Back to Eucldiean Geometry: Droz-Farny Demystified

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1. Introduction

The subject of this geometrical excursion is a theorem that has been already the focus of a recent paper here in Mathematical Reflections. Nevertheless, we keep returning to this beautiful theorem of Droz-Farny for its interesting history of complicated proofs that has made it one of the most popular gems in Euclidean Geometry. Its statement is as follows.

THEOREM A (Droz-Farny). *Two perpendicular lines are drawn through the orthocenter of a triangle. They intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.*

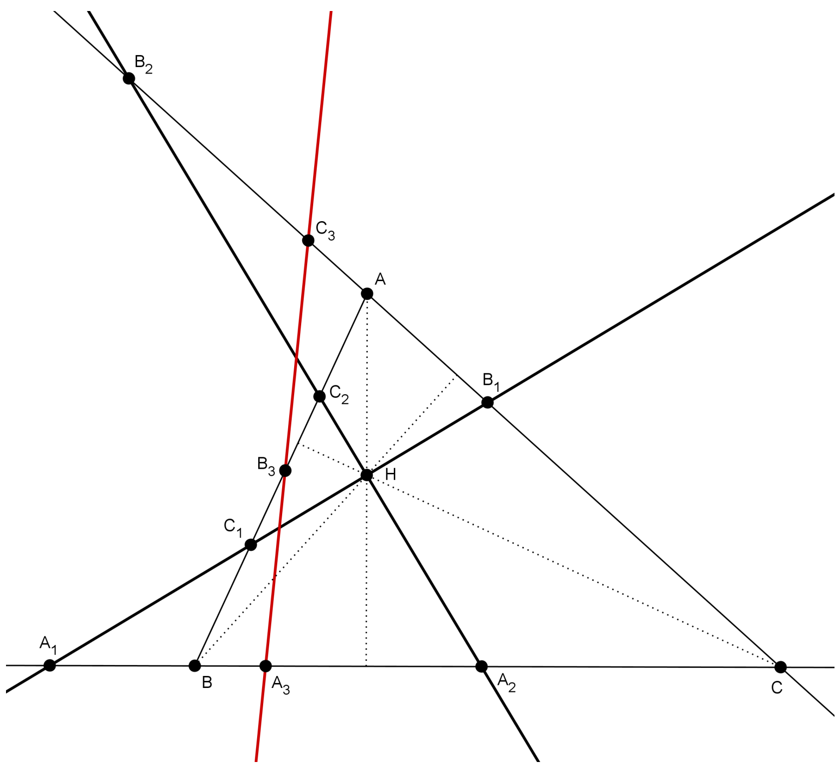

FIGURE 1.

As illustrated in Figure 1, we denote by A_1, B_1, C_1, and A_2, B_2, C_2 the intersections of the two perpendiculars d_1, d_2 with BC, CA, AB, respectively. The Droz-Farny theorem states that the midpoints A_3, B_3, C_3 of the segments A_1A_2, B_1B_2, C_1C_2 are collinear. Arnold Droz-Farny stated this without proof in [2], and, despite the simple configuration, it was only about a century later that the geometry world saw a full proof in

Sharygin's classical *Problemas de Geometria*. Years later, on the Hyacinthos forum, the topic was revived and several computational proofs were given by N. Reingold [6], D. Grinberg [4], and M. Stevanovic [8]. In 2004, J. -L. Ayme finally concretized in some sense these proofs by presenting the first synthetic solution (see [1]) - which, although beautiful, did not seem quite natural for it involved some tricky additional constructions. Generalizations and variations around the simple hypothesis have also appearead since then. See for example [3]. The reader is invited to consult Ayme's paper for a more detailed history and for a brief, yet very interesting, bibliographical note on Arnold Droz-Farny himself.

We will now pass to what we bring new to this picture!

2. Main Result

In this section we will prove the following simple result that generalizes Droz-Farny's theorem and which we think lies at the heart of the theorem itself.

