XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. Solutions. First day. 8 form

Ratmino, July 30, 2019

1. (F.Ivlev) A trapezoid with bases AB and CD is inscribed into a circle centered at O. Let AP and AQ be the tangents from A to the circumcircle of triangle CDO. Prove that the circumcircle of triangle APQ passes through the midpoint of AB.

   Solution. Let O' be the circumcenter of triangle OCD. Then AO' is a diameter of circle APQ. Since O' lies on the perpendicular bisector of segment AB, the midpoint of segment AB also lies on this circle (fig/ 8.1).

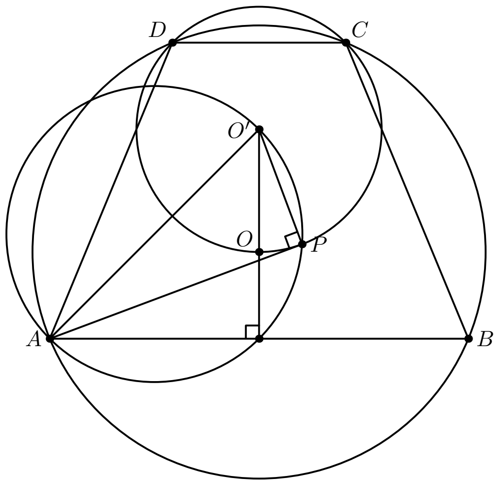


Fig. 8.1.

2. (P.Ryabov) A point M inside triangle ABC is such that $AM = AB/2$ and $CM = BC/2$. Points C_0 and A_0 lying on AB and CB respectively are such that $BC_0 : AC_0 = BA_0 : CA_0 = 3$. Prove that the distances from M to C_0 and to A_0 are equal.

   Solution. Let K, L, U, and V be the midpoints of segments AB, BC, AM and MC respectively. Then since AMK, CML are isosceles triangles, and KU, LV are medial lines of triangles ABM, CBM respectively, we have $MA_0 = LV = BM/2 = KU = MC_0$ (fig. 8.2).

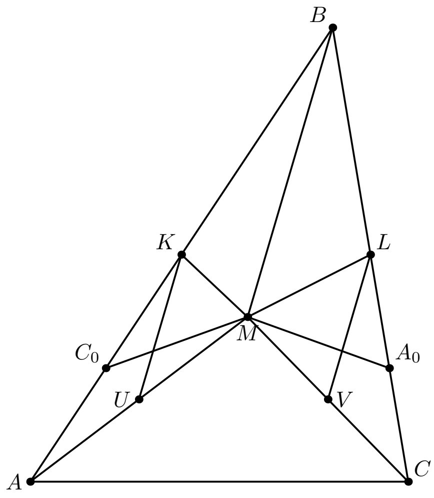


Fig. 8.2.

3. (M.Plotnikov) Construct a regular triangle using a plywood square. (*You can draw lines through pairs of points lying on the distance not greater than the side of the square, construct the perpendicular from a point to a line if the distance between them does not exceed the side of the square, and measure segments on the constructed lines equal to the side or to the diagonal of the square.*)

   Solution. Let the side of the square equal one. First we show how to construct the midpoint of any segment PQ such that the length of PQ does not exceed one. Construct any line ℓ through P such that ℓ is distinct from PQ and the angle between ℓ and PQ is distinct from $90°$. Let R be the foot of the perpendicular from Q onto ℓ. Let the line through P perpendicular to PR and the line through Q perpendicular to QR meet at S. Then line RS bisects segment PQ.

   Now to solve the problem draw two perpendicular lines through A. Plot two segments $AB = AC = 1/2$ onto them. (First plot $AB' = AC' = 1$, then halve them.) Draw line BC and the line through C perpendicular to it. Construct D so that $\angle BCD = 90°$ and $CD = 1/2$. Draw line BD and the line through D perpendicular to it. Construct E and F so that $\angle BDE = \angle BDF = 90°$ and $DE = DF = 1/2$. Draw lines BE and BF. Then triangle BEF is the desired equilateral triangle with base $EF = 1$, and with altitude and median $BD = \sqrt{3}/2$.

   Remark. We can replace the segment with length $1/2$ by an arbitrary segment with sufficiently small length, for example $3 - 2\sqrt{2}$.

4. (M.Didin, I.Frolov) Let O and H be the circumcenter and the orthocenter of an acute-angled triangle ABC with $AB < AC$. Let K be the midpoint of

AH. The line through K perpendicular to OK meets AB and the tangent to the circumcircle at A at points X and Y respectively. Prove that $\angle XOY = \angle AOB$.

Solution. Since $\angle OKY = \angle OAY = 90°$, points K and A lie on the circle with diameter OY, i.e. $\angle OYX = \angle OAK = \angle B - \angle C$. Now let M be the midpoint of BC. Then $KHMO$ is a parallelogram, i.e. the corresponding sidelines of triangles AKX and CMH are perpendicular. Therefore these triangles are similar, and $KX/OK = KX/HM = AK/CM = OM/CM$. Thus the right-angled triangles OKX and CMO are similar, and $\angle OXK = \angle COM = \angle A$ (fig 8.4). Hence $\angle XOY = 2\angle C = \angle AOB$.

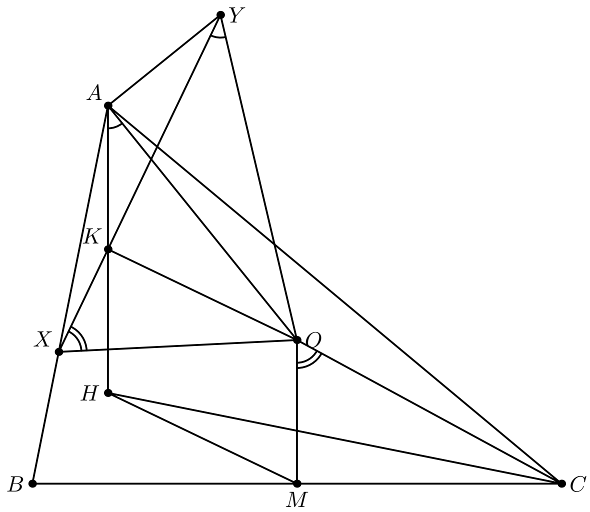


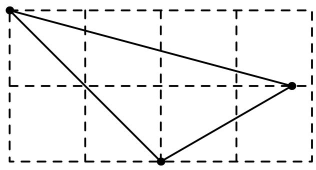
Fig. 8.4.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Solutions. Second day. 8 form

Ratmino, July 31, 2019

5. (M.Volchkevich) A triangle having one angle equal to 45° is drawn on the chequered paper (see.fig.). Find the values of its remaining angles.



