$V_n(r)$ monotone decreasing means

$$\frac{V_{2n}(r)}{V_{2n+1}(r)} > 1 \quad \text{and} \quad \frac{V_{2n-1}(r)}{V_{2n}(r)} > 1.$$

That is, $r < \min(a_n/2, b_n/\pi)$, where

$$a_n = \frac{3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)} \quad \text{and} \quad b_n = \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}.$$

(Note that in this case, $r^2 < a_n b_n / 2\pi = (2n+1)/2\pi$, hence $n + (1/2) > \pi r^2$.) Now $a_n = \prod_{i=1}^n (1 + 1/2i)$ and $b_n = \prod_{i=1}^n (1 + 1/(2i-1))$, hence both are strictly increasing and unbounded (cf [4], p. 32), so for each $r > 0$ there is an N such that $V_n(r)$ decreases for $n > N$. Direct calculation shows, for example, that $V_n(1)$ increases for $n \leqslant 5$ and decreases for $n > 5$, and $V_5(1) = 8\pi^2/15$.

In conclusion, we remark that analogous results are true for the n-area $\sigma_n(r)$ of the n-dimensional sphere $S^n(r)$, and similar arguments may be used since

$$\sigma_{n-1}(r) = \frac{n}{r} V_n(r).$$

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More on Incircles

HÜSEYIN DEMIR
CEM TEZER
Middle East Technical University
Ankara, Turkey

The contents of this note came into being during the authors' search for a "synthetic" proof of the following result by H. Demir (FIGURE 1):

"Consider a triangle ABC and points P, Q on the line segment BC. If the incircles of the subtriangles ABP and AQC are congruent then the incircles of the subtriangles ABQ and APC are congruent."

(Notice that the requirements of the "Five Circle Theorem" ([2]) are partly redundant.)

Singularly enough, this question turned out to be less accessible than a more general result which was conjectured at the very outset of our investigations and later proved by means of the methods which will constitute the body of the present work:

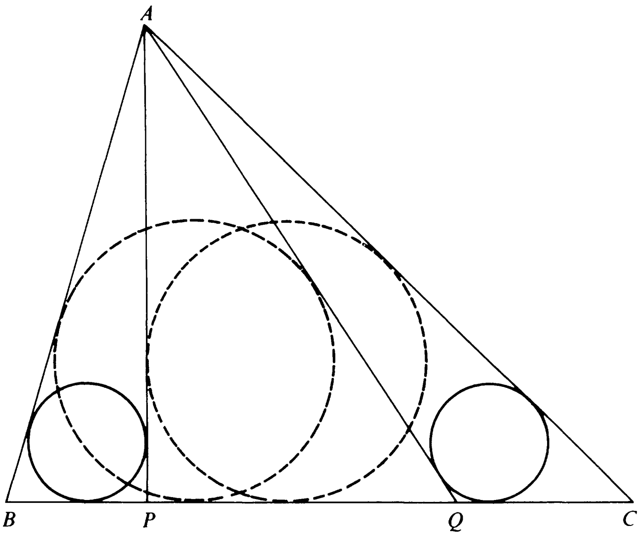

FIGURE 1

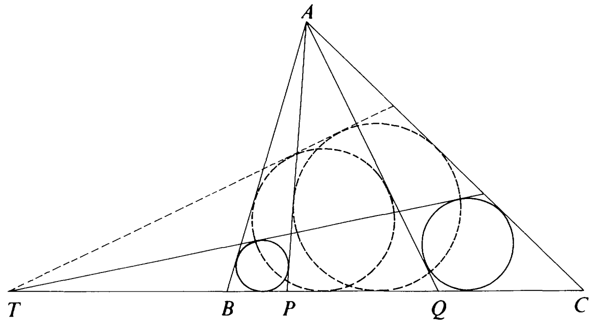

FIGURE 2

PROPOSITION 1 (FIGURE 2). *Consider a triangle ABC and points P, Q on the line segment BC. If T is the external homothety center of the incircles of the subtriangles ABP and AQC, then T is also the external homothety center of the incircles of the subtriangles ABQ and APC.*

It is not difficult to see that Proposition 1 implies the above mentioned result by H. Demir.

