Two Applications of the Generalized Ptolemy Theorem

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1. INTRODUCTION. The classical theorem of Ptolemy (used for computations by Claudius Ptolemaeus of Alexandria, 2nd century AD, but probably known even before him [**7**, pp. 123–124]) states that if A, B, C, D are, in this order, four points on the circle O, then $|AB| \times |CD| + |AD| \times |BC| = |AC| \times |BD|$. This result includes Pythagoras' Theorem when applied to a rectangle, and yields other important trigonometric identities when applied to special cyclic quadrilaterals $ABCD$. For example, if the diagonal AC is a diameter, $|AC| = 1$, $\angle DAC = \alpha$, and $\angle BAC = \beta$, then Ptolemy's Theorem gives $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$. In a more general context, Ptolemy's Theorem is the inequality $|AB| \times |CD| + |AD| \times |BC| \geq |AC| \times |BD|$, where equality occurs if and only if A, B, C, D lie (in this order) on a circle or on a line. This formulation of the theorem can be viewed as the "image under inversion" of the triangle inequality [**11**, p. 89].

Ptolemy's Theorem can be generalized to the situation where the points A, B, C, D are replaced by four circles that touch O. Our goal is to show how the Generalized Ptolemy Theorem can be used to prove two results in plane geometry.

The first result, Theorem 1, is a generalization of a theorem that was originally proposed in 1938, as a MONTHLY problem, by the French geometer Victor Thébault [**15**]. Thébault's Theorem remained an open problem (allegedly a tough one, see [**10**, p. 70–71]) for some 45 years, until it was proved in 1983 by Taylor [**16**]. Taylor's proof used analytic geometry and involved lengthy computations. It was therefore cited only in abbreviated form. A shorter (trigonometric) proof for Thébault's Theorem was found in 1986 by Turnwald [**17**], and a synthetic proof (which also generalizes the problem) was found by Stärk in 1989 [**14**]. Another (still lengthy) proof based on analytic geometry, in which computer algebra software is employed for carrying out the computations, was recently proposed by Shail [**12**]. Our generalization of Thébault's Theorem is the following.

Theorem 1 (External Thébault Theorem). *Let $\triangle ABC$ be a triangle, let (O, R) be its circumcircle, and let D be a point on BC such that $\angle BDA = \theta$; see Figure 2b. Let (O_1, r_1) be a circle touching BC (produced) at P, touching AD (produced) at M, and touching O (externally). Let (O_2, r_2) be a circle touching BC (produced) at Q, touching AD (produced) at N, and touching O (externally). Let (I_a, r_a) be the excircle of $\triangle ABC$, touching the side BC and the sides AB, AC (produced). Then:*

- *(a)* $r_a = r_1\cos^2(\theta/2) + r_2\sin^2(\theta/2)$.
- *(b) The lines l_1 and l_2 passing through P and Q, respectively, parallel to AD, are tangent to the excircle (I_a, r_a).*
- *(c) O_1, O_2, and I_a are collinear.*
- *(d)* $O_1I_a/O_2I_a = \tan^2(\theta/2)$.

Theorem 1 is a dual (an external form) of Thébault's Theorem. In the latter, (O_1, r_1) and (O_2, r_2) touch the sides BD and CD (at internal points on the respective sides) and touch O *internally*; see Figure 2a. (I, r) is the incircle of $\triangle ABC$, and the statement claims that $r = r_1 \cos^2(\theta/2) + r_2 \sin^2(\theta/2)$; O_1, O_2, and I are collinear; and $O_1I/O_2I = \tan^2(\theta/2)$. Part of the original formulation of the problem in [**15**] (namely, that $r_1 + r_2 = r^2 \sec^2(\theta/2)$) was erroneous.

Special cases of Thébault's Theorem have appeared as problems. The case in which the touching points M and N coincide was discussed in [**13**]. The author used the Generalized Ptolemy Theorem to prove this simplified version of Thébault's Theorem, noticed the natural relation to the more general case, and stated that he was not able to extend this technique to prove it. The case in which $\triangle ABC$ is isosceles and $\theta = 90°$ appeared in [**3**, problem 2.3.5]. The case in which $\theta = 90°$ appeared in [**4**]. The solutions [**5**] and [**6**] considered the general case and gave references to proofs of Thébault's Theorem published (in Dutch) prior to Taylor's proof.