Answer. 30° and 105°.

First solutions. Denote the points as in the fig. 8.5.

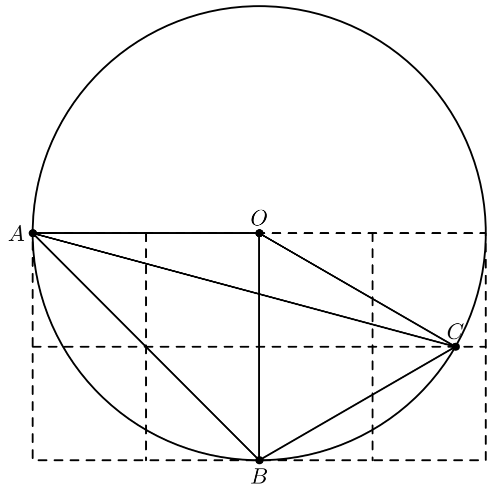


Fig. 8.5.

Since $\angle A < \angle OAB = 45^\circ = \angle OBA < \angle B$, we obtain that $\angle C = 45^\circ$. Since $OA = OB$, and $\angle AOB = 90^\circ = 2\angle ACB$, we obtain that O is the circumcenter of ABC, and $OC = OB$. But C lies on the perpendicular bisector to segment BO, thus $OC = BC$, triangle OBC is equilateral, and $\angle BOC = 60^\circ$. Hence $\angle A = 30^\circ$, $\angle B = 105^\circ$.

Second solution. Let M be the midpoint of AB. Then $\angle CMB = 45^\circ$ and we obtain that triangles ABC and CBM are similar. Hence $AB/BC = \sqrt{2}$ and by the sine law $\angle A = 30^\circ$.

6. (K.Knop) A point H lies on the side AB of regular pentagon $ABCDE$. A circle with center H and radius HE meets the segments DE and CD at points G and F respectively. It is known that $DG = AH$. Prove that $CF = AH$.

Solution. Let F' lie on segment CD so that $CF' = AH$. Then quadrilaterals $AHGE$ and $CF'HB$ are congruent by three equal sides and two equal angles, thus $HF' = HG$. To see that F' coincides with F, which would solve the problem, it suffices to verify that the second common point of line CD with the circle lies outside segment CD. To this end, prove that $\angle DCH$ is right.

Note that there exists a unique pair of points H and G lying on AB and ED respectively and such that $AH = DG$ and $HE = HG$. In fact, when H moves to A, and G moves to D, then the angle GEH increases, and the angle EGH decreases, therefore the equality $HE = HG$ is obtained in the unique position. Now let K be the common point of diagonals AD and CE, let the line passing through K and parallel to AE meet AB at H', and let the line passing through K and parallel to CD meet ED at G' (fig.8.6). Then $\angle DG'K = \angle DKG' = 72°$, i.e. $DG' = DK = EK = AH'$. Also $KH' = EA = CD = KC$ and $\angle G'KC = \angle G'KH' = 144°$. Therefore triangles CKG' and $H'KG'$ are congruent, i.e. $G'H' = G'C = H'E$ and H', G' coincide with H, G. Also $HC \perp GK \parallel CD$, q.e.d.

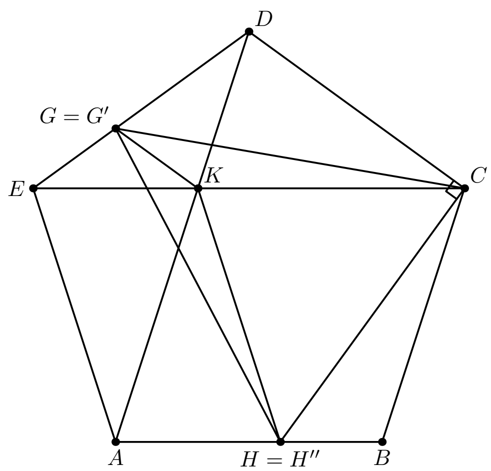


Fig. 8.6.

7. (P.Ryabov, T.Ryabova) Let points M and N lie on the sides AB and BC of triangle ABC in such a way that $MN \parallel AC$. Points M' and N' are the reflections of M and N about BC and AB respectively. Let $M'A$ meet BC at X, and $N'C$ meet AB at Y. Prove that A, C, X, Y are concyclic.

Solution. Let A' be the reflection of A about BC, and let C' be the reflection of C about AB. Let AA_1 and CC_1 be altitudes of triangle ABC. By Menelaus theorem for triangle $A'BA_1$ and line AXM', we obtain that $BX : XA_1 = 2 \cdot (BM' : M'A') = 2 \cdot (BM : MA)$. Similarly, $BY : YC_1 = 2 \cdot (BN : NC)$. Since $MN \parallel AC$, we have $BM : MA = BN : NC$, so $BX : XA_1 = BY : YC_1$, thus $XY \parallel A_1C_1$, and we are done (fig. 8.7).

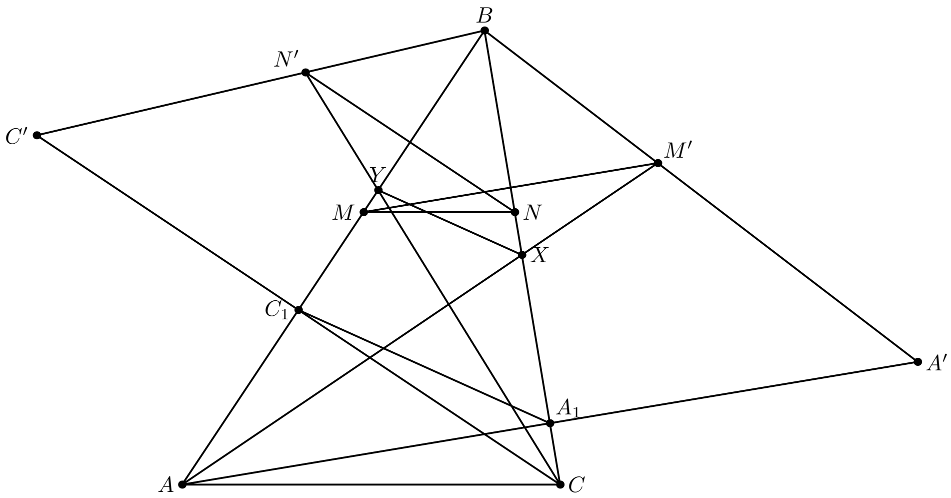


Fig. 8.7.