The authors found that Proposition 1 and several other interesting results could be obtained merely by a careful elucidation of necessary and sufficient conditions for a convex quadrangle to be circumscriptible.

1. Circumscriptible Quadrangles

In the following we consider a convex quadrangle $ABCD$, that is, a quadrangle which encloses a convex region that has as its boundary, the union of line segments AB, BC, CD, DA. Such a quadrangle will be said to be *circumscriptible* if there exists a circle lying in the convex region enclosed by the quadrangle, touching each side

AB, BC, CD, DA. We shall further exclude the triangle degeneracy by forbidding any three points from among A, B, C, D to be collinear.

The following simple result is quite standard (see p. 135 in [1] for a similar situation) and forms the basis of all our subsequent observations. The proof, which will be left to the reader as a mild challenge can be effected by repeated applications of the congruence of line segments that have one endpoint in common and are tangent to a circle at the other.

LEMMA (FIGURE 3). *Given a triangle AEF and points B, D on the line segments AE, AF, respectively, let ED, FB intersect in C. The following statements are equivalent*:

(i) *The convex quadrangle $ABCD$ is circumscriptible.*

(ii) $|AE| - |AF| = |CE| - |CF|$

(iii) $|BE| + |BF| = |DE| + |DF|$.

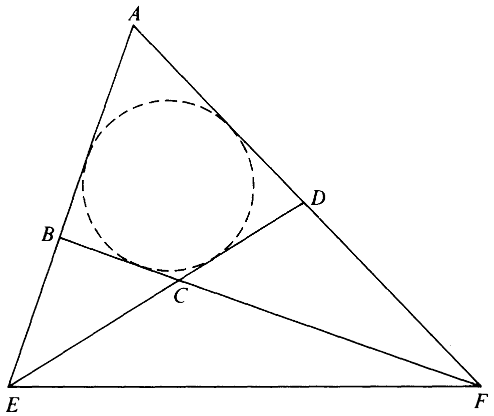

FIGURE 3

2. On the Third Incircle

Let AEF be a triangle, K, B and M, D be points on the line segments AE and AF, respectively, K, M lying nearer A than B, D (FIGURES 4, 5). Let EM and FK, EM and FB, ED and FB, ED and FK intersect in L, P, C, Q, respectively.

PROPOSITION 2 (FIGURE 4). *If any two from among the quadrangles $AKLM$, $ABCD$, $LPCQ$ are circumscriptible, then so is the third.*

Proof. Assume without loss of generality that $AKLM$ and $LPCQ$ are circumscriptible. By the Lemma

$$|AE| - |AF| = |LE| - |LF|$$

as $AKLM$ is circumscriptible and

$$|LE| - |LF| = |CE| - |CF|$$

as $LPCQ$ is circumscriptible. Therefore,

$$|AE| - |AF| = |CE| - |CF|.$$

Consequently, $ABCD$ is circumscriptible.

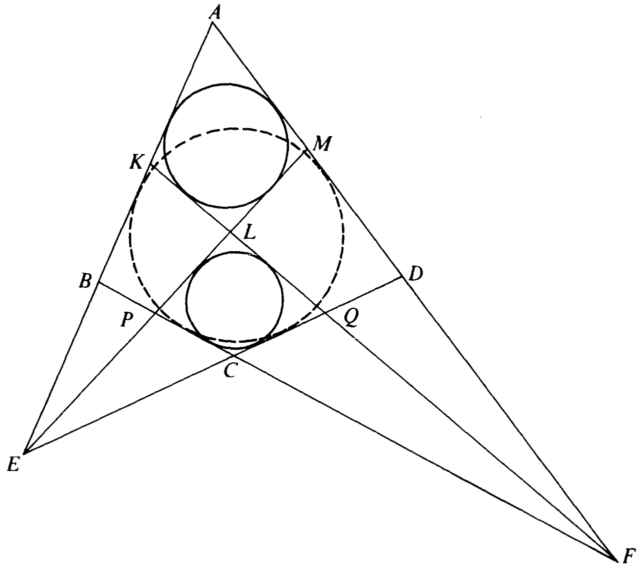

FIGURE 4

PROPOSITION 3 (FIGURE 5). *If any two from among the quadrangles KBPL, ABCD, MLQD are circumscriptible, then so is the third.*

