XVII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (I.Kukharchuk, 8) Let ABC be a triangle with $\angle C = 90^\circ$. A line joining the midpoint of its altitude CH and the vertex A meets CB at point K. Let L be the midpoint of BC, and T be a point of segment AB such that $\angle ATK = \angle LTB$. It is known that $BC = 1$. Find the perimeter of triangle KTL.

   Answer. 1.

   Solution. Let M, N be the reflections of L about AB and AC respectively. Then $AM = AL = AN$ and $\angle MAN = 2\angle BAC$. Since AK and AL are the medians of right-angled triangles ABC and ACH, we obtain that $\angle CAK = \angle LAB$ and $\angle NAK = \angle NAC + \angle CAK = \angle CAL + \angle LAB = \angle BAC$. Thus AK bisects angle MAN, and $KM = KN$ (fig. 1). On the other hand, $KM = KT + TL$, i.e. the perimeter of triangle KTL equals $KM + KL = NL = BC = 1$.

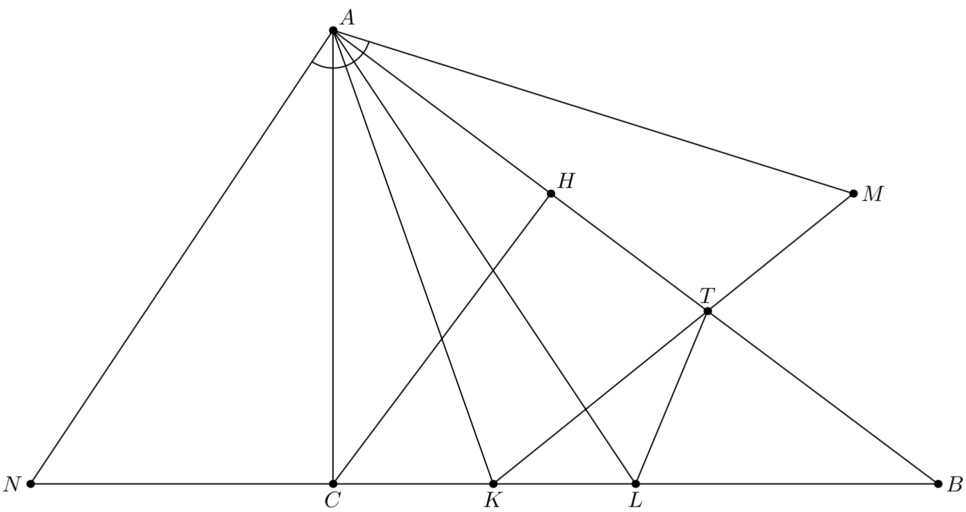

Fig. 1

2. (D.Shvetsov, 8) A perpendicular bisector to the side AC of triangle ABC meets BC, AB at points A_1 and C_1 respectively. Points O, O_1 are the circumcenters of triangles ABC and A_1BC_1 respectively. Prove that $C_1O_1 \perp AO$.

   Solution. We consider the case when triangle ABC is acute, and the other cases when it is right or obtuse are analogous. Since $\angle AOC = 2\angle ABC = 2(180^\circ - \angle A_1BC_1) = \angle A_1O_1C_1$, we get that triangles AOC and $C_1O_1A_1$ are similar and identically oriented. Thus each pair of corresponding sides in them forms the same angle (fig. 2). Since corresponding sides AC and A_1C_1 are perpendicular, so are corresponding sides AO and C_1O_1.

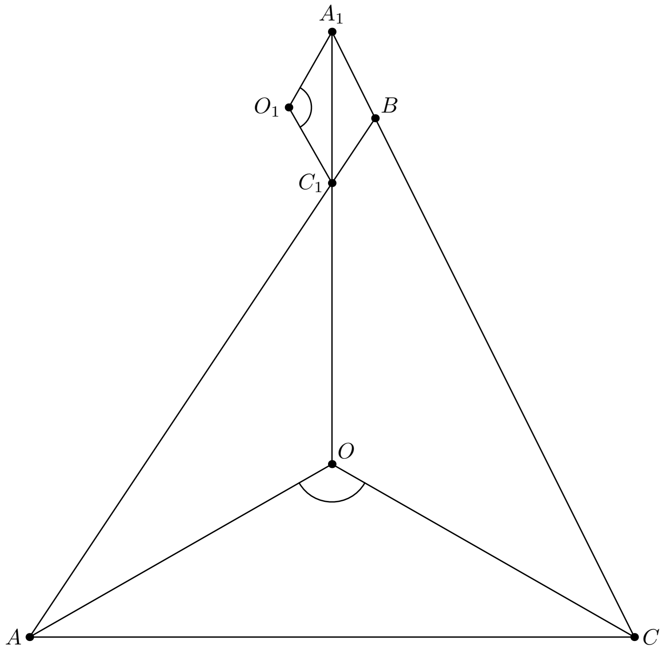

Fig. 2

3. (D.Shvetsov, 8) Altitudes AA_1, CC_1 of acute-angled triangle ABC meet at point H; B_0 is the midpoint of AC. A line passing through B and parallel to AC meets B_0A_1, B_0C_1 at points A', C' respectively. Prove that AA', CC', and BH concur.

   Solution. Let BB_1 be the altitude from B. By Thales' theorem, line AA' divides segment BB_1 in ratio $BA' : AB_1$, and line CC' divides the same segment in ratio $BC' : CB_1$ (fig. 3). We want to prove that these two ratios are equal. Again by Thales' theorem, $BA' : CB_0 = BA_1 : A_1C$ and $BC' : AB_0 = BC_1 : C_1A$. Since $AB_0 = CB_0$, the required assertion becomes equivalent to $AB_1 : B_1C = (AC_1 : C_1B) \cdot (BA_1 : A_1C)$. This follows immediately by Ceva's theorem for AA_1, BB_1, and CC_1.

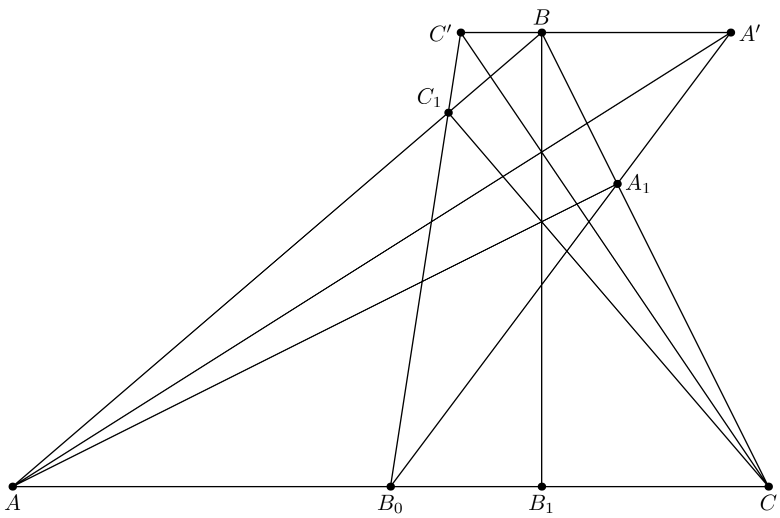

Fig. 3

4. (Tran Quang Hung, 8) Let $ABCD$ be a square with center O, and P be a point on the minor arc CD of its circumcicle. The tangents from P to the incircle of the square meet CD at points M and N. The lines PM and PN meet segments BC and AD respectively at points Q and R. Prove that the median of triangle OMN from O is perpendicular to the segment QR end equals to its half.

   Solution. The lines PR and PQ are sidelines of a square having the same circumcircle and incircle as $ABCD$. Hence the rotation by 90° around O maps M and Q onto R and N respectively, i.e., $OM = OP$, $ON = OQ$, and $\angle POM = \angle NOQ = 90°$. Thus if S is a vertex of parallelogram $MONS$, then the triangles OMS and POQ are congruent and their corresponding sidelines are perpendicular (fig. 4).

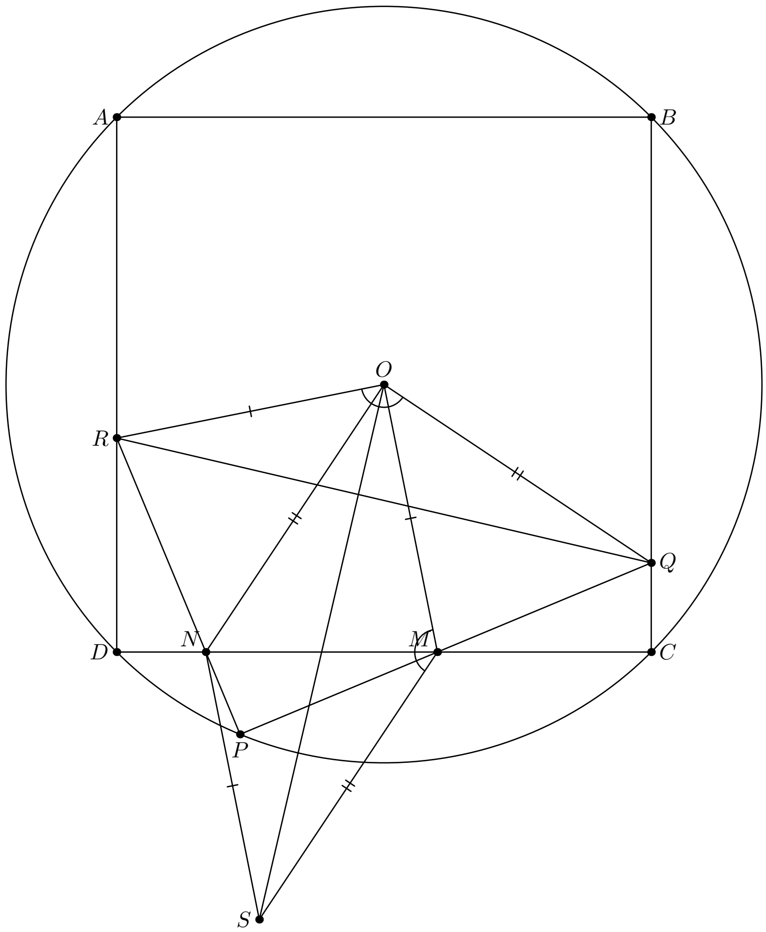

Fig. 4

   Comment. Also it is easy to see that $\angle RON = \angle NOM = \angle MOQ = 45°$.

5. (M.Saghafian, 8–9) Five points are given in the plane. Find the maximum number of similar triangles whose vertices are among these five points.

