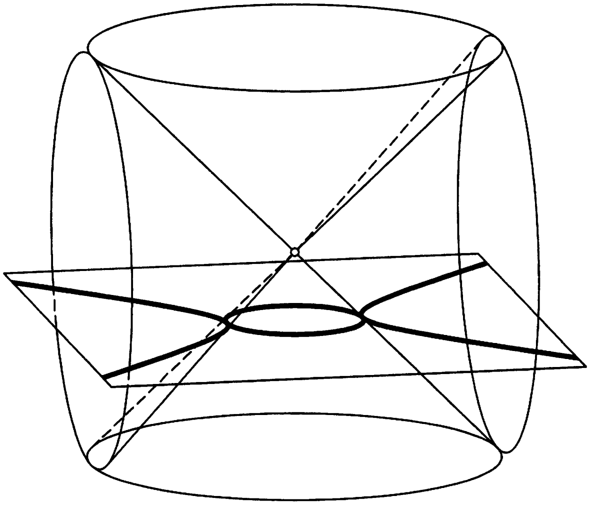

FIGURE 5.7

26. In fact, this is a reformulation of the previous problem. Indeed, O is equidistant from the opposite sides of the quadrilateral formed by the lines AX, AY, BX, BY (in this order). The condition that C, Z and O lie on a straight line is equivalent to O lying on the Gauss line. The quadrilateral in question is not a trapezoid, and therefore it is either inscribed, and then the angle O is right, or it is circumscribed, and then the lengths of the polygonal lines are equal. The proof of the converse statements is not difficult.

27. Since for any two circles two of the four intersection points are infinite (and imaginary), the desired assertion follows at once from the three conics theorem.

28. *Hint.* The points in question belong to the locus of the centers of the conics from the pencil generated by the vertices of the quadrilateral.

29. *Hint.* The axes of the parabolas circumscribed about the quadrilateral are perpendicular if and only if the quadrilateral is inscribed.

30. *Hint.* Use the dual to the theorem on four conics.

31. By Theorem 3.21, one of the eccentricities must be larger than 1 and the corresponding conic is an ellipse, whereas the other eccentricity must be smaller than 1, and therefore the conic is a hyperbola. Henceforth we refer to them as the ellipse and the hyperbola. Let F' be the projection of F to l and S the midpoint of the segment FF'. Let E_1 and E_2 be the intersection points of the line FF' with the ellipse and H_1 and H_2 with the hyperbola. Since the eccentricities of the ellipse and the hyperbola are reciprocal, the points E_1 and E_2 are symmetric to H_1 and H_2 with respect to S. Therefore the centers of the ellipse and the hyperbola are symmetric with respect to S. Those centers are the midpoints of E_1E_2 and H_1H_2, respectively. Therefore the reflections of F in the midpoints of E_1E_2 and H_1H_2 (these points are symmetric with respect to S) are symmetric with respect to F'. But these two points are, obviously, the other foci of our conics.

32. Suppose the locus in question is not empty. Then $\angle BAP = \angle BCP$ and $\angle DAP = \angle DCP$ and therefore $\angle A = \angle C$. Similarly, $\angle B = \angle D$, i.e., $ABCD$ is a parallelogram. On the other hand, if $ABCD$ is a parallelogram and the radii of the circumcircles of the triangles ABP and BCP are equal, then the equality $\angle BAP = \angle BCP$ implies that P lies on the equilateral hyperbola centered at the midpoint of the segment AC and passing through A, B and C. Since D lies on the same hyperbola, P satisfies the condition.

33. Since Q lies on the circumcircle of the triangle ABC, its reflection in the center of the equilateral hyperbola circumscribed about the quadrilateral $QABC$ is the orthocenter of that triangle. Therefore P is the orthocenter of the triangle ABC. By construction, it is also the center of the circumcircle. But the orthocenter and the center of the circumcircle coincide only if the triangle is equilateral.