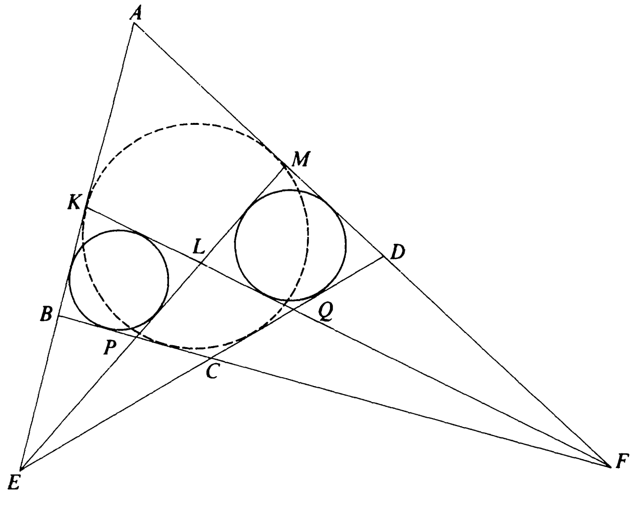

FIGURE 5

Proof. Assume without loss of generality, that $KBPL$ and $MLQD$ are circumscriptible. By the Lemma

$$|BE|+|BF|=|LE|+|LF|$$

as $KBPL$ is circumscriptible. Similarly

$$|LE|+|LF|=|DE|+|DF|$$

as $MLQD$ is circumscriptible. Combining these equations we obtain

$$|BE|+|BF|=|DE|+|DF|.$$

Therefore, $ABCD$ is circumscriptible.

Proof of Proposition 1 (FIGURE 6). Let Γ_1, Γ_2 be the incircles of the subtriangles ABP, AQC, respectively, with external homothety center T. Let the second common tangent of Γ_1, Γ_2 through T intersect AB, AP, AQ, AC in B', P', Q', C', respectively. Let Γ be the incircle of the triangle $AP'Q'$ and the second tangent to Γ through T intersect AB, AP, AQ, AC in B'', P'', Q'', C'', respectively. As $BPP'B'$ and $P'Q'Q''P''$ are circumscriptible so is $BQQ''B''$. Similarly as $QCC'Q'$ and $P'Q'Q''P''$ are circumscriptible so is $PCC''P''$. The incircles of $BQQ''B''$ and $PCC''P''$ are incircles of the subtriangles ABQ and APC respectively. TC'' is obviously their second common tangent through T.

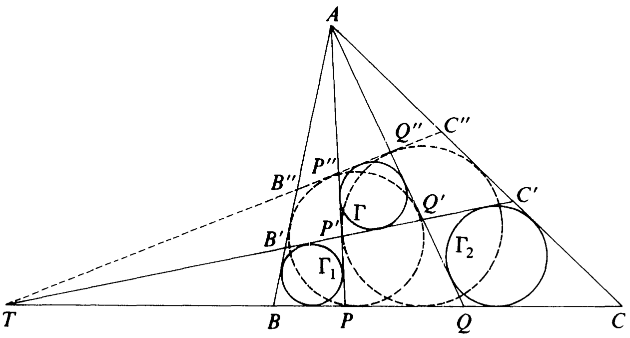

FIGURE 6

3. On the Fourth Incircle

By Propositions (2), (3) it is not difficult to see that if any three of the quadrangles $AKLM$, $KBPL$, $LPCQ$, $MLQD$ are circumscriptible, then so is the fourth (FIGURES 4, 5). A method which is traditionally ascribed to Fedorov but was possibly known to other and earlier mathematicians provides a straightforward proof of a much more general result and a simple relation between the inradii of the quadrangles in question. For a source on Fedorov's method we refer the reader to §78 in [3].