8. (N.Beluhov) What is the least positive integer k such that, in every convex 1001-gon, the sum of any k diagonals is greater than or equal to the sum of the remaining diagonals?

Answer. $k = 499000$.

Solution. Let $AB = 1$. Consider a convex 1001-gon such that one of its vertices is at A and the remaining 1000 vertices are within ε of B, where ε is small. Let $k + \ell$ equal the total number $\frac{1001\cdot 998}{2} = 499499$ of diagonals. When $k \geq 498501$, the sum of the k shortest diagonals is approximately $k-498501 = 998-\ell$ and the sum of the remaining diagonals is approximately ℓ. Therefore, $\ell \leq 499$ and so $k \geq 499000$.

We proceed to show that $k = 499000$ works. To this end, colour all $\ell = 499$ remaining diagonals green. To each green diagonal AB apart from, possibly, two last ones, we will assign two red diagonals AC and CB so that no green diagonal is ever coloured red and no diagonal is coloured red twice.

Suppose that we have already done this for $0 \leq i \leq 498$ green diagonals (thus forming i red-green triangles) and let AB be up next. Let D be the set of all diagonals emanating from A or B and distinct from AB; we have that

$|D| = 2 \cdot 997 = 1994$. Every red-green triangle formed thus far has at most two sides in D and there are $499 - (i + 1)$ green diagonals distinct from AB for which the triangles are not constructed. Therefore, the subset E of all as-of-yet uncoloured diagonals in D contains at least $1994 - 2i - (499 - (i+1)) = 1496 - i$ elements.

When $i \leq 498$, we have that $|E| \geq 998$. The total number of endpoints distinct from A and B of diagonals in D, however, is 999. Therefore, there exist two diagonals in E having a common endpoint C and we can assign AC and CB to AB or no two diagonals in E have a common endpoint other than A and B, but if so then there are two diagonals in E that intersect. Otherwise, at least one of the two vertices adjacent to A (say a) is cut off from B by the diagonals emanating from A and at least one of the two vertices adjacent to B (say b) is cut off from A by the diagonals emanating from B (and $a \neq b$). This leaves us with at most 997 suitable endpoints and at least 998 diagonals in E, a contradiction.

By the triangle inequality, this completes the solution.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. Solutions. First day. 9 form

Ratmino, July 30, 2019

1. (V.Protasov) A triangle OAB with $\angle A = 90^\circ$ lies inside a right angle with vertex O. The altitude of OAB from A is extended beyond A until it intersects the side of angle O at M. The distances from M and B to the second side of angle O are equal to 2 and 1 respectively. Find the length of OA.

   Answer. $\sqrt{2}$.

   First solution. Let S be the projection of B onto line OM. Then quadrilateral $ABOS$ is *cyclic*, so $\angle OAS = \angle OBS = 90^\circ - \angle BOM = \angle OMA$, thus triangles AOS and MOA are similar (fig. 9.1). Therefore $OA^2 = OS \cdot OM = 1 \cdot 2$, and $OA = \sqrt{2}$.

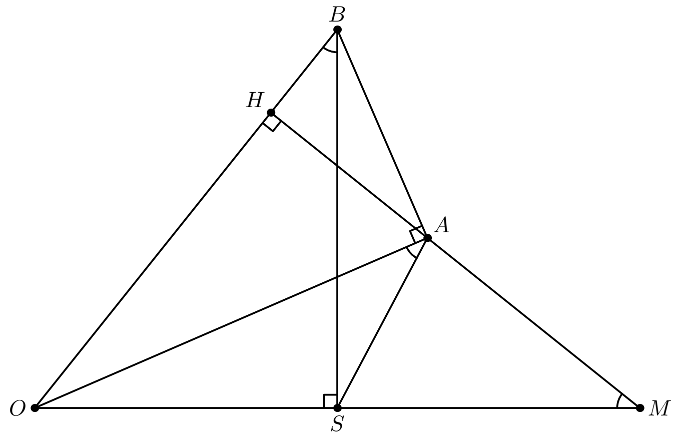


Fig. 9.1.

   Second solution. Let AH be the altitude of the triangle. Then $BHSM$ is a cyclic quadrilateral, therefore $OH \cdot OB = OS \cdot OM = 2$. But $OH \cdot OB = OA^2$ by the property of a right-angled triangle.

2. (D.Prokopenko) Let P lie on the circumcircle of triangle ABC. Let A_1 be the reflection of the orthocenter of triangle PBC about the perpendicular bisector to BC. Points B_1 and C_1 are defined similarly. Prove that A_1, B_1, and C_1 are collinear.

   First solution. Let H be the orthocenter of triangle ABC. Let P move along the circumcircle of ABC with a constant velocity. Then points A_1,

B_1 and C_1 move along circles BHC, CHA and AHB respectively with the same velocity. So it suffices to find a particular case when A_1, B_1, C_1 and H are collinear; then they will always be collinear. For example, the special case when AP is a diameter is very easy to verify.

Second solution. Let P' be the point opposite to P on the circumcircle of ABC. Then A_1 is the reflection of P' about BC, therefore A_1 lies on the Steiner line of P'. Similarly we obtain that B_1 and C_1 lie on the same line.

3. (I.Kukharchuk) Let $ABCD$ be a cyclic quadrilateral such that $AD = BD = AC$. A point P moves along the circumcircle ω of $ABCD$. The lines AP and DP meet the lines CD and AB at points E and F respectively. The lines BE and CF meet at point Q. Find the locus of Q.

Answer. A circle k passing through B, C and touching AB, CD.

Solution. Let S be the intersection point of segments AC and BD. Then S is the interior center of similarity for k and ω (because the tangent to ω at D is parallel to AB). Let ray SP meet k at Q'. We are going to prove that lines AP and BQ' meet on line CD. Then it would follow just in the same way that lines CQ' and DP meet on line AB, and, therefore, that Q' coincides with Q (fig. 9.3).