   Answer. Eight.

   Example. The center and the vertices of a square.

   Proof of the upper bound. First let us describe all configurations of four points such that the four triangles determined by them are pairwise similar. Let A, B, C, and D be such four points. Without loss of generality, there are two cases to consider.

1. Point A lies inside triangle BCD. Suppose without loss of generality that angle B is the greatest angle of triangle BCD. Then angle CAD is greater than every angle of triangle BCD, and so triangles ACD and BCD cannot be similar. We have arrived at a contradiction.

2. Quadrilateral $ABCD$ is convex. We split this case into two subcases as follows.

2.1. Diagonal AC bisects angle BAD, and similarly for vertices B, C, and D. Then $ABCD$ must be a rhombus, and so triangles ABC and BCD are similar if and only if $ABCD$ is a square.

2.2. There is at least one vertex of $ABCD$ such that the diagonal from it does not bisect the interior angle at it, say A. Then angles BAC, CAD, and BAD are pairwise distinct. Thus they must be the three angles of every one of our four similar triangles. Therefore they sum to $180°$, and so $\angle BAD = 90°$.

Since we have four similar triangles, we must also have four right angles and eight acute angles determined by our four points. Hence angles BAC, CAD, ABD, CBD and so on cannot be right because then we would get at least one obtuse angle at the corresponding vertex of $ABCD$. The only option left for angles BAD, ABC, BCD, and ADC is to be right.

In both cases 2.1 and 2.2, we obtain that if four points determine four pairwise similar triangles then these points are the vertices of a rectangle.

Now let us return to the problem. We say that a triangle is bad if it is determined by three of our five points but is not in our set of similar triangles. (In particular, three collinear points form a bad degenerate triangle.) Our goal is to prove that we have at least two bad triangles.

Observe that, given three of the vertices of a rectangle, we can always reconstruct the fourth vertex uniquely. Thus among our points there is at most one quadruple that forms a rectangle. Among the remaining four quadruples, each must contain at least one bad triangle by the above discussion. Since every bad triangle is counted in exactly two quadruples, we get at least $4/2 = 2$ bad triangles as needed.

6. (I.Kukharchuk, 8–9) Three circles Γ_1, Γ_2, Γ_3 are inscribed into an angle (the radius of Γ_1 is the minimal, and the radius of Γ_3 is the maximal) in such a way that Γ_2 touches Γ_1 and Γ_3 at points A and B respectively. Let l be a tangent at A to Γ_1. Consider circles ω touching Γ_1 and l. Find the locus of meeting points of common internal tangents to ω and Γ_3.

Answer. The circle Γ_2 and the segment AB with the exception of points A and B.

Solution. If ω touches Γ_1 and l at point A, then the common internal tangents meet on AB, and any internal point of the segment can be obtained.

Let ω touche l at point P distinct from A, and let r, r_1, r_3 be the radii of ω, Γ_1, Γ_3 respectively. Then $AP = 2\sqrt{rr_1}$, $AB = 2\sqrt{r_1 r_3}$.

The internal homothety center of ω and Γ_3 lies on segment PB. Let H be the projection of A onto this segment. Then $PH : BH = PA^2 : AB^2 = r : r_3$. Therefore H is the desired common point of tangents (fig. 6). Clearly H lies on Γ_2, and any point of this circle distinct from A and B can be obtained in such a way.

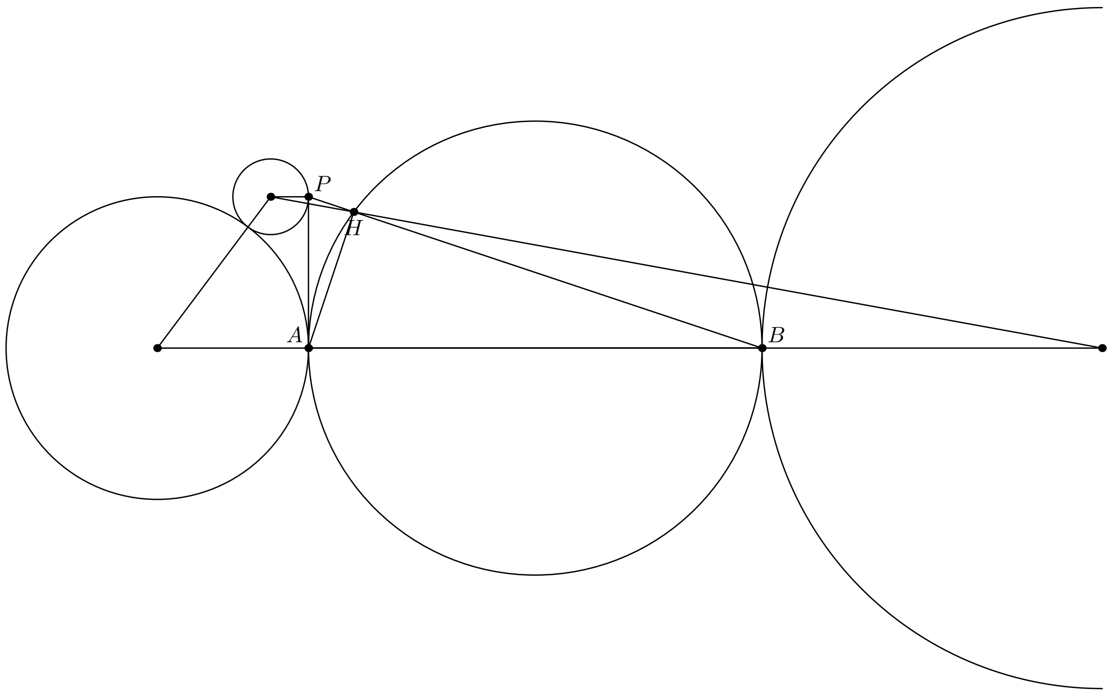

Fig. 6

7. (Tran Quang Hung, 8–9) The incircle of triangle ABC centered at I touches CA, AB at points E, F respectively. Let points M, N of line EF be such that $CM = CE$ and $BN = BF$. Lines BM and CN meet at point P. Prove that PI bisects segment MN.

   Solution. By Ceva's theorem for triangle IMN, it suffices to prove that line BM divides segment IN in the same ratio as line CN divides segment IM. (Possibly in both cases the lines divide the segments externally.)

   The first ratio equals $S_{BIM} : S_{BMN}$, and the second ratio equals $S_{CIN} : S_{CMN}$.

   Observe that triangles AEF, BFN, and CEM are similar and isosceles. Thus $S_{BMN} : S_{CMN} = d(B, MN) : d(C, MN) = BF : CE = BD : DC$, where D is the tangency point of the incircle with side BC.

   Next up, again from triangles AEF, BFN, and CEM being similar we get $AB \parallel CM$ and $AC \parallel BN$. Together with $BD = BN$ and $CD = CM$, by a straightforward angle chase this implies that $DM \parallel BI$ and $DN \parallel CI$. Hence $S_{BIM} = S_{BID}$ and $S_{CIN} = S_{CID}$ (fig. 7). Therefore $S_{BIM} : S_{CIN} = S_{BID} : S_{CID} = BD : DC$. The solution is complete.

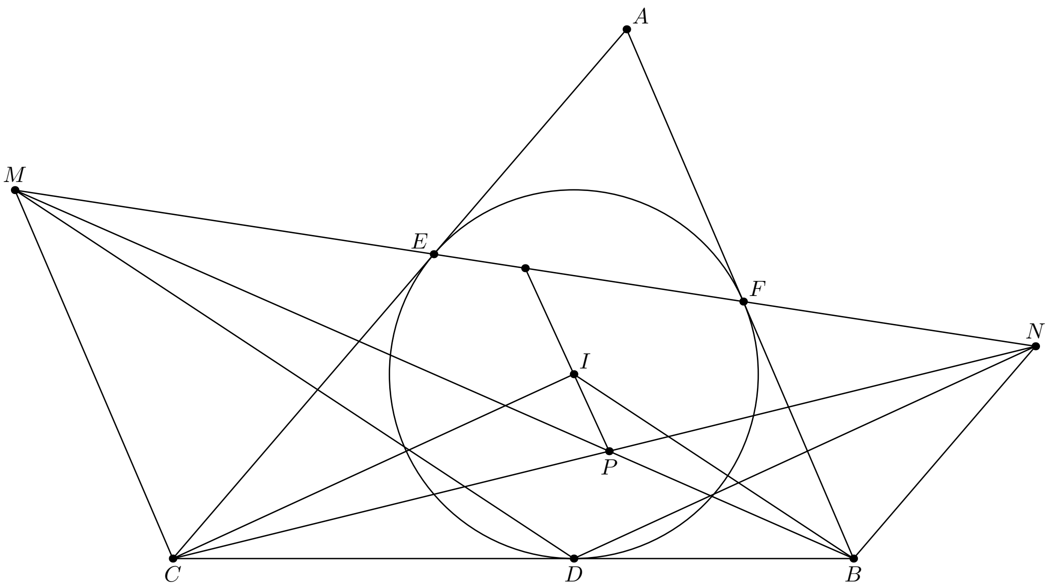

Fig. 7

8. (P.Ryabov, 8–9) Let ABC be an isosceles triangle ($AB = BC$), and l be a ray from B. Points P and Q of l lie inside the triangle in such a way that $\angle BAP = \angle QCA$. Prove that $\angle PAQ = \angle PCQ$.

   Solution. Let R be the isogonal conjugate of P in triangle ABC, and let Q' be the reflection of R across the axis of symmetry of triangle ABC. Then $\angle ABQ' = \angle CBR = \angle ABP = \angle ABQ$ and $\angle ACQ' = \angle CAR = \angle BAP = \angle ACQ$. Therefore points Q and Q' coincide. Consequently, $\angle CAQ = \angle ACR = \angle BCP$ (fig. 8). Thus (we consider the case when points B, P, and Q appear in this order along ray l; the opposite case when they appear in the order B, Q, and P, is analogous) $\angle PAQ = \angle A - \angle BAP - \angle CAQ = \angle C - \angle ACQ - \angle BCP = \angle PCQ$ as needed.