The method of Fedorov concerns the tangency of cycles (circles with orientation) and directed lines. In this connection tangency is expected to respect orientation. For instance, in FIGURE 7 the tangent directed line a goes "with" the cycle, that is, respects its orientation, whereas b does not. The essential idea is to assign to each cycle in the plane a point in space and to each pair of directed lines in the plane (except for well-isolated cases) a line in space in a one-to-one manner such that two

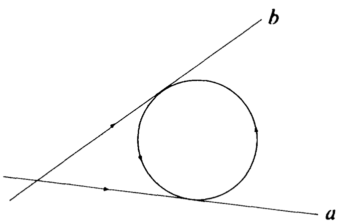

FIGURE 7

pairs of directed lines in the plane have a common tangent cycle if and only if the corresponding pair of lines in space intersect.

This "assignment" or "correspondence" is possibly best described by identifying space with $\mathbb{R}^3$ and the plane with $\mathbb{R}^2$. We assign the point $(a, b, \pm r)$ to the circle $(x-a)^2+(y-b)^2=r^2$, taking the plus sign if the circle is counterclockwise oriented, taking the minus sign if the circle is clockwise oriented. Consider, on the other hand, a pair of directed lines which is not a pair of parallel directed lines with the same direction. It can be routinely checked that the set of points in space corresponding to the cycles tangent to the given pair of directed lines in the plane is a line in space. To each pair of directed lines in the plane which is not a pair of parallel directed lines with the same direction, we assign the line in space specified as above. A simple inspection corroborates that these assignments fulfill the requirements put forth at the beginning of this section.

PROPOSITION 4 (FIGURE 8). *Consider the triangle* A_1EF *and points* B_1, A_2, B_2 *on the line segment* A_1E, *points* D_1, A_4, D_4 *on the line segment* A_1F *in order of increasing distance from* A_1. *Let* ED_1 *intersect* FB_1, FA_2, FB_2 *in* C_1, D_2, C_2; *let* EA_4 *intersect* FB_1, FA_2, FB_2 *in* B_4, A_3, B_3; *and let* ED_4 *intersect* FB_1, FA_2, FB_2 *in* C_4, D_3, C_3, *respectively. The quadrangles* $A_iB_iC_iD_i$, $i=1,2,3,4$ *are circumscriptible if any three of them are. If* $A_iB_iC_iD_i$ *are circumscriptible with inradii* r_i, $i=1,2,3,4$, *then*

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}.$$

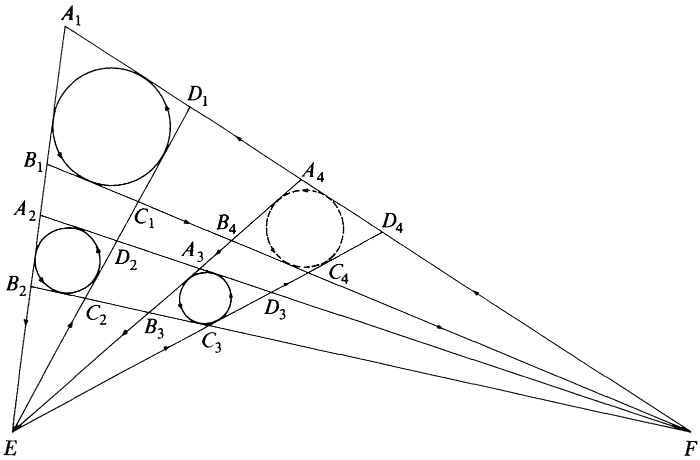

FIGURE 8

Proof. Let $A_iB_iC_iD_i$ be circumscriptible for $i=1,2,3$. Choosing suitable orientations for the circles and lines in question we obtain points K_i in space corresponding to the incircles of $A_iB_iC_iD_i$, $i=1,2,3$, and the points E, F in space corresponding to the point-circles E, F in the plane (FIGURE 9). Then K_1, K_2, E are collinear and K_2 lies between K_1 and E. Similarly K_2, K_3, F are collinear and K_3 lies between K_2 and F. Therefore K_1, K_2, K_3, E, F are coplanar and EK_3 intersects K_1F in a point K_4 between K_1 and F. Hence, $A_4B_4C_4D_4$ is circumscriptible.

