A Short Proof of Lamoen's Generalization of the Droz-Farny Line Theorem

Cosmin Pohoata and Son Hong Ta

Abstract

We give a short proof of a slightly more general version of the Droz-Farny line theorem mentioned by Floor van Lamoen in [5].

1. The Droz-Farny line theorem and Lamoen's generalization

In 1899, Arnold Droz-Farny discovered the following beautiful result, known nowadays as the Droz-Farny line theorem:

Theorem 1 (Droz-Farny). *If two perpendicular straight lines are drawn through the orthocenter of a triangle, they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.*

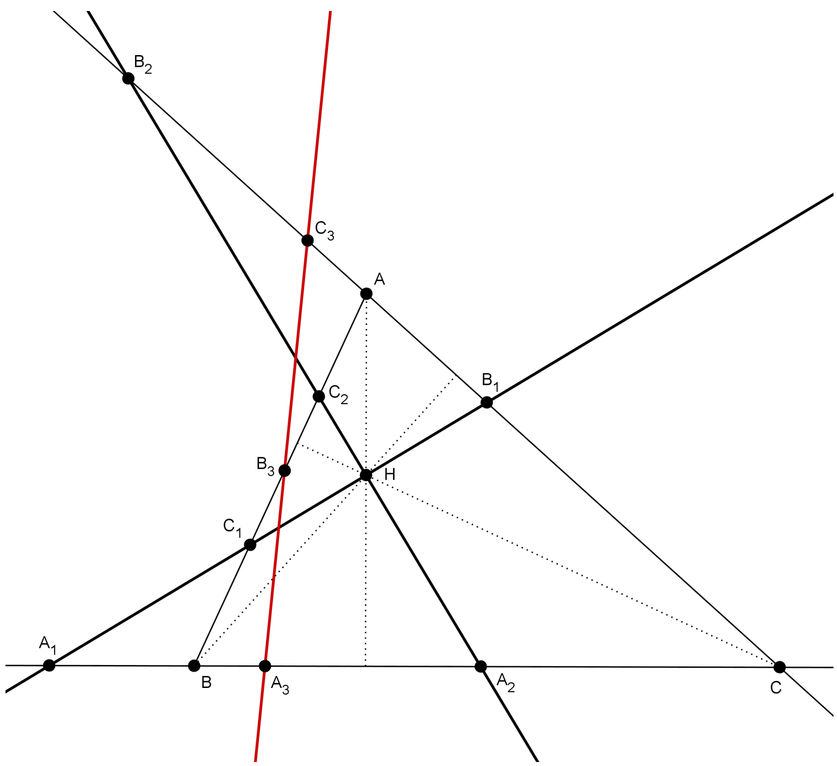

FIGURE 1.

As illustrated in Figure 1, we have denoted by A_1, B_1, C_1, and A_2, B_2, C_2 the intersections points of the two perpendicular lines d_1, d_2 with the sidelines BC, CA, and AB, respectively. The Droz-Farny line theorem states that the midpoints A_3, B_3, C_3 of the segments A_1A_2, B_1B_2, C_1C_2 are collinear. Despite of the simple configuration, the first known proof is the analytical one from [7]. Years later, on the Hyacinthos forum, several proofs were given by N. Reingold [6], D. Grinberg [2], [3], [4] and M. Stevanovic [8]. In 2004,

J. -L. Ayme ends this sequence of proofs by presenting a beautiful synthetic approach [1]. A month before the apparition of Ayme's article, Lamoen [5] mentioned, without proof, the following generalization:

Theorem 2 (Lamoen). If the midpoints of the intercepted segments are replaced by three points A_3, B_3, C_3 dividing into the same ratio the corresponding segments A_1A_2, B_1B_2, and C_1C_2, then A_3, B_3, C_3 remain collinear.

2. Proof of Theorem 2

Denote by e, f the lines through the orthocenter H parallel to AB, AC, respectively. Furthermore, denote by x, y the lines through the vertex A parallel to the lines d_1, d_2, and let X, Y be the intersection points of the sideline BC with x, and y, respectively.