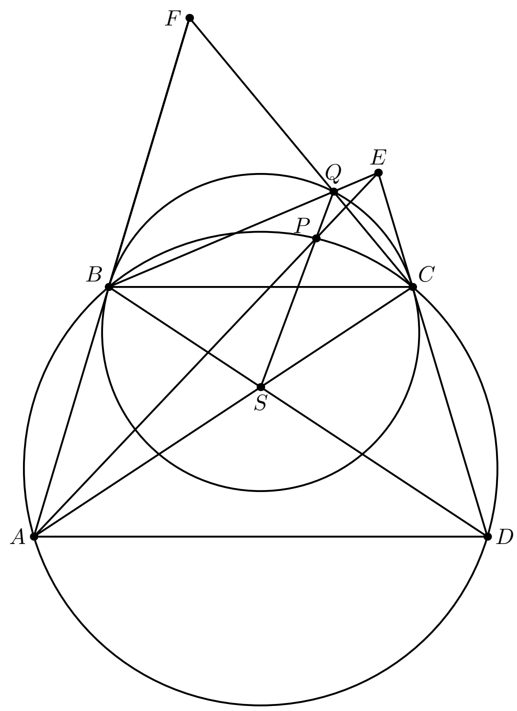


Fig. 9.3.

Let line SP meet ω for the second time at R, let lines AB and CD meet at T, and let line BD meet k for the second time at U. Let lines BQ' and AP meet line CD at E' and E'', respectively. We need to show that E' and E'' coincide. We will use cross-ratios.

We have that $(C, D, T, E') = (BC, BD, BT, BE') = (C, U, B, Q') = (A, B, D, R) = (C, D, B, P) = (AC, AD, AB, AP) = (C, D, T, E'')$ (the second equality is obtained by the projection from B onto k, and the third one follows from the homothety with center S between k and ω). This completes the solution.

Remark. We can also prove that B, Q' and E are collinear by the following way. Let R' be the common point of k and the line SP distinct from Q'. Since S is the homothety center of k and ω, we have $AP \parallel CR'$ and $BQ' \parallel DR$. Hence $\angle Q'CE = \angle Q'BC = \angle Q'R'C = \angle Q'PE$, i.e. $PQ'EC$ is a cyclic quadrilateral. Also $\angle BQ'R = \angle Q'RD = \angle PCE$. Therefore $\angle PQ'E + \angle BQ'P = 180°$, q.e.d.

4. (V.Protasov) A ship tries to land in the fog. The crew does not know the direction to the land. They see a lighthouse on a little island, and they understand that the distance to the lighthouse does not exceed 10 km (the precise distance is not known). The distance from the lighthouse to the land equals 10 km. The lighthouse is surrounded by reefs, hence the ship cannot approach it. Can the ship land having sailed the distance not greater than 75 km? (The waterside is a straight line, the trajectory has to be given before the beginning of the motion, after that the autopilot navigates the ship according to it.)

   Answer. Yes, it can.

   Solution. Let the ship be at point K, the lighthouse be at point M, and K' be the point of ray KM such that $KK' = 10$ km. To guarantee the attainment of the land, the convex hull of the trajectory has to contain the disc centered at M with radius KK', but since the position of M on segment KK' is not known, this convex hull has to contain the union of all such disks centered at KK'. It is clear that this condition is also sufficient.

   Let ω, ω' be circles centered at K, K' respectively with radii equal to KK', let CC' and DD' be the common tangents to these circles, X be the point of line CC' such that $\angle XKC = 30°$, XA be the tangent to ω, B be the midpoint of arc $C'D'$ lying outside ω, and Y be the projection of B onto CC' (fig. 9.4). Then the trajectory $KXADD'BY$ satisfies the condition, and its length equals $10(\sqrt{3} + 2\pi/3 + 1 + \pi/2 + 1) < 74$ km.

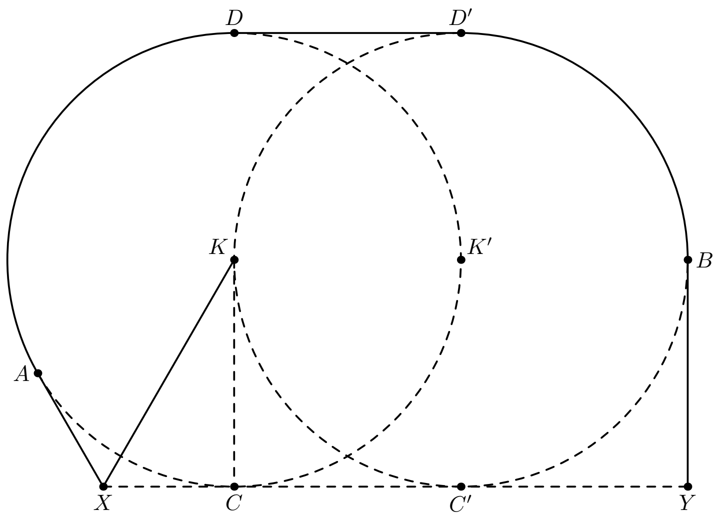


Fig. 9.4.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9 form

Ratmino, July 31, 2019

5. (A.Akopyan). Let R be the inradius of a circumscribed quadrilateral $ABCD$. Let h_1 and h_2 be the altitudes from A to BC and CD respectively. Similarly h_3 and h_4 are the altitudes from C to AB and AD. Prove that
$$\frac{h_1+h_2-2R}{h_1h_2} = \frac{h_3+h_4-2R}{h_3h_4}.$$
Solution. Let a be the length of the tangent from A to the incircle, and define b, c and d similarly. Then, by calculating the area of $ABCD$ in three different ways, we obtain $h_1(b+c)+h_2(c+d) = h_3(a+b)+h_4(a+d) = 2R(a+b+c+d)$. Multiply both sides of the desired identity by $a+b+c+d$. Then the left-hand side numerator simplifies to $h_1(a+d)+h_2(a+b)$, and similarly for the right-hand side. So we are left to prove that $(a+b)/h_1 + (a+d)/h_2 = (b+c)/h_3 + (c+d)/h_4$. This is clear since we have that $h_1(b+c) = h_3(a+b)$ by calculating the area of triangle ABC in two different ways, and similarly for h_2 and h_4.