Our second result generalizes the following well-known property: *Let the circles c_1, c_2 touch each other externally and touch the circle c internally, and let their common internal tangent intersect c at the point X. Then, the tangent to c that passes through X is parallel to the common external tangent of c_1 and c_2 that lies on the same side of c_1, c_2 as X.* We extend this geometric situation to the case where c_1 and c_2 are *disjoint.*

Theorem 2 (The Parallel Tangent Theorem). *Let c be a given circle, and let c_1 and c_2 be two disjoint circles touching c: (a) internally (as in Figure 3a); (b) externally (as in Figure 3b); (c) from different sides (as in Figure 3c). Denote the common (a) internal; (b) internal; (c) external tangents of c_1 and c_2 by L_1 and L_2, respectively, and denote their points of intersection with c by B and A, respectively. Let L_3 be a common (a) external; (b) external; (c) internal tangent to c_1 and c_2 that lies on the same side as A and B. Then AB is parallel to L_3.*

Before proving Theorems 1 and 2, we first review the Generalized Ptolemy Theorem.

2. ON THE THE GENERALIZED PTOLEMY THEOREM. The Generalized Ptolemy Theorem was first stated by John Casey as early as 1881 [**1**] (in [**3**, p. 120], the statement is dated 1857), although there is some indication [**3**, p. 120] that it was known in Japan even before Casey. The complete statement of the Generalized Ptolemy Theorem involves several cases, and Casey's original statement did not sufficiently restrict the discussion of these cases; see [**1**, p. 101] and remarks in [**2**, Theorem 45, pp. 37–38]. On the other hand, other authors only partially formulate the Generalized Ptolemy Theorem. For example, see [**2**] and the related comments in [**8**, p. 124] and [**7**, p. 129].

The first complete discussion of the Generalized Ptolemy Theorem is probably due to Johnson [**8**, pp. 121–127].

Notations: We identify circles by giving their center, or by giving their center and radius. Suppose that O is a nondegenerate circle and that the two circles O_1 and O_2, neither of which includes the other, touch O at two distinct points. The notation $\widehat{O_1O_2}$ denotes the length of (1) the common external tangent of O_1 and O_2 (i.e., O_1 and O_2 lie on the same side of this tangent) if both of circles touch O from the same side; (2) the common internal tangent of O_1 and O_2 (i.e., O_1 and O_2 lie on opposite sides of this tangent) if O_1 and O_2 touch O from opposite sides; (3) the tangent to O_2, extended from O_1, if the circle O_1 is degenerate (i.e., zero radius); or (4) the line segment O_1O_2 if both O_1 and O_2 are degenerate circles.

Theorem 3 (The Generalized Ptolemy Theorem). *Let O be a nondegenerate circle, and let A, B, C, D be, in this order, four distinct points on O. Let (O_1, R_1), (O_2, R_2), (O_3, R_3), (O_4, R_4) be four circles touching O at the points A, B, C, D, respectively. Then*

$$\widehat{O_1O_2} \times \widehat{O_3O_4} + \widehat{O_1O_4} \times \widehat{O_2O_3} = \widehat{O_1O_3} \times \widehat{O_2O_4}. \qquad (1)$$

A proof based on inversion is found in [**8**, pp. 121–127]. We give an elementary proof in the Appendix.

Remarks:

1. The Generalized Ptolemy Theorem involves six configurations, depending on the relative positions of the four circles with respect to the circle O, i.e., depending on whether they touch O externally or internally; see Figure 1.

   An additional configuration arises when the four circles O_1, O_2, O_3, O_4 include O (i.e., when O touches them internally). The analogous result remains true in this case as well. However, it is not true when *some but not all* of the four circles include O.
2. The theorem is valid with any combination of degenerate circles. In particular, Ptolemy's Theorem is the special case in which the circles O_1, O_2, O_3, O_4 are degenerate.

   A different generalization of the Ptolemy theorem involves four circles O_1, O_2, O_3, and O_4 that touch the fifth circle O externally, *and also* touch a sixth circle internally [**9**].
3. Relation (1) remains true (but is less interesting) when O is a line or a point.
4. The converse of the Generalized Ptolemy Theorem is also true: if the relation (1) holds for four circles O_1, O_2, O_3, and O_4, then there exists a fifth circle O that touches all of them (with the appropriate combination of internal/external tangency).