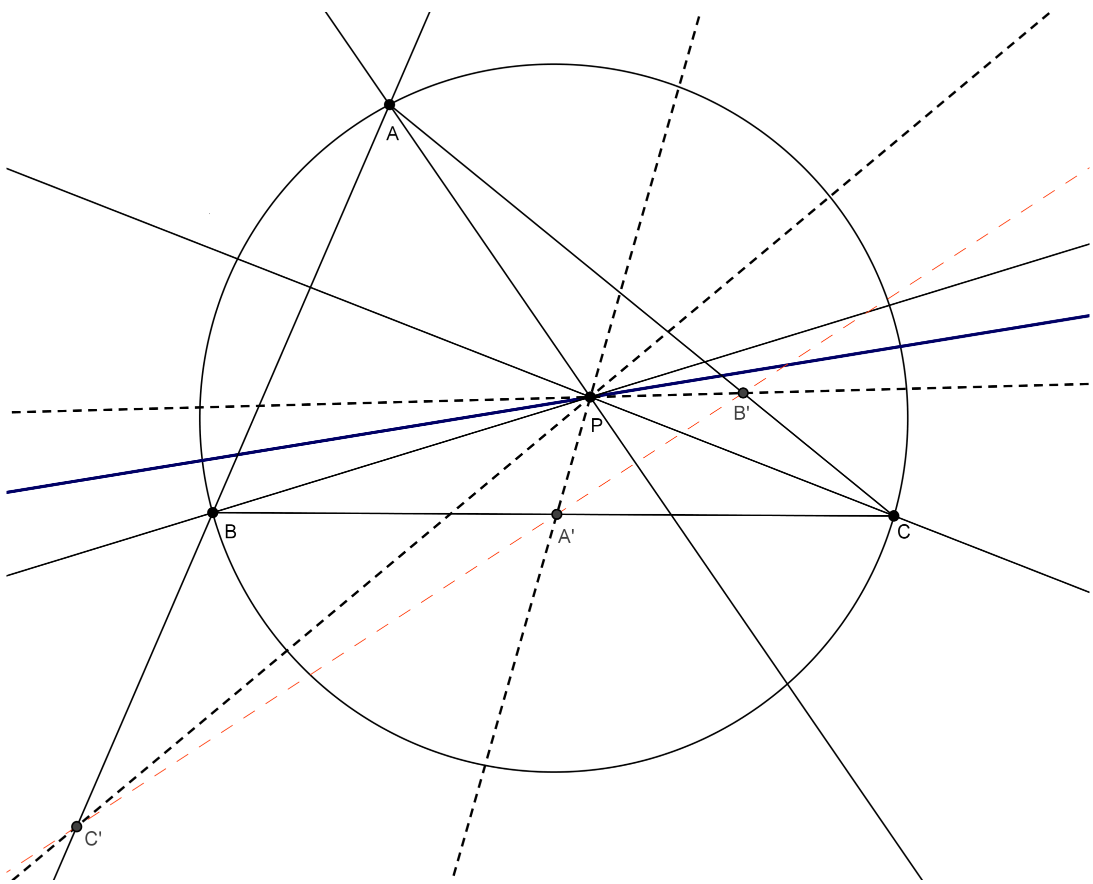

FIGURE 2.

THEOREM B. Let P be a point in the plane of a given triangle ABC. If A', B', C' are the points where the reflections of lines PA, PB, PC in a given line through P meet the sides BC, CA, AB, then A', B', and C' are collinear.

Surprise, surprise! This is Problem 5 from the recent USA Mathematical Olympiad (and Problem 6 of the USA Junior Mathematical Olympiad). So, stop for a few seconds and try to figure out how this is related to Droz-Farny!

In any case, we would like to prolong the suspense by passing to the proof of this result rather than offering more details at this point. We first need a basic preliminary result.

Lemma. The circumcenter of the triangle determined by the reflections of a point P across the sidelines of a triangle ABC is the isogonal conjugate of the point with respect to triangle ABC.

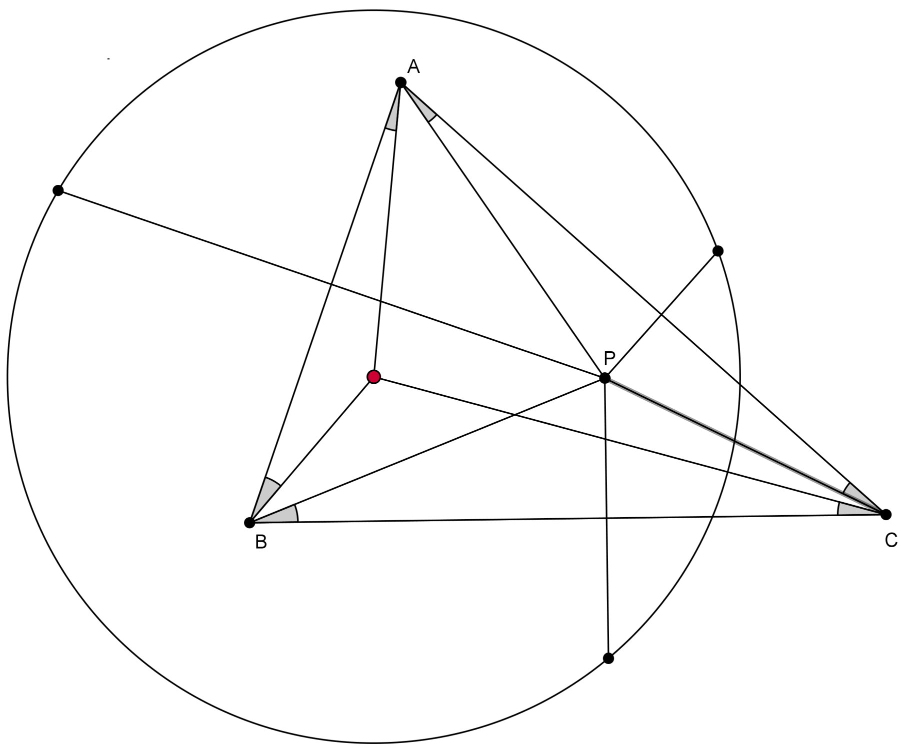

FIGURE 3.