6. (M.Saghafian) A non-convex polygon has the property that every three consecutive its vertices form a right-angled triangle. Is it true that this polygon has always an angle equal to 90° or to 270°?

First solution. (N.Beluhov). Let $A = (0,1)$, $B = (1,0)$, $C = (1,1)$, $D = (2,0)$, $E = (2,1)$, $F = (3,0)$, and let G be the intersection point of line BE and the line through F perpendicular to AF (fig. 9.6). Then heptagon $ABCDEFG$ does work.

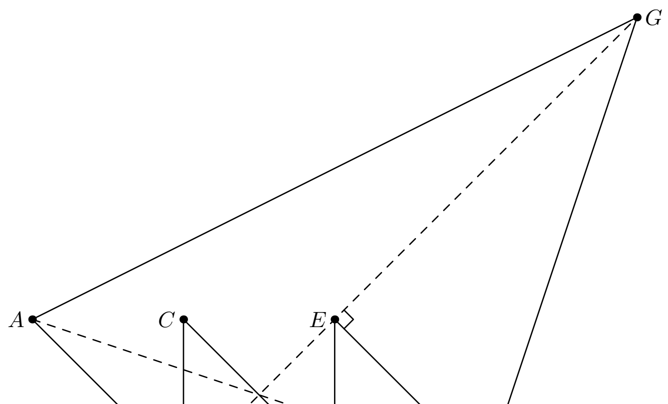


Fig. 9.6.

Second solution. Take a rectangle with sides equal to 2 and $\sqrt{3}$, and on each its side construct outside it a trapezoid with the ratio of sides equal to $1:1:1:2$, in such a way that the smallest base of the trapezoid coincides with the side of the rectangle. Every three consecutive vertices of obtained 12-gon form a triangle with angles equal to 30°, 60° and 90°, but every angle of 12-gon is equal to 60° or to 330°.

Third solution. (Found by the participants of the olympiad.) Fix two points A_4, A_5 and some point A_3 lying on the circle with diameter A_4A_5 and such that $A_3A_4 < A_3A_5$. Let A_2 be an arbitrary point inside triangle $A_3A_4A_5$ such that $\angle A_3A_2A_4 = 90^\circ$, and A_1 be such that $A_3A_1 \parallel A_4A_2$ and $\angle A_4A_1A_5 = 90^\circ$. If A_2 lies near the segment A_4A_5 we have $\angle A_1A_2A_5 < 90^\circ$. And if the angle between A_2A_4 and the tangent to circle $A_3A_4A_5$ at A_3 is small we have $\angle A_1A_2A_5 > 90^\circ$. Hence there exists such position of A_2 that $\angle A_1A_2A_5 = 90^\circ$. The corresponding pentagon $A_1A_2A_3A_4A_5$ is the required one.

7. (F.Yudin) Let the incircle ω of triangle ABC touch AC and AB at points E and F respectively. Points X, Y of ω are such that $\angle BXC = \angle BYC = 90^\circ$. Prove that EF and XY meet on the medial line of ABC. t **First solution.** Let A_0, B_0, C_0 be the midpoint of BC, CA, AB respectively. Let EF meet B_0C_0, A_0B_0, and A_0C_0 at points Z, M, and N respectively. Then M and N are the projections of C and B to the bisectors of angles B and C respectively, hence M and N lie on the circle $BXYC$. Also since $A_0C_0 \parallel AC$ and $A_0B_0 \parallel AB$ we obtain that $ZE/ZN = ZB_0/ZC_0 = ZM/ZF$, i.e. the powers of Z with respect to ω and the circle $BXYC$ are equal, thus Z lie on XY.

   Second solution. Let I be the incenter of triangle ABC, let H be the orthocenter of triangle BIC, let k be the circle with diameter IH, and let Γ be the circle with diameter BC.

   Observe that line XY is the radical axis of circles ω and Γ.

   Let K and L be the projections of B and C onto lines CI and BI respectively. It is well-known that K and L lie on line EF. Therefore, line EF is the radical axis of circles k and Γ.

   We are left to show that the midline ℓ of triangle ABC opposite to A is the radical axis of circles k and ω.

Let M and N be the projections of A onto lines BI and CI, respectively. It is well-known that M and N lie on line ℓ. We are going to show that the powers of M with respect to circles k and ω are equal. Then we would have similarly that the powers of N with respect to circles k and ω are equal as well.

Observe that the polar of A with respect to ω is line EF, which passes through L. So the polar of L with respect to ω passes through A. On the other hand, the polar of L with respect to ω is perpendicular to IL. So the polar of L with respect to ω is line AM. Consequently, the polar of M with respect to ω is line CL. Let MP and MQ be the tangents from M to ω. Then P and Q lie on line CL (fig.9.7). Therefore, $ML \cdot MI = MP^2$ and so the powers of M with respect to circles k and ω are equal, as needed.

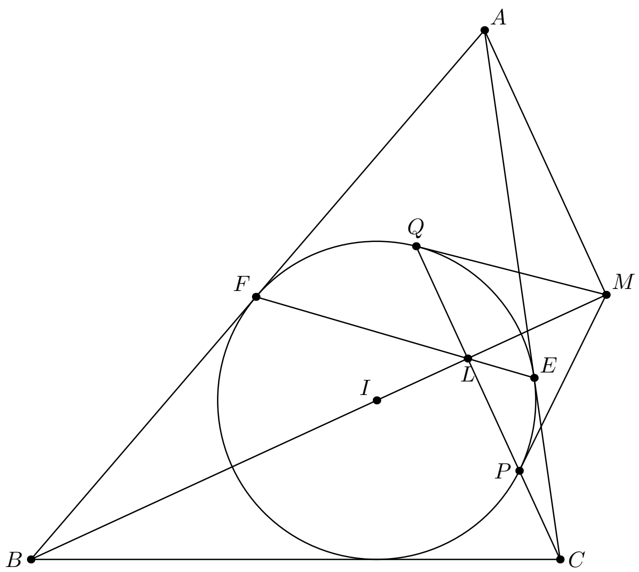


Fig. 9.7.

8. (I.Frolov) A hexagon $A_1A_2A_3A_4A_5A_6$ has no four concyclic vertices, and its diagonals A_1A_4, A_2A_5, and A_3A_6 concur. Let l_i be the radical axis of circles $A_iA_{i+1}A_{i-2}$ and $A_iA_{i-1}A_{i+2}$ (the points A_i and A_{i+6} coincide). Prove that l_i, $i = 1, \dots, 6$, concur.