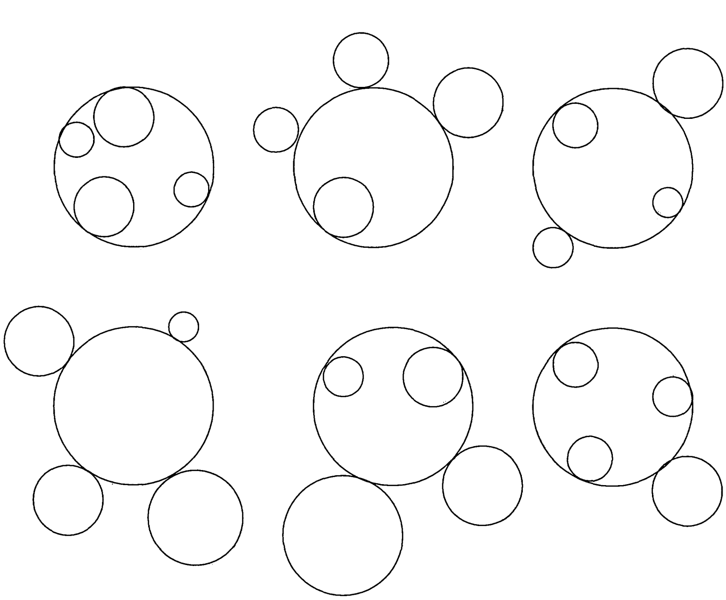

Figure 1. The six configurations of internal/external tangency for the Generalized Ptolemy Theorem.

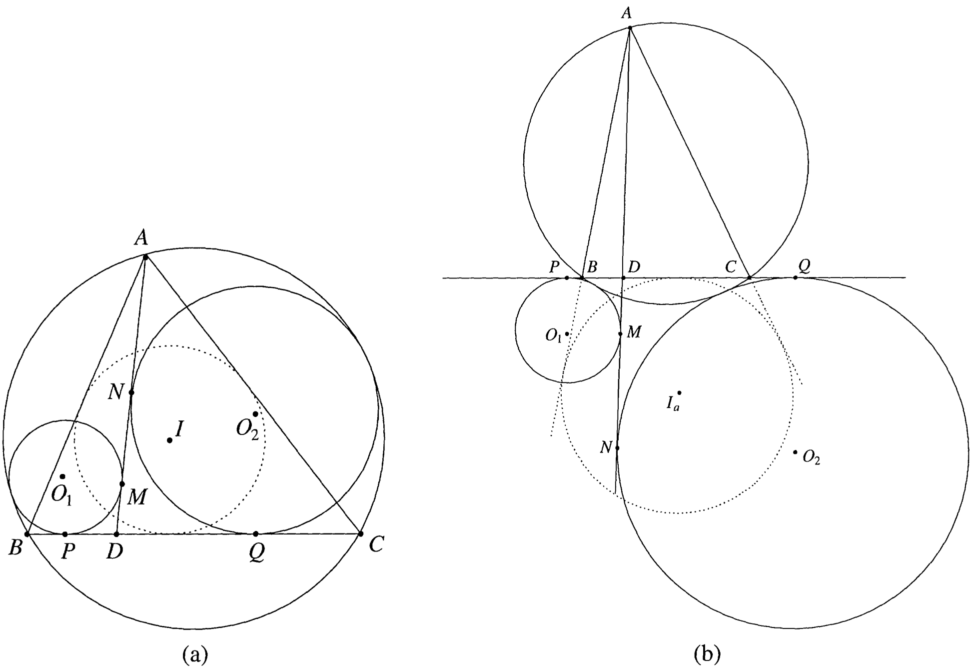

Figure 2. (a) Thébault's Theorem; (b) External Thébault Theorem.

3. PROOF OF THEOREM 1 (EXTERNAL THÉBAULT'S THEOREM). Denote $|AB| = c$, $|BC| = a$, $|CA| = b$, $|DP| = x$, $|DQ| = y$. Use the Generalized Ptolemy Theorem with A, B, O_1, C, and the identity $|DM| = |DP|$ to obtain

$$c(|CD| + x) + b(x - |BD|) = a(|AD| + x),$$

which, after rearranging, reduces to

$$x(c + b - a) = a|AD| + b|BD| - c|CD|. \tag{2}$$

Similarly, using the Generalized Ptolemy Theorem with A, B, O_2, C, we get

$$y(c + b - a) = a|AD| + c|CD| - b|BD|. \tag{3}$$

Adding (2) and (3) gives

$$(c + b - a)(x + y) = 2a|AD|.$$

Since $\cot(\theta/2) = r_1/x$ and $\tan(\theta/2) = r_2/y$, we obtain

$$(c + b - a)\left(r_1 \cot\frac{\theta}{2} + r_2 \tan\frac{\theta}{2}\right) = 2a|AD|. \tag{4}$$