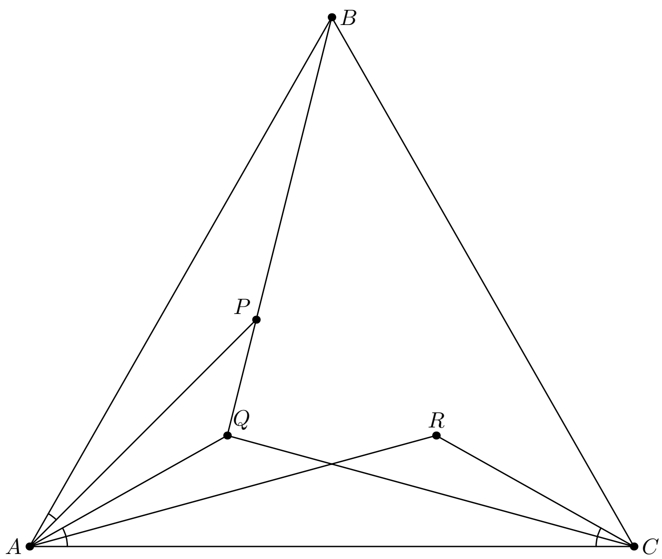

Fig. 8

9. (I.Kukharchuk, 8–9) Points E and F lying on sides BC and AD respectively of a parallelogram $ABCD$ are such that $EF = ED = DC$. Let M be the midpoint of BE, and MD meet EF at G. Prove that $\angle EAC = \angle GBD$.

   Solution. Let lines AD and BG meet at H. Since lines BE and FH are parallel, and GM is a median in triangle BEG, we also have that GD is a median in triangle GFH. Then, since lines CE and DFH are parallel and $CD = DE = EF$, we get that all three of these segments are equal to segment CH as well.

   Consider triangles ACE and BHD. We already know that $CE = DH$. Since $AB = CD = DE$ and $AD \parallel BE$, quadrilateral $ABED$ is an isosceles trapezoid, and so also $AE = BD$. Finally, since $AB = CD = CH$ and $AH \parallel BC$, quadrilateral $ABCH$ is an isosceles trapezoid as well, and so $AC = BH$ too.

   Thus triangles ACE and BHD are congruent (fig. 9). Therefore $\angle CAE = \angle HBD = \angle GBD$ as needed.

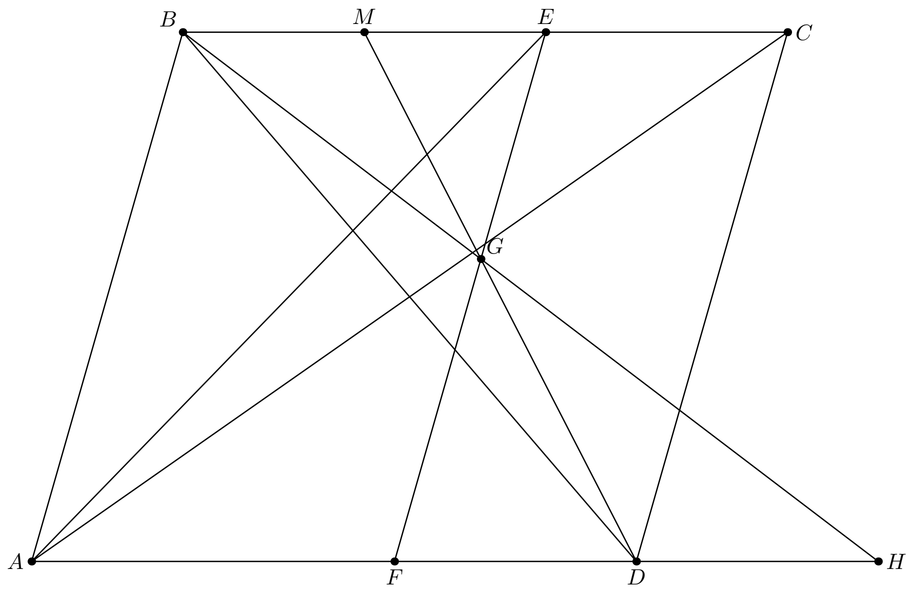

Fig. 9

10. (L.Emelyanov, 8–9) Prove that two isotomic lines of a triangle cannot meet inside its medial triangle. (*Two lines are isotomic lines of triangle ABC if their common points with BC, CA, AB are symmetric with respect to the midpoints of the corresponding sides.*)

    First solution. Let our triangle be ABC, and let the midpoints of its sides be K, L, and M, with A opposite to K, B opposite to L, and C opposite to M.

    Let ℓ be one of our two lines. If ℓ does not intersect the interior of triangle ABC then there is nothing to prove. Otherwise ℓ must meet two of segments AB, BC, and CA in interior points; say, AB and AC at points P and Q. Take points P' and Q' so that M and L are the midpoints of segments PP' and QQ' respectively. Then the isotomic conjugate of line ℓ is line $P'Q'$ which we denote ℓ'.

    Suppose that P and Q lie in segments AM and AL respectively. Then ℓ does not meet the interior of triangle KLM, and we are done. Analogously, if P and Q lie in segments

BM and CL respectively then P' and Q' lie in segments AM and AL respectively, and ℓ' does not meet the interior of triangle KLM instead.

Without loss of generality, we are left to consider the case when P lies in segment AM and Q lies in segment CL. Then P' lies in segment BM and Q' lies in segment AL. We claim that in this case lines ℓ and ℓ' meet inside triangle ALM.

Let lines ℓ and ℓ' meet segment LM at points X and Y respectively (fig. 10). To prove our claim, it suffices to show that points L, X, Y, and M appear in this order along line LM.

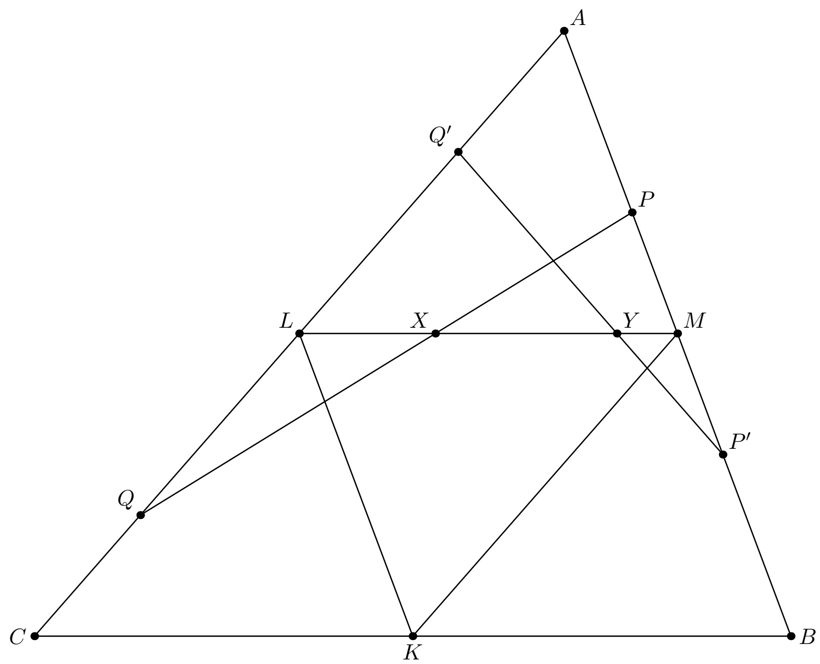

Рис. 10

By Menelaus' theorem for triangle ALM and line ℓ, we get that $LX : XM = (LQ : QA) \cdot (AP : PM)$. Analogously, by Menelaus' theorem for triangle ALM and line ℓ' we get that $LY : YM = (LQ' : Q'A) \cdot (AP' : P'M) = (LQ : Q'A) \cdot (AP' : PM)$. Since $QA > Q'A$ and $AP < A'P$, we conclude that $LX : XM < LY : YM$, and the solution is complete.

Second solution. Suppose that the common point S of given lines lies inside the medial triangle. Then there exists an affine map transforming A, B, C, S into A', B', C', and the circumcenter of triangle $A'B'C'$. The images of the given lines are also isotomic, hence they are symmetric about the perpendicular bisector to any side. Clearly this is impossible.

Comment. This reasoning also yields that two isotomic lines cannot meet inside any angle vertical to an angle of medial triangle.

11. (A.Zaslavsky, 8–9) The midpoints of four sides of a cyclic pentagon were marked, after this the pentagon was erased. Restore it.

Solution. Let K, L, M, N be the given midpoints of sides AB, BC, CD, DE of pentagon $ABCDE$ inscribed into circle Ω centered at O. Construct parallelograms $NMLP$ and

$KLMQ$. The triangles KLP and ACE are homothetic with center B and coefficient $1/2$. Therefore the segment BO is a diameter of the circumcircle of triangle KLP. Similarly O lies on the circumcircle of triangle MNQ (fig. 11). This we can construct the point O, the vertex B, and finally the desired pentagon.

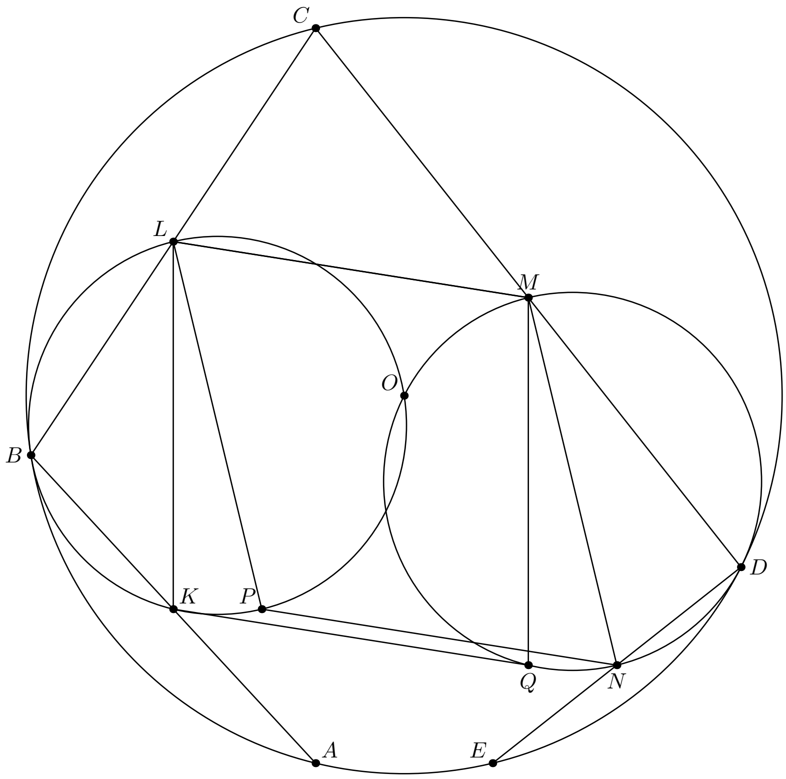

Fig. 11

Comment. The circles KLP and MNQ meet at two points, and the pentagon can be constructed using each of them. If both pentagons are convex the problem has two solutions. Otherwise the desired pentagon is unique.