Solution. Let $A_1, \dots, A_5$ be fixed and A_6 move along the line passing through A_3 and the common point of the diagonals of quadrilateral $A_1A_2A_4A_5$. Then the center O of circle $A_1A_2A_5$ is fixed, and the center O' of circle $A_1A_3A_6$ moves along the perpendicular bisector to segment A_1A_3 in such a way that the correspondence between A_6 and O' is projective (because $\angle O'A_1A_6 = \pi/2 - \angle A_6A_3A_1 = const$). Since the radical axis l_1 is perpendicular to OO', we obtain that the correspondence between A_6 and l_1 is also projective, thus the correspondence between lines l_1 and l_2 rotating around A_1 and A_2 is projective too. Therefore the common point of these lines moves along some conic. Since both lines coincide with A_1A_2, when A_6 meets the circle $A_1A_2A_3$, this conic degenerates to A_1A_2 and another line passing through A_3. Also when A_6 meets the circle $A_2A_3A_5$ then the common point lies on l_3, therefore it lies on l_3 for all positions of A_6. So l_1, l_2 and l_3 concur. Similarly we obtain that three remaining radical axes pass through the same point.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. Solutions. First day. 10 form

Ratmino, July 30, 2019

1. (A.Dadgarnia) Given a triangle ABC with $\angle A = 45°$. Let A' be the antipode of A in the circumcircle of ABC. Points E and F on segments AB and AC respectively are such that $A'B = BE$, $A'C = CF$. Let K be the second intersection of circumcircles of triangles AEF and ABC. Prove that EF bisects $A'K$.

   Solution. Let K' be the reflection of A' about the line EF. Since $\angle BA'E = \angle CA'F = 45°$, we have that $\angle EK'F = \angle EA'F = 45°$, and thus $AK'EF$ is a cyclic quadrilateral. Then $\angle K'EB = \angle K'FC$. Furthermore, $K'E : EB = A'E : EB = \sqrt{2} = A'F : FC = K'F : FC$, thus triangles $K'EB$ and $K'FC$ are similar. Then $\angle BK'C = 45°$, so $AK'BC$ is a cyclic quadrilateral and K' coincides with K (fig. 10.1).

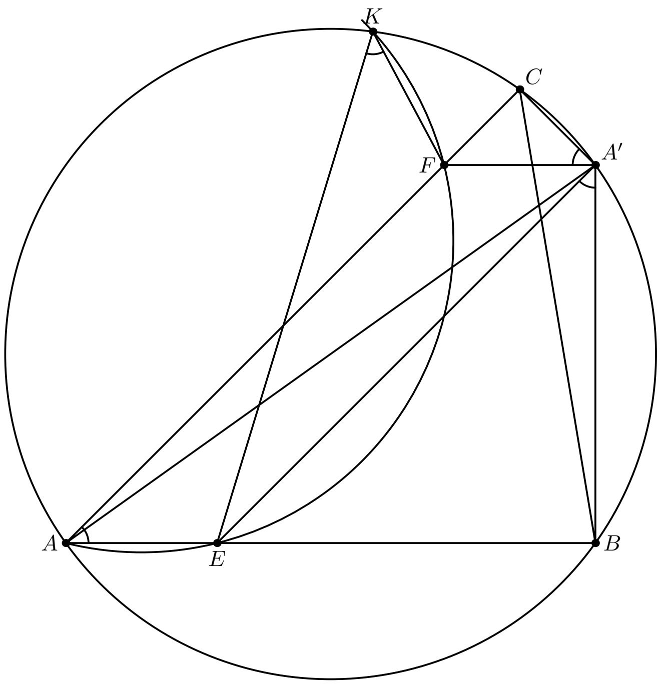


Fig. 10.1.

Second solution. Let O be the circumcenter of ABC. Note that EBA' and FCA' are isosceles right-angled triangles, thus $AEA'F$ is a parallelogram, and O is the midpoint of EF. Also O lies on the perpendicular bisector to AK, but O does not coincide with the circumcenter of $AKEF$. Therefore

$EF \parallel AK$, i.e. EF is the medial line of triangle $AA'K$, which yields the required assertion.

2. (F.Ivlev) Let A_1, B_1, C_1 be the midpoints of sides BC, AC and AB of triangle ABC, AK be its altitude from A, and L be the tangency point of the incircle γ with BC. Let the circumcircles of triangles LKB_1 and A_1LC_1 meet B_1C_1 for the second time at points X and Y respectively and γ meet this line at points Z and T. Prove that $XZ = YT$.

   Solution. Since $BC \parallel B_1C_1$, both of KB_1XL and A_1LYC_1 are isosceles trapezoids. Then $\angle BLX = \angle CKB_1 = \angle BA_1C_1 = \angle CLY$, thus X and Y are symmetric with respect to line IL, where I is the incenter of triangle ABC (fig. 10.2). It is clear that Z and T are also symmetric with respect to IL, therefore $XZ = YT$, as needed.

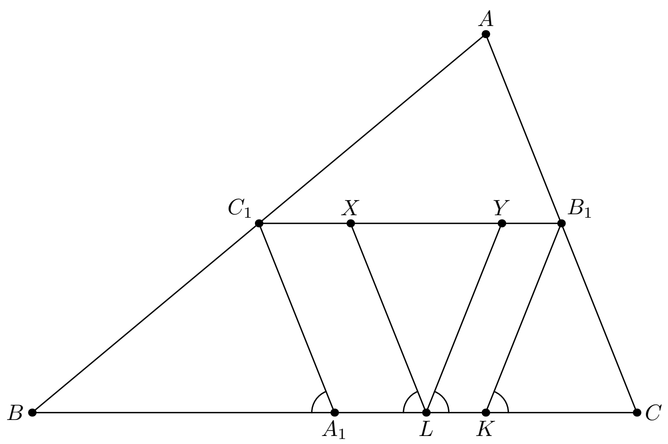


Fig. 10.2.

3. (A.Bhattacharya) Let P and Q be isogonal conjugates inside triangle ABC. Let ω be the circumcircle of ABC. Let A_1 be a point on arc BC of ω satisfying $\angle BA_1P = \angle CA_1Q$. Points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1, and CC_1 are concurrent.