Multiplying (4) by $\sin(\theta/2)\cos(\theta/2)$ gives

$$(c + b - a)\left(r_1 \cos^2\frac{\theta}{2} + r_2 \sin^2\frac{\theta}{2}\right) = a|AD|\sin\theta = 2S_{ABC}$$

and, finally,

$$r_1 \cos^2\frac{\theta}{2} + r_2 \sin^2\frac{\theta}{2} = \frac{2S_{ABC}}{c+b-a} = r_a.$$

This completes part (a) of Theorem 1.

To prove (b) we denote the intersection point of the line l_1 with AB (produced) by R (not shown in Figure 2). Using the Generalized Ptolemy Theorem with A, B, O_1, C gives

$$b|BP| + c(a + |BP|) = a(|AD| + x + |BP|),$$

or, after rearranging,

$$|BP|(b+c-a) = a(|AD| + x - c).$$

Since $\triangle BPR \sim \triangle BDA$, we have $|PR|/|AD| = |BP|/|BD| = |BR|/|BA|$, and therefore

$$|PR| = \frac{|BP| \times |AD|}{x}, \qquad |BR| = \frac{|BP| \times c}{x}.$$

We denote the exradius of $\triangle PBR$, with respect to the side BR, by r_a'. To complete the proof of (b), it is sufficient to show that $r_a' = r_a$. To this end, we note that

$$r_a' = \frac{2S_{PRB}}{|PB| + |PR| - |BR|}.$$

Now,

$$\frac{S_{PRB}}{S_{ABD}} = \frac{|BP|^2}{x^2} \quad \Rightarrow \quad S_{PRB} = \frac{|BP|^2}{x^2} \times S_{ABD}.$$

On the other hand, $S_{ABD}/S_{ABC} = x/a$, which implies

$$S_{PRB} = \frac{|BP|^2}{ax} \times S_{ABC}.$$

Finally, we get

$$r_a' = \frac{2|BP|^2 \times S_{ABC}}{ax\left(\frac{|BP|\times|AD|}{x} + |BP| - \frac{c\times|BP|}{x}\right)} = \frac{|BP| \times 2S_{ABC}}{a(|AD| + x - c)} = \frac{2S_{ABC}}{c+b-a} = r_a,$$

which completes our proof of part (b).

Let T be the foot of the perpendicular from I_a to BC. From (b), we have $\angle TPI_a = (\pi - \theta)/2$, and $\angle TQI_a = \theta/2$. Thus, to prove (c), it is enough to show that

$$\frac{|PT|}{|QT|} = \frac{(r_1 - r_a)}{(r_a - r_2)}.$$

We have $|PT| = r_a \tan(\theta/2)$ and $|QT| = r_a \cot(\theta/2)$, and finally, using (a), we obtain

$$\frac{r_1 - r_a}{r_a - r_2} = \frac{r_1\cos^2\frac{\theta}{2} + r_1 \sin^2\frac{\theta}{2} - r_a}{r_a - r_2\sin^2\frac{\theta}{2} - r_2\cos^2\frac{\theta}{2}} = \frac{(r_1 - r_2)\sin^2\frac{\theta}{2}}{(r_1 - r_2)\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2} = \frac{|PT|}{|QT|}.$$

The verification of (d) is straightforward. ∎

Remark: This method of proof can be used to prove Thébault's Theorem (compare with [**5**] or [**6**]). It then adds the following propetry (analogous to part (b) of Theorem 2), which extends the original Thébault's Theorem: the lines l_1 and l_2 that are parallel to AD and pass through the tangency points P and Q, respectively, are tangent to the incircle (I, r); see Figure 2b.

4. PROOF OF THEOREM 2 (THE PARALLEL TANGENT THEOREM). We prove here case (a) (Figure 3a), in which the two circles c_1 and c_2 touch c internally. Proofs for the other two cases (Figures 3b, 3c) are similar.

Let K_1 and T_1 denote the points where L_1 touches c_1 and c_2, respectively. Let K_2 and T_2 denote the points where L_2 touches c_1 and c_2, respectively.