12. (E.Bakaev, 8–10) Suppose we have ten coins with radii $1, 2, 3, \ldots, 10$ cm. We can put two of them on the table in such a way that they touch each other, after that we can add the coins in such a way that each new coin touches at least two of previous ones. The new coin cannot cover a previous one. Can we put several coins in such a way that the centers of some three coins are collinear?

    Solution. Let us try to find a construction with four coins (clearly three coins are insufficient). Place the coins with radii a, b, x, and y so that all pairs among them are externally tangent, except for pair x and y. Let O_r be the center of the coin with radius r. By the cosine theorem for triangle $O_aO_bO_x$, we get $\cos\angle O_bO_aO_x = ((a+b)^2+(a+x)^2-(b+x)^2)/2(a+b)(a+x)$. Analogously, $\cos\angle O_bO_aO_y = ((a+b)^2+(a+y)^2-(b+y)^2)/2(a+b)(a+y)$. For centers O_a, O_x, and O_y to be collinear, it is necessary and sufficient that these two cosines sum to zero. This is equivalent to $(b-a)(ax+ay+2xy) = a(a+b)(2a+x+y)$. By trial and error, we find the solution $a = 2$, $b = 5$, $x = 3$, and $y = 8$. (Then the two cosines equal 1/7 and -1/7, fig. 7.)

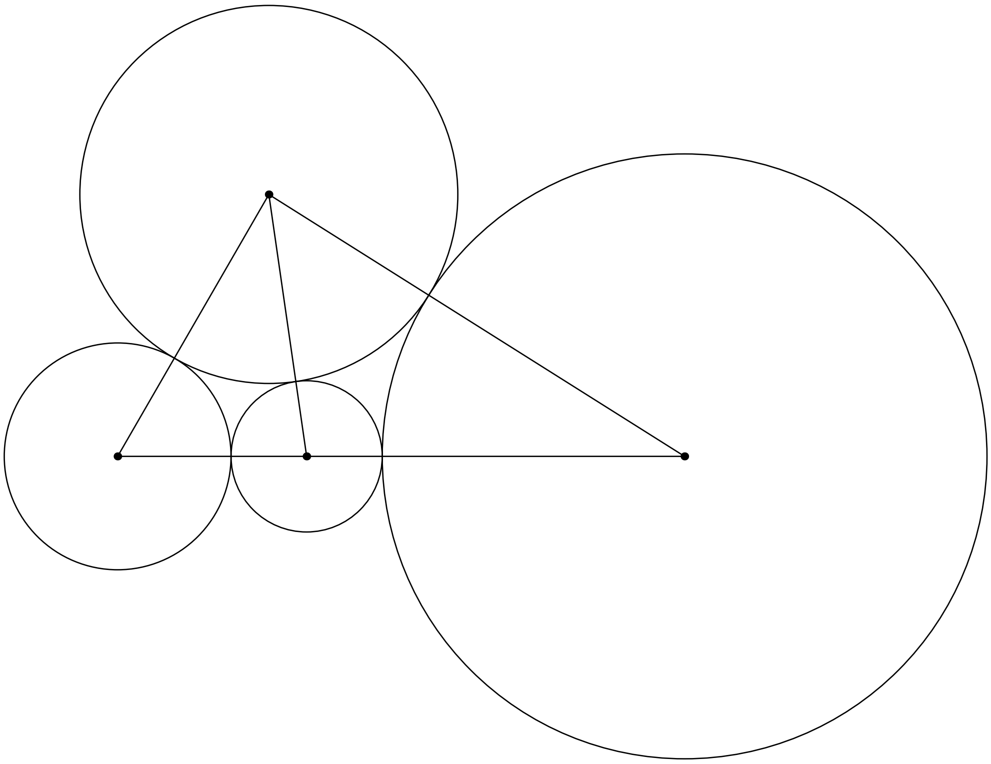

Fig. 12

Comments (N.Beluhov). 1. An exhaustive computer search shows that there is only one more configuration with four coins that works. It is given by $a = 3$, $b = 8$, $x = 5$, and $y = 9$, where the two cosines equal 1/11 and -1/11. (There are many other ordered quadruples a, b, x, and y that solve our Diophantine equation, but none of them have pairwise distinct numbers.)

2. With coins of radii $1, 2, \ldots, 16$, there is the following construction which does not involve any trial and error. Place the coins with radii 1, 2, 4, 8, and 16 so that the following pairs are externally tangent: 1—2, 1—4, 2—4, 2—8, 4—8, 4—16, and 8—16. Then triangles $O_1O_2O_4$, $O_2O_4O_8$, and $O_4O_8O_{16}$ are pairwise similar, and so $\angle O_1O_4O_{16} = \angle O_1O_4O_2 + \angle O_2O_4O_8 + \angle O_8O_4O_{16} = 180^\circ$ because it equals the sum of the interior angles of any one of our three similar triangles.

13. (A.Mudgal, 9–11) In triangle ABC with circumcircle Ω and incenter I, point M bisects arc BAC and line $\overline{AI}$ meets Ω at $N \neq A$. The excircle opposite to A touches side $\overline{BC}$ at point E. Point $Q \neq I$ on the circumcircle of $\triangle MIN$ is such that $\overline{QI} \parallel \overline{BC}$. Prove that the lines $\overline{AE}$ and $\overline{QN}$ meet on Ω.

Solution. Let ω be the mixtilinear incircle opposite to A. (That is, the circle tangent to segments AB and AC and also to Ω internally.) Let ω and Ω touch at T. We use two well-known lemmas about the mixtilinear incircles:

Lemma 1. Rays AE and AT are isogonal in angle A.

Lemma 2. Points M, I, and T are collinear.

From these two lemmas, we can derive the problem statement rather easily as follows. Let line AE meet Ω again at X. By Lemma 1, lines BC and TX are parallel. Thus lines

IQ and TX are parallel as well. Using Lemma 2 we conclude that $\angle MNQ = \angle MIQ = \angle MTX = \angle MNX$. Therefore X lies on NQ as needed (fig. 13).

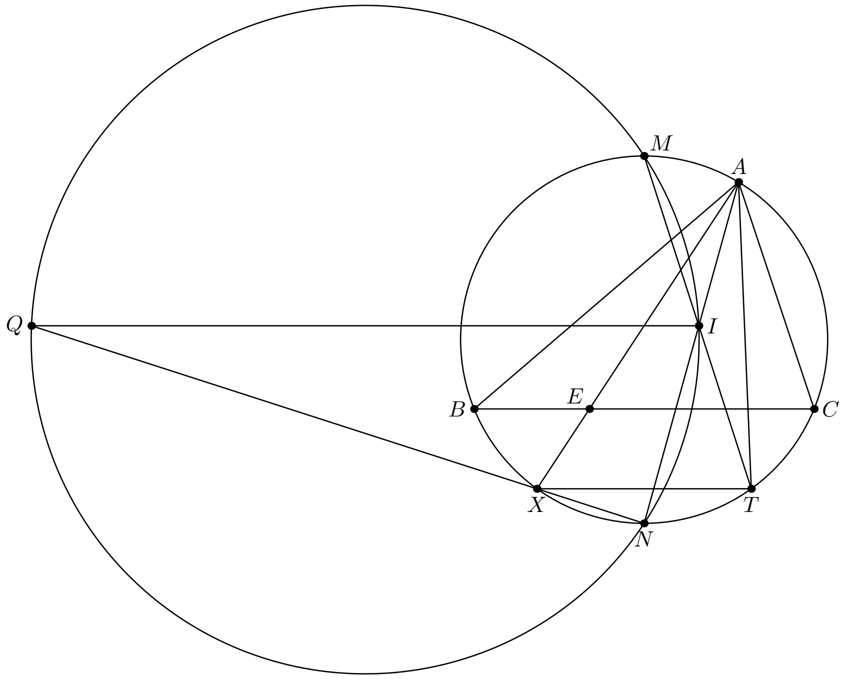

Fig. 13

14. (P.Agarwal, 9–11) Let γ_A, γ_B, γ_C be excircles of triangle ABC, touching the sides BC, CA, AB respectively. Let l_A denote the common external tangent to γ_B and γ_C distinct from BC. Define l_B, l_C similarly. The tangent from a point P of l_A to γ_B distinct from l_A meets l_C at point X. Similarly the tangent from P to γ_C meets l_B at Y. Prove that XY touches γ_A.

 Solution. Lemma. Let ω be any circle, and let m and n be two lines tangent to it. Consider the mapping f from m to n defined as follows: For every point P of m, $f(P) = Q$ is the intersection point of n with the second tangent from P to ω. Then f is projective.

 Proof. Let O be the center of ω. Then the oriented angle that goes counterclockwise from line OP to line OQ is constant when P varies over m. Thus for every four points P_1, P_2, P_3, and P_4 on m with $f(P_i) = Q_i$ for all i, we have that rotation about O maps lines OP_1, OP_2, OP_3, and OP_4 onto lines OQ_1, OQ_2, OQ_3, and OQ_4. Therefore the cross-ratios $(P_1P_2P_3P_4)$ and $(Q_1Q_2Q_3Q_4)$ are equal. This completes the proof of the Lemma.

 For the problem, define f_A as in the Lemma for circle γ_A and tangent lines l_B and l_C, with the mapping going from l_B to l_C. Define f_B and f_C cyclically. Our goal is to prove that the composition of f_A, f_B, and f_C is the identity mapping. Since, by the Lemma, this composition is projective, it suffices to find three special cases where this holds. That is, we need to find three points P on line l_B such that $f_C(f_B(f_A(P))) = P$. The points where the sides of triangle ABC meet line l_B all work.

15. (A.Mudgal, N.V.Tejaswi, 9–11) Let $APBCQ$ be a cyclic pentagon. A point M inside triangle ABC is such that $\angle MAB = \angle MCA$, $\angle MAC = \angle MBA$ and $\angle PMB = \angle QMC = 90°$. Prove that AM, BP, and CQ concur.

Solution. Let k be the circumcircle of $APBCQ$, and let line AM meet k again at N. By the given conditions, triangles AMB and CMA are similar. Since $\angle BCN = \angle BAM$ and $\angle CBN = \angle CAM$, both of them are similar to triangle CNB as well.