   First solution. Let A_1' be the the Miquel point for lines BP, BQ, CP, and CQ. Then $\angle BA_1'C = (\pi - \angle BPC) + (\pi - \angle BQC) = \pi - \angle A$ (fig.10.3), hence A_1' lies on ω. Also A_1' is the center of the similarity that maps B onto P and Q onto C. (Then it is also the center of the similarity that maps B onto Q and P onto C). Thus $\angle BA_1'P = \angle CA_1'Q$ and A_1' coincides with A_1 (a unique point A_1 satisfies to $\angle BA_1P = \angle CA_1Q$ because when the point moves along the arc BC one of these angles increases and the second one decreases). Then since triangle A_1BP is similar to triangle A_1QC, and triangle A_1BQ is similar to triangle A_1PC, we have $BA_1 : A_1C = (BA_1 : A_1P) \cdot (PA_1 : A_1C) = (BQ : PC) \cdot (BP : QC) = (BP \cdot BQ) : (CP \cdot CQ)$. Express ratios $(CB_1 : B_1A)$ and $(AC_1 : C_1B)$ similarly, and then multiply all three to obtain one. It follows that the main diagonals of inscribed hexagon $AC_1BA_1CB_1$ are concurrent.

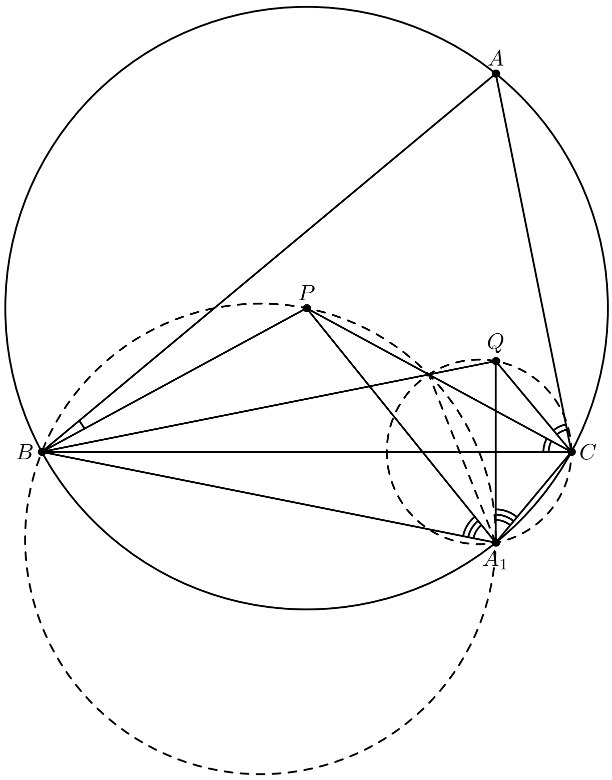


Fig. 10.3.

Second solution. Since the bisectors of angles BA_1C and PA_1Q coincide we have that the reflections of BA_1, CA_1, PA_1 and QA_1 with respect to the bisectors of angles PBQ, PCQ, BPC and BQC respectively are concurrent or parallel. Since the bisectors of angles PBQ and PCQ coincide with the bisectors of angles B and C of triangle ABC and A_1 lies on the circumcircle jf this triangle we obtain that these four lines are parallel, i.e. A_1 is isogonally conjugated with respect to quadrilateral $BPCQ$ to the infinite point of its Gauss line (i.e. coincide with the Miquel point of lines BP, BQ, CP и CQ). But the lines passing through A, B, C and parallel to the Gauss lines of quadrilaterals $BPCQ$, $APCQ$, $APBQ$ respectively concur at the point anticiomplimentar to the midpoint of PQ with respect to ABC. Therefore AA_1, BB_1, CC_1 concur at the isogonally conjugated point.

4. (L.Emelyanov) Prove that the sum of two nagelians is greater than the semiperimeter of the triangle. (A nagelian is the segment between a vertex of a triangle and the tangency point of the opposite side with the corresponding excircle.)

Solution. Let the incircle of ABC touch BC, CA, AB at points A', B', C' respectively, and let the correspondent excircles touch these sides at A'', B'', C''. Suppose that $\angle A \leq \angle B \leq \angle C$. Then $AA'' \geq BB'' \geq CC''$ and we have to prove that the sum of BB'' and CC'' is greater than the semiperimeter p. Let CH be the altitude of the triangle, and A_1 be the point of ray BA such that $BA_1 = p$ (fig.10.4.1). Then $AB'' = AA_1 = p - c$ and $p < BB'' + B''A_1$. Proving that $B''A_1 < CH$ we obtain that $p < BB'' + CH < BB'' + CC''$.

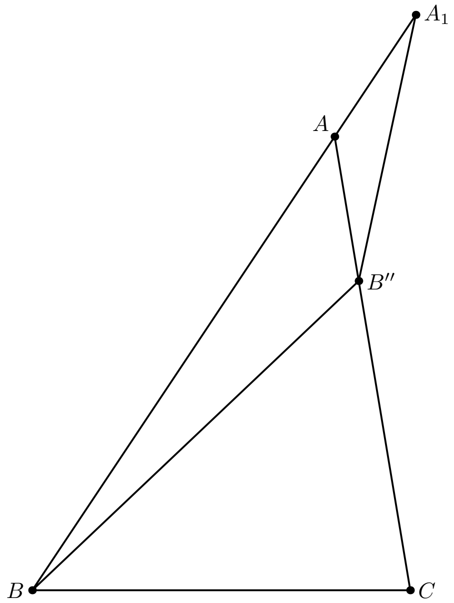


Fig. 10.4.1

Since $A_1B'' = 2(p-c)\cos\frac{\angle A}{2}$, $CH = AC\sin\angle A = 2AC\sin\frac{\angle A}{2}\cos\frac{\angle A}{2}$, we have to prove that $AC\sin\frac{\angle A}{2} > p - c$.

Let P be the projection of C to the bisector of angle A. Then P lies on segment $A'C'$ because $\angle C \geq \angle B$ (fig.10.4.2). Also $PC = AC\sin\frac{\angle A}{2}$, $A'C = p - c$ and $\angle PA'C = (\pi + \angle B)/2$, therefore CP is the greatest side of triangle $A'CP$, as needed.