Let $A_iB_iC_iD_i$ be circumscriptible with inradii r_i, $i=1,2,3,4$. Let K_1K_3, K_2K_4 intersect EF in L, M, respectively. Choose S, T on EF such that K_2S and K_4T are

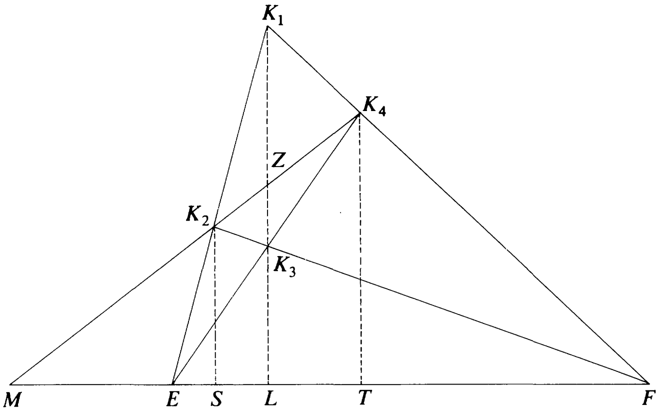

FIGURE 9

parallel to K_1K_3. And r_1, r_2, r_3, r_4 are proportional to $|K_1L|, |K_2S|, |K_3L|, |K_4T|$, respectively. Let K_1K_3 and K_2K_4 intersect in Z. Then K_1, Z, K_3, L form a harmonic division (Ch. 4 in [**5**]).

Hence,

$$\frac{2}{|LZ|} = \frac{1}{|K_1L|} + \frac{1}{|K_3L|}. \tag{1}$$

Similarly K_4, Z, K_2, M form a harmonic division. Hence,

$$\frac{2}{|MZ|} = \frac{1}{|K_2M|} + \frac{1}{|K_4M|}$$

unless M is at infinity, from which we obtain

$$\frac{2}{|LZ|} = \frac{1}{|K_2S|} + \frac{1}{|K_4T|}. \tag{2}$$

Combining (1) and (2) we obtain

$$\frac{1}{|K_1L|} + \frac{1}{|K_3L|} = \frac{1}{|K_2S|} + \frac{1}{|K_4T|}.$$

Consequently,

$$\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}.$$

The above described and employed method of assigning points of $\mathbb{R}^3$ to cycles in $\mathbb{R}^2$ is by no means the only one. We draw the attention of the reader to the "Six Circle Theorem" treated in §94.2 of [4] and in [**6**].

4. Conclusion

Our further investigations led us to a not unamusing but rather disconnected collection of lesser results. Instead of offering a list of them at the risk of incurring the impatience of our readers, we present a diagram which we like to call "the pseudolattice" (FIGURE 10). In the pseudolattice each quadrangle in which the sides are made up of the same number of segments—the pseudosquare, so to speak—is circumscriptible.

Through each lattice point there exist an ellipse and a hyperbola orthogonal to each other with common foci ∞_x, ∞_y which have the property that if they enter a pseudosquare by one vertex, they leave the same by the opposite vertex.

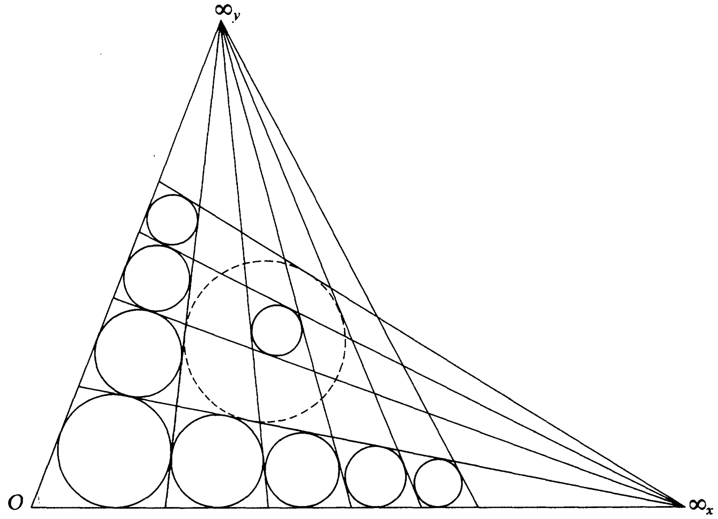

FIGURE 10

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I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto.

—Francis Bacon