By the aforementioned similarities, $AB : AC = BM : AM = BN : NC$. Thus quadrilateral $ABNC$ is harmonic and M is the midpoint of AN.

Next up, we claim that $APNQ$ is harmonic as well. To see this, let line PM meet k again at R. Then angle $NMR = 90^\circ - \angle BMN$ and $\angle AMQ = \angle AMC - 90^\circ$. Since $\angle BMN + \angle AMC = \angle BAC + \angle BNC = 180^\circ$, from this we obtain that $\angle NMR = \angle AMQ$. Together with $AM = MN$, this implies that points Q and R are symmetric with respect to the perpendicular bisector of segment AN. Therefore $\smile AQ = \smile NR$ (fig. 15), and so $\angle MPN = \angle NPR = \angle APQ = \angle ANQ = \angle MNQ$. Analogously, $\angle MQN = \angle AQP = \angle MNP$. Consequently, triangles APQ, MPN, and MNQ are pairwise similar. From here, we establish that $APNQ$ is harmonic, exactly as for $ABNC$.

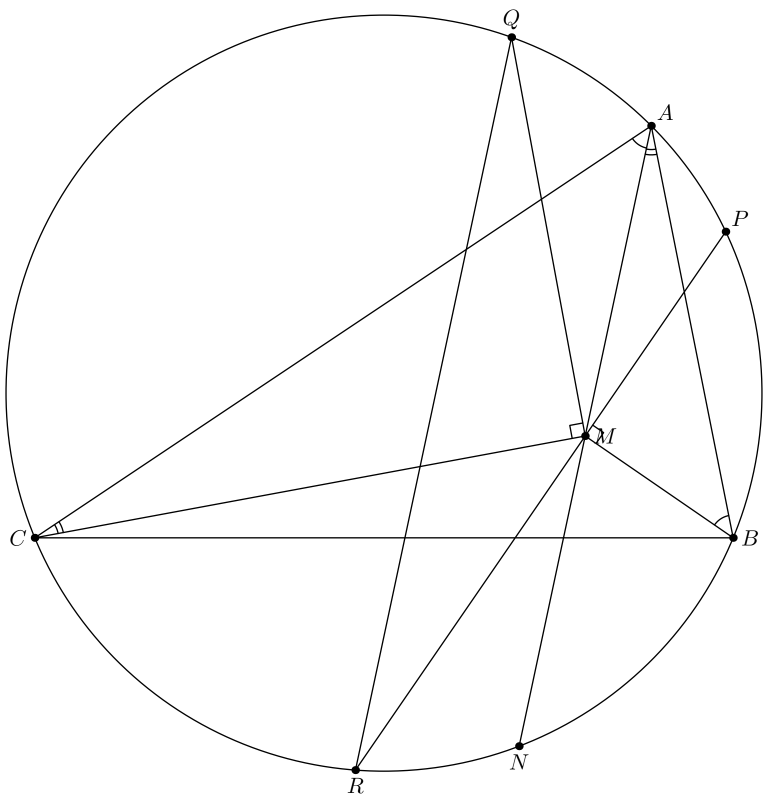

Fig. 15

We are ready to finish the solution. Line BP divides segment AN externally in ratio $S_{BAP} : S_{BNP} = (AB \cdot AP) : (BN \cdot PN)$. Analogously, line CQ divides segment AN externally in ratio $(AC \cdot AQ) : (CN \cdot QN)$. Since both of $ABNC$ and $APNQ$ are harmonic, these ratios are equal, and the solution is complete.

16. (P.Bibikov, 9–11) Let circles Ω and ω touch internally at point A. A chord BC of Ω touches ω at point K. Let O be the center of ω. Prove that the circle BOC bisects segment AK.

Solution. Let M be the midpoint of AK, and let the common tangent to ω and Ω at A meet line BC at point X. Then $XA^2 = XB \cdot XC$; furthermore $XA^2 = XM \cdot XO$ because both of XA and XK are tangents to ω, and so triangles XMA and XAO are similar. Thus $XB \cdot XC = XM \cdot XO$, and so quadrilateral $BOMC$ is cyclic (fig. 16).

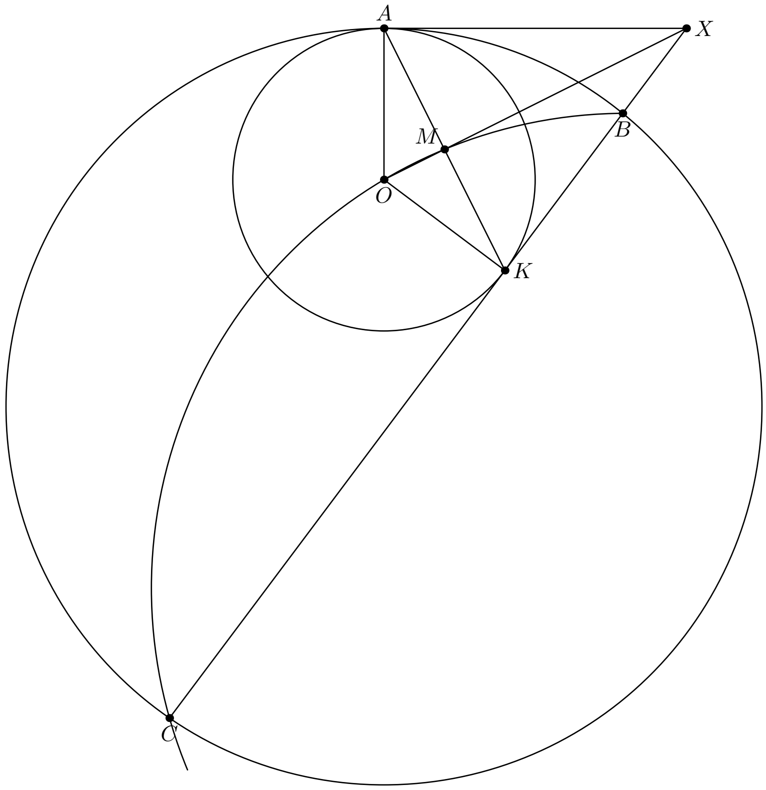

Fig. 16

17. (S.Sevastyanov, 9–11) Let ABC be an acute-angled triangle. Points A_0 and C_0 are the midpoints of minor arcs BC and AB respectively. A circle passing through A_0 and C_0 meets AB and BC at points P and S, Q and R respectively (all these points are distinct). It is known that $PQ \parallel AC$. Prove that $A_0P + C_0S = C_0Q + A_0R$.

    Solution. Suppose that $AB \neq BC$ (the opposite case is clear). Let t be the tangent to the circumcircle of ABC at B, and let X be the intersection point of lines t and A_0C_0. Observe that A_0C_0 is the radical axis of circles ABC and $PRSQ$, PQ is the radical axis of circles BPQ and $PRSQ$, and (since $PQ \parallel AC$) t is the radical axis of circles ABC and BPQ. Therefore X is the radical center of these three circles, and so X lies on PQ. Furthermore A_0C_0 forms equal angles with AC and t, i.e., bisects angle BXP. And since the incenter I of ABC is the reflection of B about A_0C_0, I lies on PQ. Now $IQ = QC$, $IA_0 = A_0C$, thus A_0Q passes through the midpoint B_0 of arc AC. Similarly C_0P passes through B_0.

    Finally $\angle RA_0P = \angle RQP = \angle C$, $\angle PRA_0 = \angle PC_0A_0 = (\pi - \angle C)/2$. Therefore $A_0P = A_0R$ (fig. 17). Similarly $C_0Q = C_0S$ which solves the problem.

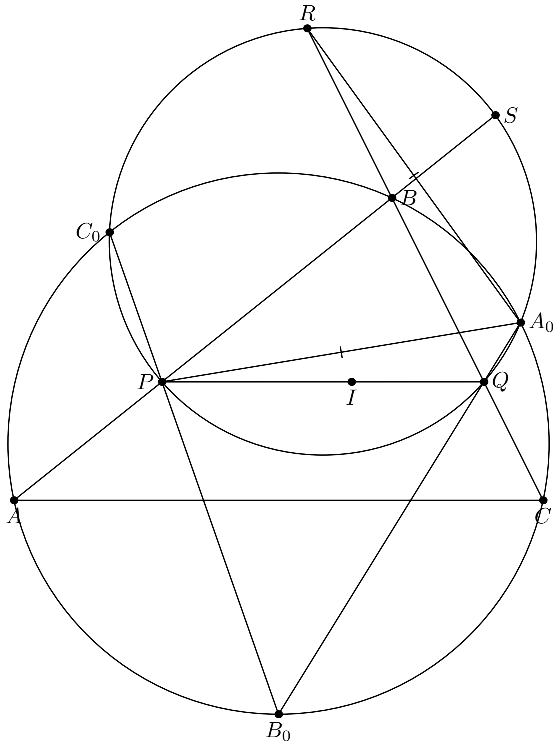

Fig. 17

18. (N.Beluhov, 10–11) Let ABC be a scalene triangle, AM be the median through A, and ω be the incircle. Let ω touch side BC at point T, and segment AT meet ω for the second time at point S. Let δ be the triangle formed by lines AM and BC and the tangent to ω at S. Prove that the incircle of triangle δ is tangent to the circumcircle of triangle ABC.

    First solution. Let D be any point on segment BC. The two circles inscribed in $\angle ADB$ and $\angle ADC$ and internally tangent to the circumcircle of triangle ABC are known as the *Thebault circles* for cevian AD. We will make use of two theorems about Thebault circles (see [1]).

    1. The second common exterior tangent of two Thebault circles is also tangent to ω.

    2. Let E be the intersection point of segment AD and the second common exterior tangent of two Thebault circles. Then point E lies on the tangent to ω parallel to side BC.

    Let U be the point of ω opposite to T. We have to prove that the tangents to ω at U and S meet on AM. Let the tangent at U meet the tangent at S and the line AT at points V and W respectively. Since $\angle USW = \pi/2$ and $UV = VS$, we obtain that V is the midpoint of UW. But AU meets BC at its touching point with the excircle symmetric to T with respect to M, thus AM passes through V (fig. 18).