34. Let K, L, M and N be the midpoints of the segments AB, BC, CD and AD, respectively. Let O be the center of the circumcircle and H_d the orthocenter of the triangle ABC. The proof of Theorem 4.1 implies that P is the midpoint of the segment DH_d. It is easy to see that the line OK is parallel to the line AH_d, which, in turn, is parallel to the line PM. Similarly, the line OL is parallel to PN. This means that P and O are symmetric with respect to the center of the parallelogram $KLMN$. But the center of that parallelogram is the centroid of the quadrilateral $ABCD$.

35. Suppose the points lie on a conic. Apply a projective transformation taking the line $A'B'$ to the line at infinity, and then apply an affine transformation to make C' the orthocenter of the triangle ABC. Then the conic passing through the given points is an equilateral hyperbola, i.e., the directions determined by A' and B' are perpendicular. Since C' is the orthocenter of the triangle ABC, there is a circle (possibly, imaginary) with respect to which the triangle ABC is self-polar. As the lines $A'C'$ and $B'C'$ are perpendicular, the triangle $A'B'C'$ is also self-polar with respect to that circle.

Conversely, suppose triangles ABC and $A'B'C'$ are self-polar with respect to a conic. Transform that conic to a circle and C' to its center. Then C' becomes the orthocenter of the triangle ABC, and A' and B' become points at infinity with perpendicular directions. Therefore the equilateral hyperbola passing through A, B, C, C' with asymptotes parallel to those directions is the desired conic.

36. Suppose a conic passing through the points A, B, C and O intersects the line at infinity at points P and Q. Similar to the previous problem, one proves that the triangle POQ is self-polar with respect to the same conic α as the triangle ABC. Hence the fixed points of the involution induced on the line at infinity by the pencil $ABCO$ are the points of intersection with α. But this means that the midpoint conic passing through those points is homothetic to α.

37. *Hint.* Brianchon's theorem implies that the diagonals of the quadrilateral circumscribed about a conic and the lines connecting the tangency points of the conic with its opposite sides intersect at a single point. Applying this observation to the quadrilateral formed by the sides of the triangle ABC and the line at infinity, we have the desired assertion.

38. *Answer.* The center of the homothety is the centroid of the triangle, and the coefficient is 2.

39. The preceding problem implies that the midpoints of the segments connecting the centroid with the vertices lie on the Steiner ellipse. Therefore that ellipse has six common points with the conic of the centers (that this is indeed a conic follows from Theorem 3.17). Those points are the midpoints of the sides and of the segments.

40. Let I_a, I_b and I_c be the centers of the excircles of the triangle ABC. Since ABC is the orthotriangle of the triangle $I_aI_bI_c$, the polars of P with respect to the equilateral hyperbolas passing through I_a, I_b and I_c pass through P'. By Theorem 3.9, the lines AA', BB' and CC' intersect at a single point, which is the pole of the line PI, where I is the center of the incircle of the triangle ABC, with respect to the conic $I_aI_bI_cIP$. But that conic is an equilateral hyperbola so that the obtained point lies on PP'.

41. Any point in the plane can be the perspector of a conic inscribed in the given triangle, and such a conic is unique. It is clear that the conic changes continuously as the perspector moves. Therefore the perspectors of ellipses lie inside the circumscribed Steiner ellipse, whereas the perspectors of hyperbolas lie outside. Now apply Theorem 4.7. Under the isotomic conjugation, the interior of the triangle will transform into itself, and the points of the segment bounded, for example, by the side AB of the triangle and the arc of the ellipse subtended by it and not containing the third vertex C will transform into the points of the angle vertical to the angle C. Accordingly, the set of centers of the inscribed ellipses is the interior of the midline triangle and the three angles vertical to its angles.

42. Let X and Y be the tangency points of the ellipse and the parallel tangents, T the tangency point of the ellipse and the circle, U and V the intersection points of the tangent to the ellipse at T and the parallel tangents, and U' and V' the intersection points of the parallel tangents and some other tangent to the ellipse. Arguing as in the proof of the last theorem, we see that $XU + UV + VY < XU' + U'V' + V'Y$, i.e., U and V lie on an ellipse confocal with the given ellipse. Moreover, OU and OV, where O is the center of the circle, are tangent to that ellipse and $\angle UOV = 90°$. Therefore O lies on the circle centered at the center of the ellipse whose radius equals the sum of the semiaxes of the ellipse.

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