Selected Problems by Nikolai Beluhov

1. The point F lies inside $\triangle ABC$ and is such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. Let $A_1 = AF \cap BC$, $B_1 = BF \cap CA$, $C_1 = CF \cap AB$. Show that the Euler lines of the triangles AFB_1, BFC_1, CFA_1 form an equilateral triangle of perimeter $AA_1 + BB_1 + CC_1$. [*Matematika+*, 2006]

2. Let $ABCDEF$ be a non-convex hexagon which has no parallel sides and in which $AB = DE$, $BC = EF$, $CD = FA$, $\angle FAB = 3\angle CDE$, $\angle BCD = 3\angle EFA$, $\angle DEF = 3\angle ABC$. Show that the lines AD, BE, CF are concurrent. [*Matematika*, 2009; *Kvant*, 2009]

3. The two circles ω_1 and ω_2 meet in A and B and their common external tangents meet in O. The line l through O meets ω_1 and ω_2 in the points P and Q closer to O. Let $M = AP \cap BQ$, $N = AQ \cap BP$, and let $C \in l$ be such that $CM = CN = a$. Show that a remains constant when l varies. [*Matematika*, 2009]

4. Let $ABCD$ be a circumscribed quadrilateral and let l be an arbitrary line through A which intersects the broken line BCD. Let l meet the lines BC and CD in M and N. Let the incenters of $\triangle ABM$, $\triangle MCN$, $\triangle NDA$ be I_1, I_2, I_3, respectively. Show that the orthocenter of $\triangle I_1I_2I_3$ lies on l. [IMO Shortlist, 2009; *Kvant*, 2010]

5. The point P on the side BC of $\triangle ABC$ is such that $2\angle BAP = 3\angle PAC$. Show that

$$AB^2 \cdot AC^3 > AP^5.$$

[Spring Tournament, 2009]

6. Two perpendicular lines l_1 and l_2 pass through the orthocenter H of an acute-angled $\triangle ABC$. The lines of the sides of $\triangle ABC$ cut two segments from each of the lines l_1 and l_2 – one segment which lies inside the triangle, and another one which lies outside. Show that the product of the two inner segments equals the product of the two outer ones. [Sharygin Olympiad, 2012; with Emil Kolev]

7. The incircle and the ex-circle opposite A of $\triangle ABC$ touch the segment BC in M and N. If $\angle BAC = 2\angle MAN$, then show that $BC = 2MN$. [Sharygin Olympiad, 2009]

8. The incircle ω of $\triangle ABC$ touches BC, CA, AB in A_1, B_1, C_1, respectively. The triangle $A'B'C'$ is the reflection of $\triangle A_1B_1C_1$ in an arbitrary line l passing through the center of ω. Show that the lines AA', BB', CC' are concurrent. [Bulgarian National Olympiad, 2009]

9. An equilateral trangle δ is inscribed in an acute-angled triangle ABC. Show that the incenter of $\triangle ABC$ lies inside δ. [IMO Shortlist, 2010]

10. Given is a convex quadrilateral $ABCD$. Let $E = AC \cap BD$ and let EK, EL, EM, EN be the internal angle bisectors through E in $\triangle AEB$, $\triangle BEC$, $\triangle CED$, $\triangle DEA$, respectively. Show that the medians through A, B, C, D in $\triangle NAK$, $\triangle KBL$, $\triangle LCM$, $\triangle MDN$, respectively, are concurrent. [Unpublished, 2009]

11. Given is a triangle ABC. Its circumcircle is drawn and three points A_1, B_1, C_1 are marked on its sides BC, CA, AB, respectively, following which the triangle itself is erased. Show that the triangle can be recovered from the remaining figure if and only if the lines AA_1, BB_1, CC_1 are concurrent. [Sharygin Olympiad, 2010]

12. Let $ABCD$ be a circumscribed quadrilateral. Let $E = AC \cap BD$ and let I_a, I_b, I_c, I_d be the incenters of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, $\triangle ABC$, respectively. Show that the segments I_aI_c and I_bI_d meet in the center of a circle which passes through the incenters of $\triangle AEB$, $\triangle BEC$, $\triangle CED$, $\triangle DEA$. [*Kvant*, 2010]

13. Does there exist a linear function f of five variables such that, for any triangle ABC of circumradius R, inradius r, and exradii r_a, r_b, r_c, we have

$$f(R, r, r_a, r_b, r_c) = 0?$$

[Unpublished, 2009]

14. In $\triangle ABC$, AL_a and AM_a are an internal and an external angle bisector. Let ω_a be the circle symmetric to the circumcircle of $\triangle AL_aM_a$ with respect to the midpoint of BC. The circle ω_b is defined analogously. Show that the circles ω_a and ω_b are tangent if and only if $\triangle ABC$ is right-angled. [Sharygin Olympiad, 2010]

15. The incircle of $\triangle ABC$ touches its sides in A_1, B_1, C_1, respectively. Let the projections of the orthocenter of $\triangle A_1B_1C_1$ on the lines AA_1 and BC be P and Q. Show that the line PQ bisects the segment B_1C_1. [Bulgarian IMO TST, 2012]

16. $\triangle ABC$ and $\triangle A_1B_1C_1$ are two equal, oppositely oriented equilateral triangles of side 1. What is the least possible length of the longest one of the segments AA_1, BB_1, CC_1? [Autumn Tournament, 2012]