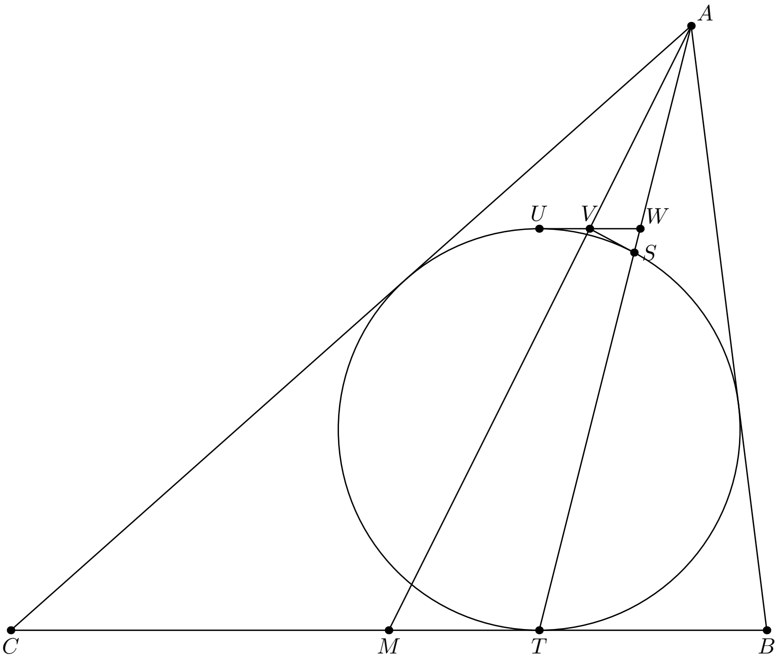

Fig. 18

Second solution. Let σ be the incircle of triangle δ and, for any point X outside of σ, let t_X be the length of the tangent from X to σ. Suppose without loss of generality that $AB < AC$. By Casey's theorem for the circumcircle of triangle ABC, circles A, B, and C of zero radius, and circle σ, it suffices to prove that $at_A + bt_B - ct_C = 0$.

Let the tangent t to ω parallel to side BC meet segments AM and AT at points K and P respectively. Let TQ be a diameter in ω. (Thus Q is also the tangency point of t and ω.) As in the first solution we obtain that segment SK is a median to the hypotenuse in right-angled triangle PSQ. Therefore $\angle KSQ = \angle KQS$ and so segment KS is tangent to ω. Let lines BC and KS meet at L. Then δ coincides with triangle KLM.

We have that

$$\begin{aligned}
2t_A &= 2(AK + t_K) = 2AK + (KL + KM - LM) = 2AK + KS + KM - MT,\\
2t_B &= 2(t_L - BL) = (KL + LM - KM) - 2BL\\
&= (KL - BL) + (LM - BL) - KM = KS + BT + BM - KM, \text{ and}\\
2t_C &= 2(CM + t_M) = 2CM + (KM + LM - KL) = BC + KM + MT - KS.
\end{aligned}$$

Therefore

$$\begin{aligned}
&2(at_A + bt_B - ct_C) =\\
&\qquad = 2aAK - (-a+b+c)KM + (a+b+c)KS - (a+c)MT +\\
&\qquad\qquad + b\cdot\frac{a-b+c}{2} + b\cdot\frac{a}{2} - ca.
\end{aligned}$$

We have that $KP = KS$ and

$$AK : AM = KP : MT = r : r_A,$$

where r and r_A are the radii of ω and ω_A respectively, since the homothety with center A that maps ω onto ω_A also maps triangle AKP onto triangle AMT.

On the other hand, for the lengths of the tangents from A to circles ω and ω_A, we get that

$$r : r_a = (-a + b + c) : (a + b + c).$$

Therefore

$$\begin{aligned} AK : KM &= (-a + b + c) : 2a \text{ and} \\ KS : MT &= (-a + b + c) : (a + b + c), \end{aligned}$$

and so

$$\begin{aligned} 2aAK - (-a + b + c)KM + (a + b + c)KS - (a + c)MT = \\ = (-2a + b)MT = (-2a + b) \cdot \frac{b - c}{2}. \end{aligned}$$

We are left to prove that

$$(-2a + b) \cdot \frac{b - c}{2} + b \cdot \frac{a - b + c}{2} + b \cdot \frac{a}{2} - ca = 0,$$

and this is indeed a correct identity for arbitrary real numbers a, b, and c. The solution is complete.

19. (Tran Quang Hung, 10–11) A point P lies inside a convex quadrilateral $ABCD$. Common internal tangents to the incircles of triangles PAB and PCD meet at point Q, and common internal tangents to the incircles of triangles PBC and PAD meet at point R. Prove that P, Q, R are collinear.

Solution. Let the incircles of triangles APB, BPC, CPD, and DPA be ω_1, ω_2, ω_3, and ω_4.

The case when P is the intersection point of lines AC and BD is clear because then points P, Q, and R coincide. Thus in the sequel we assume that P lies outside of at least one of lines AC and BD. Then $P \neq Q$ and $P \neq R$.

Lemma 1. Let Γ_1 be any circle inscribed in angle APB, and let Γ_3 be any circle inscribed in angle CPD. Let the two common interior tangents of Γ_1 and Γ_3 meet at X. Then points P, Q, and X are collinear.

Proof. Let the two common interior tangents of Γ_1 and ω_3 meet at T. Observe that P is the exterior homothety center of ω_1 and Γ_1, Q is the interior homothety center of ω_1 and ω_3, and T is the interior homothety center of Γ_1 and ω_3. By the three homothety centers theorem, points P, Q, and T are collinear. Furthermore, since P lies outside of at least one of lines AC and BD, we have $P \neq T$, and so lines PQ and PT are both well-defined and coincide.

Next up, observe that P is the exterior homothety center of ω_3 and Γ_3 and X is the interior homothety center of Γ_1 and Γ_3. By the three homothety centers theorem, points P, T, and X are collinear. Furthermore, since P lies outside of at least one of lines AC

and BD, we have $P \neq X$, and so lines PT and PX are both well-defined and coincide. This completes the proof of Lemma 1.

Analogously to Lemma 1, let Γ_2 be any circle inscribed in angle BPC and let Γ_4 be any circle inscribed in angle DPA. Let the two common interior tangents of Γ_2 and Γ_4 meet at Y. Then points P, R, and Y are collinear. Furthermore, from the proof of Lemma 1 we know that both lines PR and PY are well-defined and coincide.

Therefore to solve the problem it suffices to find four circles Γ_1, Γ_2, Γ_3, and Γ_4 such that points P, X, and Y are collinear.

Choose points A', B', C', and D' on rays PA, PB, PC, and PD such that $PA' = PB' = PC' = PD'$, and let Γ_1, Γ_2, Γ_3, and Γ_4 be tangent to rays PA, PB, PC, and PD at points A', B', C', and D'.

Lemma 2. Let lines $A'C'$ and $B'D'$ meet at Z. Then for this choice of circles Γ_1, Γ_2, Γ_3, and Γ_4, we get that both of X and Y coincide with Z.

Proof. We prove that X coincides with Z, and for Y the proof is analogous. Let line $A'C'$ meet Γ_1 and Γ_3 for the second time at U and V, and let line $B'D'$ meet Γ_3 for the second time at W. We claim that triangles $A'UB'$ and $VC'W$ are homothetic.

We consider the case when point U is on the greater arc $A'B'$, point V is on the lesser arc $C'D'$, and point W is on the greater arc $C'D'$ (fig. 19). All other configurations are analogous.

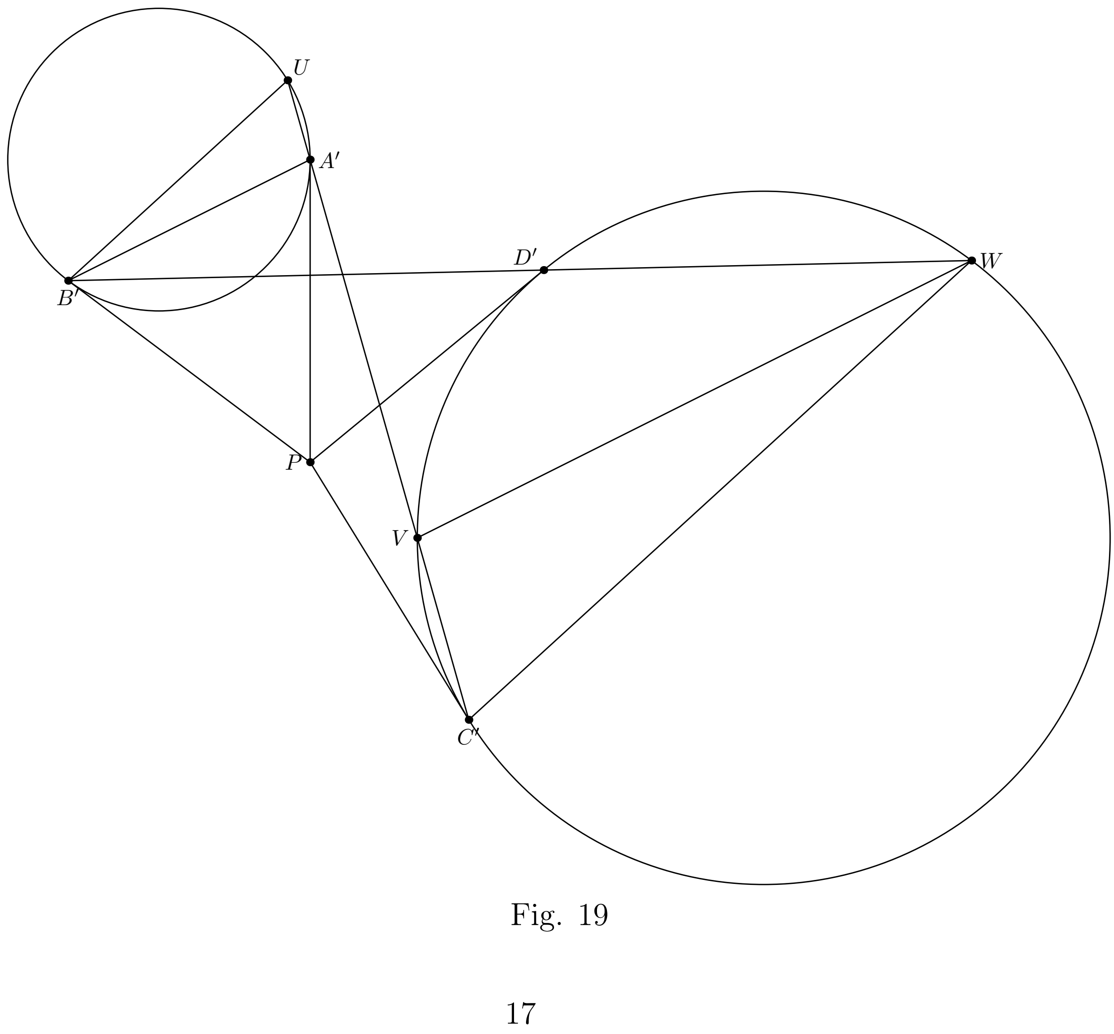

Fig. 19

Clearly, sides $A'U$ and $C'V$ are parallel as they lie on the same line.

