XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XXI Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade (at the start of the correspondence round) have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. **If a problem has an explicit answer, this answer must be presented distinctly**. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The **solutions** for the problems (in Russian or in English) must be **delivered not before December 1, 2024 and not later than on March 1, 2025**. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, and follow the instructions.

Attention:

1. The solution of each problem (and of each part of it if any) must be contained in a **separate** pdf, doc, docx or jpg file. If the solution is contained in several files then pack them to an **archive** (zip or rar) and load it.
2. We recommend to prepare the paper using computer or to scan it rather than to photograph it. **In all cases, please check readability of the file before uploading**.
3. If you upload the solution of some problem more than once then only the last version is retained in the checking system. **Thus if you need to change something in your solution then you have to upload the whole solution again.**
4. After uploading, log in to the server, open the loaded file and check its correctness.

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round will be held in July–August 2025 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before the round. The cutoff is determined after the examination of the papers of the correspondence round according to the number of participants with any given score. The graduates who are winners of the correspondence round will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** up to June 1, 2025. If you want to know your detailed results, please apply to **geomshar@yandex.ru** after publication of the list.

1. (8) Let I be the incenter of a triangle ABC, D be an arbitrary point of segment AC, and A_1, A_2 be the common points of the perpendicular from D to the bisector CI with BC and AI respectively. Define similarly the points C_1, C_2. Prove that B, A_1, A_2, I, C_1, C_2 are concyclic.

2. (8) Four points on the plane are not concyclic, and any three of them are not collinear. Prove that there exists a point Z such that the reflection of each of these four points about Z lies on the circle passing through three remaining points.

3. (8) An excircle centered at I_A touches the side BC of a triangle ABC at point D. Prove that the pedal circles of D with respect to the triangles ABI_A and ACI_A are congruent.

4. (8) Let AL be the bisector of a triangle ABC, X be an arbitrary point on the external bisector of angle A, the lines BX, CX meet the perpendicular bisector to AL at points P, Q respectively. Prove that A, X, P, Q are concyclic.

5. (8) Let M be the midpoint of the cathetus AC of a right-angled triangle ABC ($\angle C = 90^\circ$). The perpendicular from M to the bisector of angle ABC meets AB at point N. Prove that the circumcircle of triangle ANM touches the bisector of angle ABC.

6. (8–9) One bisector of a given triangle is parallel to one sideline of its Nagel triangle. Prove that one of two remaining bisectors is parallel to another sideline of the Nagel triangle.

7. (8–9) Let I, I_a be the incenter and the A-excenter of a triangle ABC; E, F be the touching points of the incircle with AC, AB respectively; G be the common point of BE and CF. The perpendicular to BC from G meets AI at point J. Prove that E, F, J, I_a are concyclic.

8. (8–9) The diagonals of a cyclic quadrilateral $ABCD$ meet at point P. Points K and L lie on AC, BD respectively in such a way that $CK = AP$ and $DL = BP$. Prove that the line joining the common points of circles ALC and BKD passes through the mass-center of $ABCD$.

9. (8–9) The line ℓ passing through the orthocenter H of a triangle ABC ($BC > AB$) and parallel to AC meets AB and BC at points D and E respectively. The line passing through the circumcenter of the triangle and parallel to the median BM meets ℓ at point F. Prove that the length of segment HF is three times greater than the difference of FE and DH.

10. (8–9) An acute-angled triangle with one side equal to the altitude from the opposite vertex is cut from paper. Construct a point inside this triangle such that the square of the distance from it to one of the vertices equals the sum of the squares of distances to to the remaining two vertices. No instruments are available, it is allowed only to fold the paper and to mark the common points of folding lines.

11. (8–10) A point X is the origin of three rays such that the angle between any two of them equals 120°. Let w be an arbitrary circle with radius R such that X lies inside it, and A, B, C be the common points of the rays with this circle. Find $\max(XA + XB + XC)$.