In other words, this point is the intersection of the reflections of PA, PB, PC in the corresponding internal angle bisectors of triangle ABC. The proof is immediate, a simple consequence of the fact that the vertices A, B, C of the original triangle lie on the line bisectors of the triangle determined by the reflections of P across the sidelines. For more details, the reader is advised to consult for example [5].

Now, we move to the proof itself!

Proof of Theorem B. We split it into two parts according to the position of P with respect to the circumcircle of ABC.

First, if P lies on the circumcircle, then everything is a simple angle chase! Indeed, since $\angle A'BC' = \angle CBA = \angle CPA = \angle A'PC'$, it follows that points P, A', B, C' are concyclic, and similarly for P, A, B', C', and P, A', B', C. Hence $\angle CA'B' = \angle CPB' = \angle BPC' = \angle BA'C'$, so A', B', C' are collinear.

Now, if P is not on the circumcircle of triangle ABC, then let Q be its isogonal conjugate with respect to triangle ABC and let Q' be the isogonal conjugate of P with respect to triangle $AB'C'$. Note that Q is not degenerate - i.e. at infinity - as P does not lie on the circumcircle.

Claim. $Q = Q'$.

Proof of Claim. This represents the key step of the proof. To begin, note that

$$\begin{aligned}\angle BQC &= \angle BAC + \angle CPB \text{ (since } P \text{ and } Q \text{ are isogonal conjugates in } ABC\text{)}\\ &= \angle C'AB' + \angle B'PC'\\ &= \angle C'Q'B' \text{ (since } P \text{ and } Q \text{ are isogonal conjugates in triangle } AB'C'\text{)}.\end{aligned}$$

Denote by X, Y, Z the reflections of P in the sides BC, CA, AB, and by X' its reflection in the side $B'C'$ of triangle $AB'C'$. Then $\angle ZXY = \angle BQC$ (because QC is orthogonal to XY and QB is orthogonal to XZ), whereas $\angle ZX'Y = \angle C'Q'B'$ (because $Q'B'$ is orthogonal to $X'Y$ and $Q'C'$ is orthogonal $X'Z$). So, since $\angle C'Q'B' = \angle BQC$, we get $\angle ZXY = \angle ZX'Y$. From this it follows that X, Y, Z, and X' are concyclic. Nonetheless, the center of the XYZ-circle is Q, while the center of the $X'YZ$-circle is Q' (by the Lemma). Thus, Q and Q' do indeed coincide, proving our Claim. □

Next, in a similar way, we can deduce that Q is also the isogonal point of P with respect to triangles $A'BC'$ and $A'B'C$ (by cyclicity). Therefore,

$$\begin{aligned}\angle BC'A' &= \angle AC'A' = \angle AC'P + \angle PC'Q + \angle QC'A'\\ &= \angle QC'B' + \angle PC'Q + \angle BC'P\\ &= \angle BC'B'\\ &= \angle AC'B'.\end{aligned}$$

This means that A', B', C' are collinear, thus completing the proof. ■

3. Concluding Remarks

Simple enough, right? Other synthetic solutions using Simson's theorem or even computation approaches via Menelaus are possible. But again, what does this have to do with Droz-Farny's theorem? Let us take look at the following

Reformulation. Let γ and δ be lines passing through a given point P in the plane of a triangle ABC. Let A', B', C' and A'', B'', C'' be the intersections of γ and δ with BC, CA, AB, respectively. Furthermore, let X be the intersection of BC with the reflection of AP into the internal angle bisector of $\angle A'PA''$, and similarly, define Y and Z. Then points X, Y, Z are collinear.

Things might appear brighter now! Of course, this new statement involves the redundant definition of A', B', C', A'', B'', C'', but it points out the remarkable fact that Theorem A is nothing but a special case of Theorem B, where we take P to be the orthocenter of ABC and the lines γ and δ to be perpendicular! Why is that precisely? Lines AP, BP, CP in this case, besides being the altitudes of triangle ABC, become altitudes in the triangles $A'PA''$, $B'PB''$, $C'PC''$, respectively. Hence their reflections in the corresponding internal angle bisectors pass through the circumcenters of triangles $A'PA''$, $B'PB''$, $C'PC''$ (as the orthocenter and the circumcenter are isogonal conjugates; see [5]), i.e. they are precisely the midpoints of the interceptions from Droz-Farny!!!

References

[1] J.-L. Ayme, A synthetic proof of the Droz-Farny line theorem, *Forum Geom.*, 4 (2004) 219-224.

[2] A. Droz-Farny, Question 14111, Ed. Times 71 (1899) 89-90.

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[4] D. Grinberg, Hyacinthos messages #6128, #6141, #6245 from December 10th-11th, 2002, #7384 from July 23rd, 2003, and #9845 from June 2nd, 2004.

[5] R. A. Jonhson, *Advanced Euclidean Geometry*, Dover Publications, 2007.

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