Then (since $A'B'C'D'$ is inscribed into a circle centered at O) $\angle B'A'C' = \angle B'D'C' = \pi - \angle C'D'W = \pi - \angle C'VW$, and so sides $A'B'$ and VW are parallel as well.

Finally, $\angle UB'Z = \pi - \angle B'UZ - \angle B'ZU = \pi - \angle A'UB' - \angle A'ZB' = \pi - \angle A'B'P - \angle A'ZB' = \angle C'D'P = \angle C'WD'$. Thus sides $B'U$ and $C'W$ are parallel as well.

Consequently, triangles $A'UB'$ and $VC'W$ are indeed homothetic. Therefore Z is their homothety center and hence also the interior homothety center of Γ_1 and Γ_3. This completes the proof of Lemma 2.

Clearly, Lemma 2 implies the statement of the problem. The solution is complete.

Comment 1. From the solution it is clear that the problem can be generalised as follows: Instead of the incircles of triangles APB, BPC, CPD, and DPA, we can take any four circles inscribed in angles APB, BPC, CPD, and DPA, and the problem statement still holds.

Comment 2. Here is an alternative proof of Lemma 2. Let K, L, M, and N be the centers of Γ_1, Γ_2, Γ_3, and Γ_4. Then convex quadrilateral $KLMN$ is circumscribed about a circle Ω with center P and radius $PA' = PB' = PC' = PD'$ that touches its sides at points A', B', C', and D'. By a well-known theorem about circumscribed quadrilaterals, the diagonals KM and LN and chords $A'C'$ and $B'D'$ of Ω are concurrent, and their concurrency point divides each diagonal of $KLMN$ in the same ratio as the ratio of the tangents from the endpoints of that diagonal to Ω. The latter claim is equivalent to Lemma 2.

20. (N.Beluhov, 10–11) The mapping f assigns a circle to every triangle in the plane so that the following conditions hold. (We consider only nondegenerate triangles and circles of nonzero radius.)

(a) Let σ be any similarity in the plane and let σ map triangle Δ_1 onto triangle Δ_2. Then σ also maps circle $f(\Delta_1)$ onto circle $f(\Delta_2)$.

(b) Let A, B, C, and D be any four points in general position. Then circles $f(ABC)$, $f(BCD)$, $f(CDA)$, and $f(DAB)$ have a common point.

Prove that for any triangle Δ, the circle $f(\Delta)$ is the Euler circle of Δ.

Solution. By a well-known theorem, for any four points A, B, C, and D in general position the Euler circles of triangles ABC, BCD, CDA, and DAB have a common point. That is, the mapping which assigns to every triangle its Euler circle does indeed satisfy both (a) and (b).

Let us say that a triangle Δ is *well-behaved* if $f(\Delta)$ is the Euler circle of Δ. Our goal is to prove that all triangles are well-behaved. Given any four points A, B, C, and D in general position, we call any common point of $f(ABC)$, $f(BCD)$, $f(CDA)$, and $f(DAB)$ a *witness* for A, B, C, and D.

Lemma 1. Every equilateral triangle is well-behaved.

Proof. Let ABC be any equilateral triangle and let O be its center. By (a), rotation by $120°$ about O must preserve $f(ABC)$. Therefore O is the center of $f(ABC)$.

Let P be any witness for A, B, C, and O. Then $P \neq O$. Let Q and R be such that PQR is an equilateral triangle centered at O. By (a) and $\pm 120°$ rotation about O, both of Q

and R are witnesses for A, B, C, and O as well. Consequently, O is also the center of $f(AOB)$.

Let D be such that $ABCD$ is a rhombus and let S be any witness for A, B, C, and D. By (a), the circles $f(ABC)$ and $f(ACD)$ are symmetric with respect to line AC. Since the center O of $f(ABC)$ lies outside of line AC, this means that S lies on line AC. Similarly, S must lie on line BD as well. Therefore S is the midpoint of AC. $\square$

Lemma 2. Let ABC be any isosceles triangle with $AB = AC$. Then the center of $f(ABC)$ lies on the perpendicular bisector of side BC.

Proof. By (a) and reflection across the perpendicular bisector of side BC. $\square$

Lemma 3. Let ABC be any triangle. Then the midpoints of its sides AB, BC, and CA lie either inside or on $f(ABC)$.

Proof. Suppose for the sake of contradiction that the midpoint M of side AC lies outside of $f(ABC)$. Then there exists some line ℓ through M such that ℓ does not meet $f(ABC)$. Let D be such that $ABCD$ is a parallelogram. Then, by (a), ℓ separates $f(ABC)$ and $f(ACD)$. This contradicts (b) for A, B, C, and D. $\square$

Lemma 4. Let ABC be any isosceles triangle with $AB = AC$ and $\angle A \leq 30°$. Then $f(ABC)$ is tangent to BC at its midpoint, and it lies on the same side of line BC as point A.

Proof. Let D be such that A and D lie on the same side of line BC and triangle BCD is equilateral. Let M be the midpoint of BC, let N be the midpoint of AM, and let k be the circle on diameter MN.

By Lemma 1 and $\angle A \leq 30°$, $f(BCD)$ lies inside k, except that they are tangent at M. By Lemmas 2 and 3, k lies inside of $f(ABC)$, except that they could be tangent at M. Therefore M is the only possible witness for A, B, C, and D. $\square$

Lemma 5. Let ABC be any isosceles triangle with $AB = AC$ and $\angle A \geq 150°$. Then $f(ABC)$ is tangent to BC at its midpoint, and it lies on the same side of line BC as point A.

Proof. Let D be such that $ABDC$ is a rhombus. By Lemma 4 for ABD and ACD, the only possible witness for A, B, C, and D is the common midpoint of AD and BC. Then we finish by Lemmas 2 and 3 for ABC, just as in the proof of Lemma 4. $\square$

Lemma 6. Every acute-angled triangle ABC with $\angle A \leq 15°$, $\angle B \geq 75°$, and $\angle C \geq 75°$ is well-behaved.

Proof. Let D be the reflection of A across line BC. By Lemma 5, $75° \leq \angle B < 90°$, and $75° \leq \angle C < 90°$, we get that the projection P of A onto line BC is the unique common point of $f(ABD)$ and $f(ACD)$. By (b), P lies on $f(ABC)$. Similarly (from Lemmas 4 and 5), so do the projections Q of B onto line CA and R of C onto line AB. Therefore $f(ABC)$ is the circumcircle of triangle PQR. $\square$

Lemma 7. Let ABC be any triangle. Suppose that there exists some disc $\mathcal{D}$ such that for all $D \in \mathcal{D}$ both triangles ABD and ACD are well-behaved. Then triangle ABC is well-behaved as well.

Proof. Suppose for the sake of contradiction that ABC is not well-behaved. Then $f(ABC)$ and the Euler circle e of ABC have at most two common points.

By shrinking $\mathcal{D}$ if necessary, we can ensure that for all $D \in \mathcal{D}$ the midpoint M of AD does not lie on e, and the Euler circles e_1 and e_2 of ABD and ACD are distinct. Thus e_1 and e_2 meet at M as well as one more point N which, since the Euler circle mapping satisfies condition (b) of the problem, must lie on e.

However, by shrinking $\mathcal{D}$ even more if necessary, we can ensure that M does not lie on $f(ABC)$ and that N is not one of the common points of e and $f(ABC)$, for all $D \in \mathcal{D}$. Thus circles $f(ABC)$, $f(ABD) = e_1$, and $f(ACD) = e_2$ cannot have a common point, and we arrive at a contradiction. □

Lemma 8. Every triangle with two angles strictly smaller than $7°30'$ is well-behaved.

Proof. Consider a sufficiently small disk $\mathcal{D}$ centered at the triangle's circumcenter, and then apply Lemmas 6 and 7. □

Lemma 9. Suppose that every triangle with two angles strictly smaller than θ is well-behaved. Then every triangle with one angle strictly smaller than 2θ is well-behaved as well.

Proof. Suppose that $\angle A < 2\theta$ in triangle ABC. Then there exists a disk $\mathcal{D}$ such that $\angle BAD < \theta$ and $\angle CAD < \theta$ for all $D \in \mathcal{D}$, and, provided that $\mathcal{D}$ lies sufficiently far away from ABC, we also have $\angle ADB < \theta$ and $\angle ADC < \theta$ for all $D \in \mathcal{D}$. Now we apply Lemma 7. □

By Lemma 8 and repeated application of Lemma 9, all triangles are well-behaved. This completes the solution.

21. (D.Ratarov, 10–11) A trapezoid $ABCD$ is bicentral. The vertex A, the incenter I, the circumcircle ω and its center O are given and the trapezoid is erased. Restore it using only a ruler.

    Solution. Let AO meet ω for the second time at point M. Let ray AI meet ω at M'. Finally let $M'O$ meet ω at point M''. The angles $AM'M$ and $M'AM''$ are right, because AM and $M'M''$ are diameters, thus lines AM'' and MM' are parallel as two perpendiculars to AM'. Using only a ruler we can pass the line through I parallel to them. This line meet ω at B because $\angle AIB = \pi/2$ (AI and BI bisect angles BAD and ABC respectively). Construct line j through O and I. Using only a ruler we can construct two lines perpendicular to the diameter j. Finally construct the lines passing through A and B and parallel to these perpendiculars. They meet ω at points D and C respectively.

22. (Ju.Nesterov, V.Protasov, 10–11) A convex polyhedron and a point K outside it are given. For each point M of a polyhedron construct a ball with diameter MK. Prove that there exists a unique point on a polyhedron which belongs to all such balls.

    Solution. Let P be the point of the polyhedron nearest to K. Since the polyhedron is convex, P is defined uniquely and the polyhedron lies on the one side from the plane passing through P and perpendicular to PK. Hence the ball with diameter PK and the polyhedron have no common points distinct from P. On the other hand, P lies inside any ball with diameter KM.