12. (8–10) Circles ω_1 and ω_2 are given. Let M be the midpoint of the segment joining their centers, X, Y be arbitrary points on ω_1, ω_2 respectively such that $MX = MY$. Find the locus of the midpoints of segments XY.

13. (8–11) Each two opposite sides of a convex $2n$-gon are parallel. (Two sides are opposite if one passes $n-1$ other sides moving from one side to another along the borderline of the $2n$-gon.) The pair of opposite sides is called *regular* if there exists a common perpendicular to them such that its endpoints lie on the sides and not on their extensions. Which is the minimal possible number of regular pairs?

14. (9–11) A point D lies inside a triangle ABC on the bisector of angle B. Let ω_1 and ω_2 be the circles touching AD and CD at D and passing through B; P and Q be the common points of ω_1 and ω_2 with the circumcircle of ABC distinct from B. Prove that the circumcircles of the triangles PQD and ACD are tangent.

15. (9–11) A point C lies on the bisector of an acute angle with vertex S. Let P, Q be the projections of C to the sidelines of the angle. The circle centered at C with radius PQ meets the sidelines at points A and B such that $SA \neq SB$. Prove that the circle with center A touching SB and the circle with center B touching SA are tangent.

16. (9–11) The Feuerbach point of a scalene triangle lies on one of its bisectors. Prove that it bisects the segment between the corresponding vertex and the incenter.

17. (9–11) Let O, I be the circumcenter and the incenter of an acute-angled scalene triangle ABC; D, E, F be the touching points of its excircle with the side BC and the extensions of AC, AB respectively. Prove that if the orthocenter of the triangle DEF lies on the circumcircle of ABC, then it is symmetric to the midpoint of the arc BC with respect to OI.

18. (9–11) Let $ABCD$ be a quadrilateral such that the excircles ω_1 and ω_2 of triangles ABC and BCD touching their sides AB and BD respectively touch the extension of BC at the same point P. The segment AD meets ω_2 at point Q, and the line AD meets ω_1 at R and S. Prove that one of angles RPQ and SPQ is right.

19. (10–11) Let I be the incenter of a triangle ABC; A', B', C' be the orthocenters of the triangles BIC, AIC, AIB; M_a, M_b, M_c be the midpoints of BC, CA, AB, and S_a, S_b, S_c be the midpoints of AA', BB', CC'. Prove that M_aS_a, M_bS_b, M_cS_c concur.

20. (10–11) Let H be the orthocenter of a triangle ABC, and M, N be the midpoints of segments BC, AH respectively. The perpendicular from N to MH meets BC at point A'. Points B' and C' are defined similarly. Prove that A', B', C' are collinear.

21. (10–11) Let P be a point inside a quadrilateral $ABCD$ such that $\angle APB + \angle CPD = 180°$. Points P_a, P_b, P_c, P_d are isogonally conjugated to P with respect to the triangles BCD, CDA, DAB, ABC respectively. Prove that the diagonals of the quadrilaterals $ABCD$ and $P_aP_bP_cP_d$ concur.

22. (10–11) A circle and an ellipse with foci F_1, F_2 lying inside it are given. Construct a chord AB of the circle touching the ellipse and such that AF_1F_2B is a cyclic quadrilateral.

23. (10–11) Let us say that a subset M of the plane contains a hole if there exists a disc not contained in M, but contained inside some polygon with the boundary lying in M.

Can the plane be presented as a union of n convex sets such that the union of any $n-1$ from them contains a hole?

24. (11) The insphere of a tetrahedron $ABCD$ touches the faces ABC, BCD, CDA, DAB at D', A', B', C' respectively. Denote by S_{AB} the square of the triangle $AC'B$. Define similarly S_{AC}, S_{BC}, S_{AD}, S_{BD}, S_{CD}. Prove that there exists a triangle with sidelengths $\sqrt{S_{AB}S_{CD}}$, $\sqrt{S_{AC}S_{BD}}$, $\sqrt{S_{AD}S_{BC}}$.