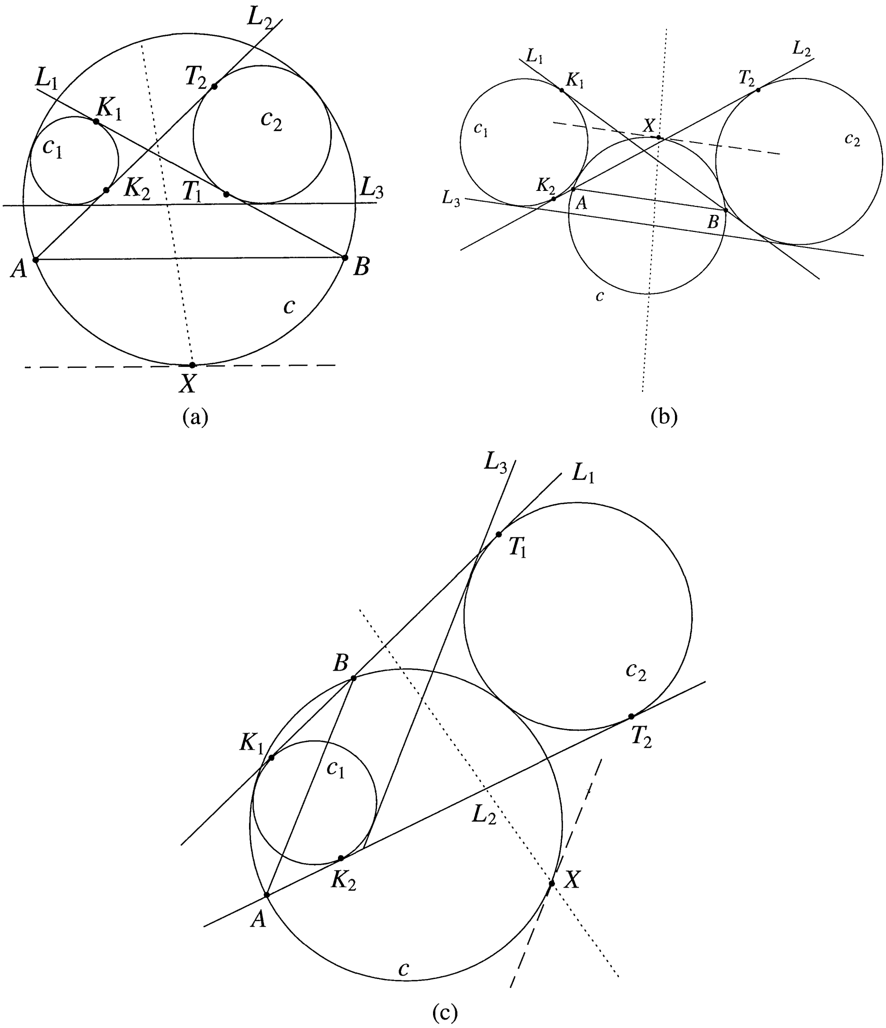

Figure 3. Parallel Tangent Theorem.

Lemma 1. *Let X be the point where c intersects the radical axis (dotted lines in Figure 3) of c_1 and c_2. Then the tangent to c that passes through X (dashed lines in*

Figure 3) is parallel to the external tangent to c_1 and c_2 that lies on the same side of AB as X. (The radical axis of two circles is the geometric locus of the points from which the tangents to these two circles have equal lengths).

Proof. Since X lies on the radical axis of c_1 and c_2, there exists an inversion through X as a center under which c_1 and c_2 are invariant. Since c is tangent to both c_1 and c_2, its image under the inversion (which is a line) is also tangent to c_1 and c_2. Hence this image is the common tangent of c_1 and c_2 that lies on the same side as X. The tangent to c that passes through X is invariant under this inversion. Since it is tangent to c, it is parallel to the image of c, namely, to the common tangent of c_1 and c_2. ■

Remark: To handle the two remaining configurations, we need to modify Lemma 1. For the case illustrated in Figure 3b, the external tangent lies on the opposite side of X. For the case illustrated in Figure 3c, we need to consider the internal tangent that lies on the same side of AB as X.