23. (A.Skopenkov, 10–11) Six points in general position are given in the space. For each two of them color red the common points (if they exist) of the segment between these

points and the surface of the tetrahedron formed by four remaining points. Prove that the number of red points is even.

Solution. Each red point is the intersection of a segment determined by two given points with the interior of a triangle determined by three other given points. Thus we can assign a segment and a triangle to each red point.

Consider any way to split our six points into two triples T_1 and T_2. There are $C_6^3/2 = 10$ such splits. We claim that for each such split there is an even number of red points P such that the triangle assigned to P is either T_1 or T_2. Since each red point corresponds to exactly one such split, this would solve the problem.

Consider an arbitrary split $T_1 = \{A, B, C\}$, $T_2 = \{P, Q, R\}$. Let line ℓ be the intersection of planes ABC and PQR. (If these two planes are parallel, then no red points can be associated with that split.) Let segment t_1 be the intersection of line ℓ with the interior of triangle T_1, and define segment t_2 analogously. (If one or both of segments t_1 and t_2 is the empty set, then no red points can be associated with that split.) The red points associated with the split $\{A, B, C\}$ and $\{P, Q, R\}$ are exactly the endpoints of t_1 and t_2 that lie in the interior of the other segment.

Let us consider all possibilities for ordering of the endpoints of t_1 and t_2 along ℓ. There are only three essentially different cases: t_1 and t_2 are disjoint (0 red points); t_1 and t_2 overlap, but none is contained inside the other (2 red points); or one of t_1 and t_2 is contained inside the other (2 red points). Each case yields an even number of red points, as needed.

24. (A.Zaslavsly, 11) A truncated trigonal pyramid is circumsribed around a sphere touching its bases at points T_1, T_2. Let h be the altitude of the pyramid, R_1, R_2 be the circumradii of its bases, and O_1, O_2 be the circumcenters of the bases. Prove that

$$R_1R_2h^2 = (R_1^2 - O_1T_1^2)(R_2^2 - O_2T_2^2).$$

First solution. Let the lateral edges of the pyramid meet at S. Consider a cone with vertex S circumscribed around the insphere. It meets the bases of the pyramid by inscribed ellipses, the foci of the first ellipse are T_1 and the common point T_2' of the base with the line ST_2, and the foci of the second ellipse are T_2 and the common point T_1' of the base with ST_1. By the generalized Euler formula (see [2])

$$R_1^2l_1^2 = (R_1^2 - O_1T_1^2)(R_1^2 - O_1{T'}_2^2), \tag{1}$$

$$R_2^2l_2^2 = (R_2^2 - O_2T_2^2)(R_2^2 - O_2{T'}_1^2), \tag{2}$$

where l_1, l_2 are the minor axes of the ellipses.

Consider now the symmetry plane of the cone. It passes through S, the center of the insphere and the major axes of the ellipses. The section of the pyramid by this plane is a trapezoid $ABCD$ circumscribed around a circle with center I and diameter h, and this circle touches the bases of the trapezoid at T_1 and T_2. It is easy to see that triangle AIT_1 is similar to IBT_2, and triangle CIT_2 is similar to IDT_1 (fig. 24), therefore

$$\frac{h^2}{4} = AT_1 \cdot BT_2 = CT_2 \cdot DT_1. \tag{3}$$

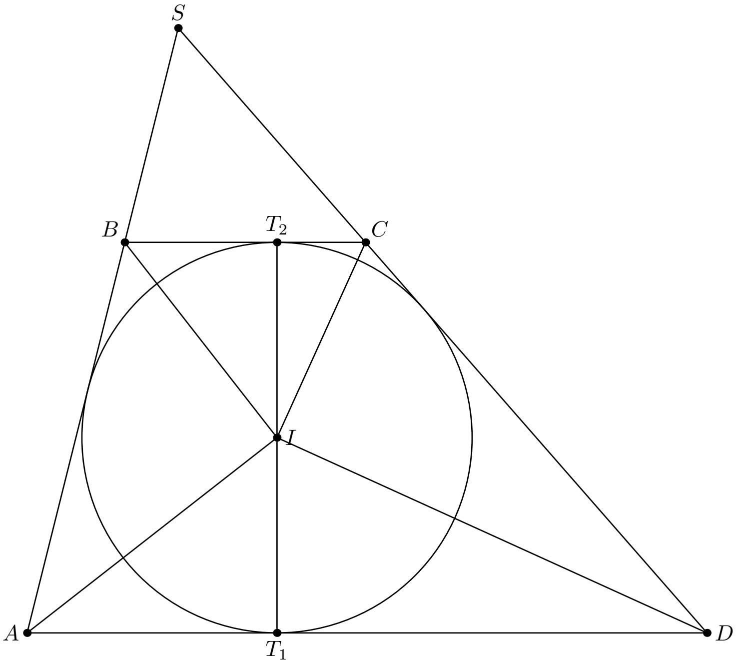

Fig. 24

Let M be the midpoint of AD. Then $AT_1 \cdot DT_1 = AM^2 - MT_1^2 = l_1^2/4$. Similarly $BT_2 \cdot CT_2 = l_2^2/4$. From this and (3) we obtain that $h^2 = l_1 l_2$. Also since the bases of the pyramid are homothetic with center S, we have $(R_1^2 - O_1T_1^2)(R_2^2 - O_2T_2^2) = (R_1^2 - O_1{T'}_2^2)(R_2^2 - O_2{T'}_1^2)$. Hence multiplying (1) and (2) we obtain the required equality.

Second solution. Let our pyramid be $A_1B_1C_1A_2B_2C_2$, and let lines A_1A_2, B_1B_2, and C_1C_2 meet at O. Suppose without loss of generality that $A_1B_1C_1$ is the larger base, so that the insphere ω of $A_1B_1C_1A_2B_2C_2$ is also the insphere of tetrahedron $OA_1B_1C_1$ and the exsphere opposite to O of tetrahedron $OA_2B_2C_2$.

Let k_1 and k_2 be the circumcircles of triangles $A_1B_1C_1$ and $A_2B_2C_2$. Observe that $R_1^2 - O_1T_1^2$ is the power of T_1 with respect to k_1, and similarly for $R_2^2 - O_2T_2^2$. Furthermore, if r is the radius of ω then, since planes $A_1B_1C_1$ and $A_2B_2C_2$ are parallel, we get that h equals $2r$. Thus our desired identity becomes $4r^2R_1R_2 = power(T_1,k_1) \cdot power(T_2,k_2)$. We rewrite this as $4r^2/R_1R_2 = (power(T_1,k_1)/R_1^2) \cdot (power(T_2,k_2)/R_2^2)$.

Let ω_O with radius r_O be the exsphere of tetrahedron $OA_1B_1C_1$ opposite O, and let ω_O touch plane $A_1B_1C_1$ at point U. Then homothety with center O maps the union of tetrahedron $OA_2B_2C_2$ and sphere ω onto the union of tetrahedron $OA_1B_1C_1$ and sphere ω_O. Consequently, $r/R_2 =_O /R_1$ and $power(T_2,k_2)/R_2^2 = power(U,k_1)/R_1^2$.

For convenience, from this point on let us write A, B, C, T, R, and k instead of A_1, B_1, C_1, T_1, R_1, and k_1. Then our desired identity becomes $4rr_O/R^2 = (power(T,k)/R^2) \cdot (power(U,k)/R^2)$. Equivalently, $4rr_OR^2 = power(T,k) \cdot power(U,k)$.

(Here $OABC$ is an arbitrary tetrahedron, k with radius R is the circumcircle of triangle ABC, r and r_O are the tetrahedron's inradius and exradius opposite O, and T and U are the tangency points of its insphere and exsphere opposite O with plane ABC.)

Let I and I_O be the centers of ω and ω_O. Let T_A and U_A be the projections of points T and U onto line BC. Since planes IBC and I_OBC are the interior and exterior angle bisectors of the dihedral angle of tetrahedron $OABC$ at edge BC, we get that $\angle IT_AT + \angle I_OU_AU = \pi/2$. Consequently, triangles IT_AT and U_AI_OU are similar. Thus $rr_O = TT_A \cdot UU_A$.

Define points T_B, U_B, T_C, and U_C analogously. Then by just the same reasoning we get also that $rr_O = TT_A \cdot UU_A = TT_B \cdot UU_B = TT_C \cdot UU_C$. Therefore points T and U are isogonal conjugates in triangle ABC. (The last statement is in fact a well-known theorem. However, we prove it anyway because the proof follows quite easily from the other steps of the solution.)

With this, we can reduce the original stereometric problem to a purely planimetric one as follows. Let points T and U be isogonal conjugates in triangle ABC (so that both of T and U lie inside the triangle), and let T_A and U_A be their projections onto side BC. Let k and R be the circumcircle and circumradius of triangle ABC. Then our desired identity becomes $4TT_A \cdot UU_A \cdot R^2 = power(T, k) \cdot power(U, k)$.

To see this, let lines BT and BU meet k for the second time at points V and W. Then $power(T, k) = BT \cdot TV$ and $power(U, k) = BU \cdot UW$. Let VX be a diameter in k. Then triangles BTT_A and XVC are similar. Consequently, $BT \cdot CV = TT_A \cdot 2R$. Analogously, $BU \cdot CW = UU_A \cdot 2R$. Thus $4TT_A \cdot UU_A \cdot R^2 = BT \cdot BU \cdot CV \cdot CW$. With this, we are only left to prove that $CV \cdot CW = TV \cdot UW$.

To this end, consider triangles CTV and CUW. We have $\angle CVT = \angle BVC = \angle A$. Analogously, $\angle CWU = \angle BWC = \angle A$. On the other hand, $\angle TCV + \angle UCW = (\angle ACT + \angle ACV) + (\angle ACU + \angle ACW) = (\angle ACT + \angle ACU) + (\angle ACV + \angle ACW) = \angle C + (\angle ABV + \angle ABW) = \angle B + \angle C$, where we firstly used the fact that rays CT and CU are isogonal in angle C, and then we used the fact that rays BV and BW are isogonal in angle B. From this, it follows that $\angle TCV = \angle CUW$ and $\angle UCW = \angle CTV$. Therefore triangles CTV and UCW are similar, and so $CV \cdot CW = TV \cdot UW$ as needed.

References

[1] V.Protasov. Touching circles, from Thebault to Feuerbach. "Kvant" 2008, N 4.

[2] "Matematicheskoje prosveschenije" 2017, N 21. Soluyion of problem 13.5.