VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the VII Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions for younger grades will not be considered).

Your work containing the solutions for the problems (in Russian or in English) should be sent not later than April 1, 2011, by e-mail to geomolymp@mccme.ru in pdf, doc or jpg files. Please, follow several simple rules:

1. Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.

2. If your work consists of several files, send it as an archive. If the size of your work exceeds 10 Mb cut it to several archives and send each of them by a separate message.

3. In the subject of the message write "The work for Sharygin olympiad", and present the following personal information in the body of your message:

- last name, first name;

- E-mail, post address, phone number;

- the current number of your grade at school;

- the number and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no e-mail access, please, send your work by regular mail to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of Sharygin.* In the title page write your personal information indicated in the item 3 above.

In your work you should start writing the solution to each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures. Solutions of computational problems have to be completed with a distinctly presented answer. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may (this isn't necessary) note the problems which you liked. Your opinion is interesting for the Jury.

Your work will be examined thoroughly, and your marks will be sent to you by the end of May 2011. Winners of the correspondence round will be invited to take part in the final round in Summer 2011 in Dubna town (near Moscow).

1. (8) Does a convex heptagon exist which can be divided into 2011 equal triangles?

2. (8) Let ABC be a triangle with sides $AB = 4$, $AC = 6$. Point H is the projection of vertex B to the bisector of angle A. Find MH, where M is the midpoint of BC.

3. (8) Let ABC be a triangle with $\angle A = 60^\circ$. The midperpendicular of segment AB meets line AC at point C_1. The midperpendicular of segment AC meets line AB at point B_1. Prove that line B_1C_1 touches the incircle of triangle ABC.

4. (8) Segments AA', BB', CC' are the bisectrices of triangle ABC. It is known that these lines are also the bisectrices of triangle $A'B'C'$. Is it true that triangle ABC is regular?

5. (8) Given triangle ABC. The midperpendicular of side AB meets one of the remaining sides at point C'. Points A' and B' are defined similarly. For which original triangles triangle $A'B'C'$ is regular?

6. (8) Two unit circles ω_1 and ω_2 intersect at points A and B. M is an arbitrary point of ω_1, N is an arbitrary point of ω_2. Two unit circles ω_3 and ω_4 pass through both points M and N. Let C be the second common point of ω_1 and ω_3, and D be the second common point of ω_2 and ω_4. Prove that $ACBD$ is a parallelogram.

7. (8–9) Points P and Q on sides AB and AC of triangle ABC are such that $PB = QC$. Prove that $PQ < BC$.

8. (8–9) The incircle of right-angled triangle ABC ($\angle B = 90^\circ$) touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, C_2 are the reflections of B_1 in lines BC, AB respectively. Prove that lines $A_1A_2andC_1C_2$ meet on the median of triangle ABC.

9. (8–9) Let H be the orthocenter of triangle ABC. The tangents to the circumcircles of triangles CHB and AHB at point H meet AC at points A_1 and C_1 respectively. Prove that $A_1H = C_1H$.

10. (8–9) The diagonals of trapezoid $ABCD$ meet at point O. Point M of lateral side CD and points P, Q of bases BC and AD are such that segments MP and MQ are parallel to the diagonals of the trapezoid. Prove that line PQ passes through point O.

11. (8–10) The excircle of right-angled triangle ABC ($\angle B = 90^\circ$) touches side BC at point A_1 and touches line AC in point A_2. Line A_1A_2 meets the incircle of ABC for the first time at point A'; point C' is defined similarly. Prove that $AC||A'C'$.

12. (8–10) Let AP and BQ be the altitudes of acute-angled triangle ABC. Using a compass and a ruler, construct a point M on side AB such that $\angle AQM = \angle BPM$.

13. a) (8–10) Find the locus of centroids for triangles whose vertices lie on the sides of a given triangle (each side contains a single vertex).

    b) (11) Find the locus of centroids for tetrahedrons whose vertices lie on the faces of a given tetrahedron (each face contains a single vertex).

14. (9) In triangle ABC, the altitude and the median from vertex A form (together with line BC) a triangle such that the bisectrix of angle A is the median; the altitude and the median from vertex B form (together with line AC) a triangle such that the bisectrix of angle B is the bisectrix. Find the ratio of sides for triangle ABC.

15. (9–10) Given a circle with center O and radius equal to 1. AB and AC are the tangents to this circle from point A. Point M on the circle is such that the areas of quadrilaterals $OBMC$ and $ABMC$ are equal. Find MA.

16. (9–10) Given are triangle ABC and line l. The reflections of l in AB and AC meet at point A_1. Points B_1, C_1 are defined similarly. Prove that

    a) lines AA_1, BB_1, CC_1 concur;

    b) their common point lies on the circumcircle of ABC;

    c) two points constructed in this way for two perpendicular lines are opposite.

17. (9–11) a) Does there exist a triangle in which the shortest median is longer that the longest bisectrix?

    b) Does there exist a triangle in which the shortest bisectrix is longer that the longest altitude?

18. (9–11) On the plane, given are n lines in general position, i.e. any two of them aren't parallel and any three of them don't concur. These lines divide the plane into several parts. What is

    a) the minimal;

    b) the maximal

    number of these parts that can be angles?

19. (9–11) Does there exist a nonisosceles triangle such that the altitude from one vertex, the bisectrix from the second one and the median from the third one are equal?

20. (9–11) Quadrilateral $ABCD$ is circumscribed around a circle with center I. Points M and N are the midpoints of diagonals AC and BD. Prove that $ABCD$ is cyclic quadrilateral if and only if $IM : AC = IN : BD$.

21. (10–11) On a circle with diameter AC, let B be an arbitrary point distinct from A and C. Points M, N are the midpoints of chords AB, BC, and points P, Q are the midpoints of smaller arcs restricted by these chords. Lines AQ and BC meet at point K, and lines CP and AB meet at point L. Prove that lines MQ, NP and KL concur.

22. (10–11) Let CX, CY be the tangents from vertex C of triangle ABC to the circle passing through the midpoints of its sides. Prove that lines XY, AB and the tangent to the circumcircle of ABC at point C concur.

23. (10–11) Given are triangle ABC and line l intersecting BC, CA and AB at points A_1, B_1 and C_1 respectively. Point A' is the midpoint of the segment between the projections of A_1 to AB and AC. Points B' and C' are defined similarly.

    (a) Prove that A', B' and C' lie on some line l'.

    (b) Suppose l passes through the circumcenter of $\triangle ABC$. Prove that in this case l' passes through the center of its nine-points circle.

24. (10–11) Given is an acute-angled triangle ABC. On sides BC, CA, AB, find points A', B', C' such that the longest side of triangle $A'B'C'$ is minimal.

25. (10–11) Three equal regular tetrahedrons have the common center. Is it possible that all faces of the polyhedron that forms their intersection are equal?