Proof of the Theorem: Denote the common length of the tangents from X to c_1 and to c_2 by t. Using the Generalized Ptolemy Theorem with A, X, B, c_1 gives

$$|AX| \times |BK_1| + |BX| \times |AK_2| = |AB| \times t, \tag{5}$$

Using the Generalized Ptolemy Theorem with A, X, B, c_2 gives

$$|AX| \times |BT_2| + |BX| \times |AT_1| = |AB| \times t. \tag{6}$$

Subtracting (5) from (6), and using the identity $|T_1K_1| = |T_2K_2|$, we get $|AX| = |BX|$. Hence, the tangent to c that passes through X is parallel to AB. Finally, invoking Lemma 1 completes the proof. ■

Because the Generalized Ptolemy Theorem includes the various configurations detailed in the statement of Theorem 3 (see Figure 1), the proof of Theorem 2 holds also for the cases where c_1 and c_2 touch c externally or from different sides.

Appendix: a Proof of Theorem 3 (Generalized Ptolemy Theorem).

Lemma 2. *Suppose that (O_1, R_1) and (O_2, R_2) are two given circles, not containing one another. Then the length of their common external tangent is*

$$\sqrt{(O_1O_2)^2 - (R_1 - R_2)^2}.$$

If the circles are disjoint, the length of their common internal tangent is

$$\sqrt{(O_1O_2)^2 - (R_1 + R_2)^2}.$$

Proof. The external tangents indeed exist, since neither circle contains the other. The internal tangents exist when the circles are disjoint. The required expression for the length of the common tangents follows from Pythagoras' Theorem. The statement holds when one or both circles are degenerate. ■

Lemma 3. *Let (O, R) be a non-degenerate circle, and suppose the two circles (O_1, R_1) and (O_2, R_2) touch it at the distinct points P_1 and P_2, respectively. Suppose that O_1 and O_2 do not include O. Then:*

(a) *If O_1 and O_2 touch O externally, then*

$$\widehat{O_1 O_2} = \frac{|P_1 P_2|}{R}\sqrt{(R+R_1)(R+R_2)}.$$

(b) *If O_1 and O_2 touch O internally, then*

$$\widehat{O_1 O_2} = \frac{|P_1 P_2|}{R}\sqrt{(R-R_1)(R-R_2)}.$$

(c) *If O_1 touches O internally and O_2 touches O externally, then*

$$\widehat{O_1 O_2} = \frac{|P_1 P_2|}{R}\sqrt{(R+R_1)(R-R_2)}.$$

Proof. Apply the cosine law to the triangles $\triangle O O_1 O_2$ and $\triangle O \widehat{P_1 P_2}$, and use Lemma 2. The given conditions guarantee that the expressions for $\widehat{O_1 O_2}$ are well defined. ■

A proof of the Generalized Ptolemy Theorem. We need to consider six cases, according to the various configurations of external and internal tangency of the four circles illustrated in Figure 1.

We demonstrate the case in which two adjacent circles touch O externally and the other two touch it internally. Ptolemy's Theorem ensures that

$$|AB| \times |CD| + |AD| \times |BC| = |AC| \times |BD|. \tag{7}$$

Multiply both sides of (7) by $\sqrt{(R+R_1)(R+R_2)(R-R_3)(R-R_4)}/R^2$. The desired equality (1) follows from Lemma 3.

Proofs of (1) for the other cases can be constructed in a similar fashion: apply Ptolemy's Theorem to the cyclic quadrilateral $ABCD$, multiply the resulting expression by the appropriate combination of $\sqrt{(R \pm R_1)(R \pm R_2)(R \pm R_3)(R \pm R_4)}/R^2$, and use Lemma 3. ■

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A Short Proof of the Lebesgue Number Theorem

Lebesgue Number Theorem. For each open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of a compact set A in a metric space (X, d) there exists a number $\delta > 0$ with the following property: corresponding to each point p of A there is a member U of $\mathcal{U}$ that contains the open metric ball $B(p, \delta)$. (Such a number δ is called a *Lebesgue number* for $\mathcal{U}$.)

Proof. For each point p of A, let $\varepsilon_p > 0$ be chosen so that $B(p, \varepsilon_p)$ lies in some U_α. Let $\{B(p_i, \varepsilon_{p_i})\}_{i=1,\dots,n}$ be a finite subcover of A by members of $\{B(p, \varepsilon_p) : p \in A\}$. Then $\{B(p_i, \varepsilon_{p_i} - k^{-1})\}_{i=1,\dots,n;\, k=1,2,3,\dots}$ is an open cover of A. This cover has a finite subcover $\{B(p_i, \varepsilon_{p_i} - k_i^{-1})\}_{i=1,\dots,n}$. Clearly

$$\delta = \min_{i=1,\dots,n} \left\{ \frac{1}{k_i} \right\}$$

is a Lebesgue number for U.

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