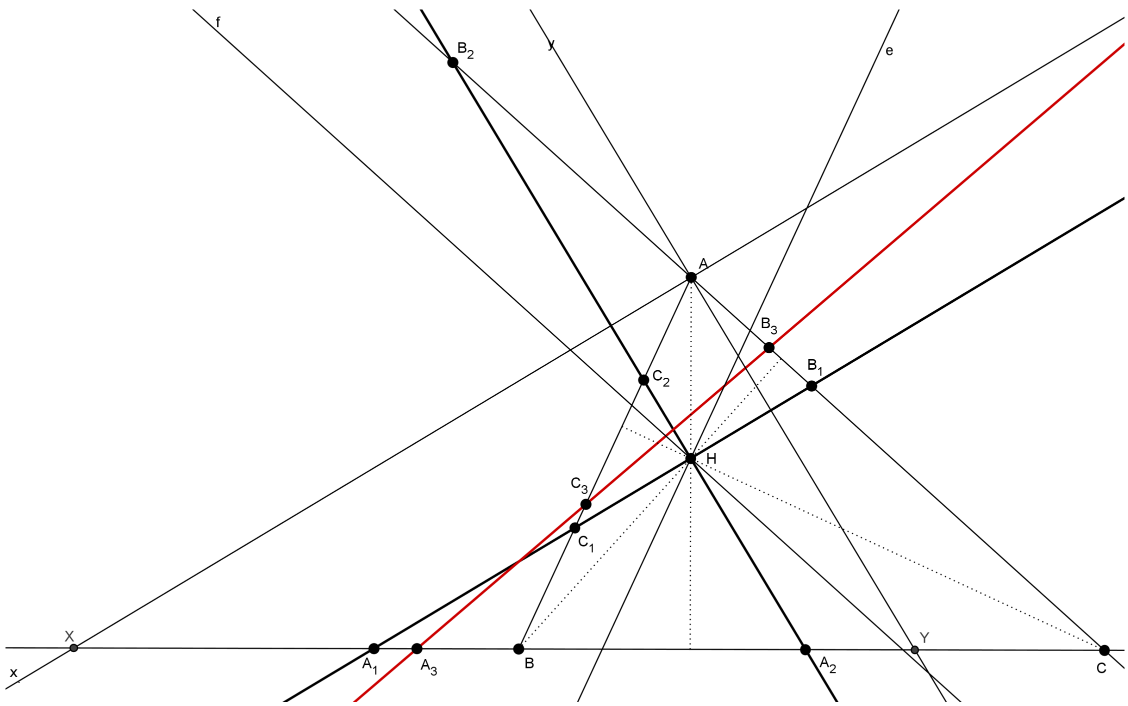

Figure 2.

Since the pencil $(HC_1,\ HC_2,\ HB,\ e)$ is the image of $(HB_2,\ HB_1,\ f,\ HC)$ under the rotation $\Psi(H, +\pi/2)$,

$$\frac{BC_1}{BC_2} = \frac{CB_1}{CB_2} \text{ if and only if } \frac{BC_1}{CB_1} = \frac{BC_2}{CB_2},$$

and thus, by multyplying with AC/AB,

$$\frac{C_1B}{AB} \cdot \frac{AC}{B_1C} = \frac{C_2B}{AB} \cdot \frac{AC}{B_2C}.$$

On other hand, since

$$\frac{C_1B}{AB} = \frac{A_1B}{XB}, \quad \frac{AC}{B_1C} = \frac{XC}{A_1C}, \quad \frac{C_2B}{AB} = \frac{A_2B}{YB}, \quad \frac{AC}{B_2C} = \frac{YC}{A_2C},$$

it follows that

$$\frac{A_1B}{A_1C} : \frac{XB}{XC} = \frac{A_2B}{A_2C} : \frac{YB}{YC},$$

which is equivalent with the congruence of the pencils $(B,\ C,\ A_1,\ X)$ and $(B,\ C,\ A_2,\ Y)$. By intersecting now $(AB,\ AC,\ AA_1,\ AX)$ with d_1 and $(AB,\ AC,\ AA_2,\ AY)$ with d_2, we deduce that

$$\frac{C_1A_1}{C_1B_1} = \frac{C_2A_2}{C_2B_2},$$

the two degenerated triangles $A_1B_1C_1$ and $A_2B_2C_2$ being similar.

For a point P denote by $\mathbf{P}$ the vector $\overrightarrow{XP}$, where X is a fixed point in plane of triangle ABC. Since $C_1A_1/C_1B_1 = C_2A_2/C_2B_2$, there exist two real numbers k, l, satisfying $k + l = 1$, such that

$$\mathbf{C_1} = k\mathbf{A_1} + l\mathbf{B_1}, \quad \mathbf{C_2} = k\mathbf{A_2} + l\mathbf{B_2}.$$

On other hand, since A_3, B_3, C_3 divide the segments A_1A_2, B_1B_2, and C_1C_2, respectively, into the same ratio, there exist two real numbers u, v, satisfying $u + v = 1$, such that

$$\mathbf{A_3} = u\mathbf{A_1} + v\mathbf{A_2}, \quad \mathbf{B_3} = u\mathbf{B_1} + v\mathbf{B_2}, \quad \mathbf{C_3} = u\mathbf{C_1} + v\mathbf{C_2}.$$

Therefore,

$$\begin{aligned} \mathbf{C_3} = u\mathbf{C_1} + v\mathbf{C_2} &= u\left(k\mathbf{A_1} + l\mathbf{B_1}\right) + v\left(k\mathbf{A_2} + l\mathbf{B_2}\right) \\ &= k\left(u\mathbf{A_1} + v\mathbf{A_2}\right) + l\left(u\mathbf{B_1} + v\mathbf{B_2}\right) \\ &= k\mathbf{A_3} + l\mathbf{B_3}. \end{aligned}$$

According to the fact that $k + l = 1$, this implies that the points A_3, B_3, C_3 are collinear. This completes the proof of Theorem 2.

References

[1] J.-L. Ayme, A synthetic proof of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004) 219-224.

[2] D. Grinberg, Hyacinthos messages 6128, 6141, 6245, December 10-11, 2002.

[3] D. Grinberg, Hyacinthos message 7384, July 23, 2003.

[4] D. Grinberg, Hyacinthos message 9845, June 2, 2004.

[5] F. v. Lamoen, Hyacinthos message 10716, October 17, 2004.

[6] N. Reingold, Hyacinthos message 7383, July 22, 2003.

[7] I. Sharygin, *Problemas de Geometria*, (Spanish translation), Mir Edition, 1986.

[8] M. Stevanovic, Hyacinthos message 9130, January 25, 2004.

Cosmin Pohoata: 318 Walker Hall, Frist Center 3533, Princeton, NJ 08544.
E-mail address: apohoata@princeton.edu

Son Hong Ta: 136 Xuan Thuy Street, Cau Giay District, Hanoi, Vietnam.
E-mail address: april.mathlinks@gmail.com