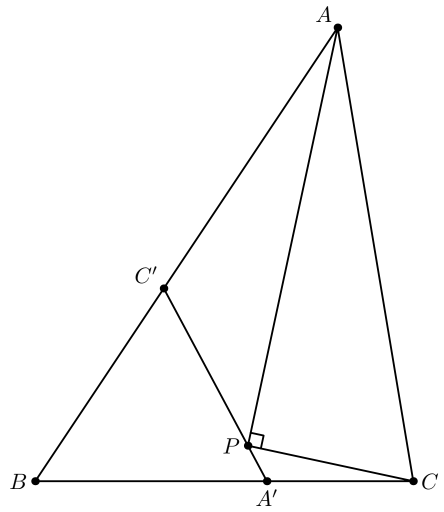


Fig. 10.4.2

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Solutions. Second day. 10 form

Ratmino, July 31, 2019

5. (D.Shvetsov) Let AA_1, BB_1, CC_1 be the altitudes of triangle ABC; and A_0, C_0 be the common points of the circumcircle of triangle A_1BC_1 with the lines A_1B_1 and C_1B_1 respectively. Prove that AA_0 and CC_0 meet on the median of ABC or are parallel to it.

Solution. Let lines AA_0 and BC meet at X and let lines CC_0 and AB meet at Y. It suffices to show that $BX : XC = BY : YA$. Observe that points A_0 and C_1 are symmetric with respect to line BB_1, as are points A_1 and C_0 (fig. 10.5). Let lines BA_0 and AC meet at Z. Then, by Menelaus theorem for triangle BCZ and line AA_0X, we obtain that $BX : XC = (BA_0 : A_0Z) \cdot (ZA : AC) = (2/AC) \cdot (BC_1 : C_1A) \cdot AB_1$. Similarly for $BY : YA$, and we are left to verify that $(BC_1 : C_1A) \cdot AB_1 = (BA_1 : A_1C) \cdot CB_1$. But this is just Ceva's theorem for the orthocenter.

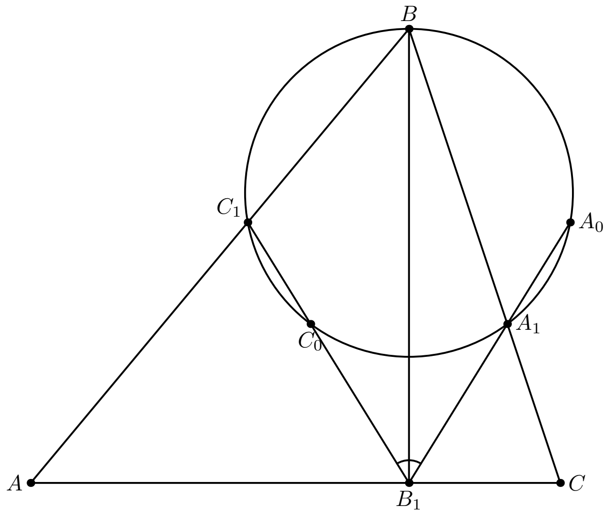


Fig. 10.5

Second solution. Since A_0, C_0 are the reflections about BH of C_1, A_1 respectively and triangles HAC_1, HCA_1 are similar we obtain that triangles HAA_0 and HCC_0 are also similar. Thus the common point of AA_0 and CC_0 coincide with the common point of circles HAC and HA_0C_0 distinct from H, i.e. with the projection of H to the median.

6. (A.Mostovoy) Let AK and AT be the bisector and the median of an acute-angled triangle ABC with $AC > AB$. The line AT meets the circumcircle of ABC at point D. Point F is the reflection of K about T. If the angles of ABC are known, find the value of angle FDA.

First solution. Let M be the midpoint of arc BC. Then $\angle MFT = \angle MKT = \angle MKC = \alpha/2 + \gamma = \angle ACM = \angle ADM = \angle TDM$, thus quadrilateral $MDFT$ is cyclic (fig/ 10/6). Then $\angle ADF = \angle TDF = \angle TMF = \angle TMK = (\beta - \gamma)/2$.

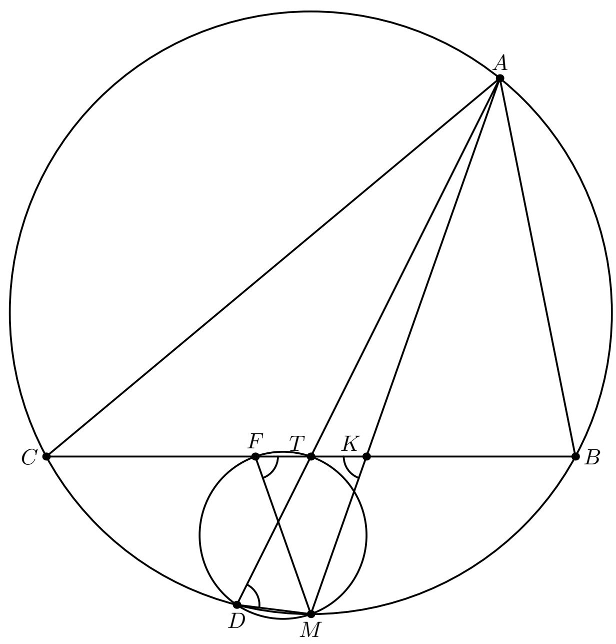


Fig. 10.6

Second solution. Let the symmedian from A meet the circumcircle at Q. Since D and Q are symmetric about the perpendicular bisector to BC, we can replace required angle FDA by equal angle KQT. Since AK and TK are bisectors of triangle AQT, we obtain that $\angle KQT = \angle KQA/2$, but the last angle equals to $\angle B = \angle C$, because the ray QK meets the circumcircle at a point forming an isosceles trapezoid with the vertices of the given triangle.

7. (Tran Quang Hung) Let P be an arbitrary point on side BC of triangle ABC. Let K be the incenter of triangle PAB. Let the incircle of triangle PAC touch BC at F. Point G on CK is such that $FG \parallel PK$. Find the locus of G.

Solution. Lemma. In triangle ABC, let I_B and I_C be the excenters opposite to B and C. Let the excircle opposite to B touch line BC at T and let ℓ be the line through T parallel to BI_C. Let P be any point on line BC and let line PI_C meet line ℓ at Q. Then $CQ \perp PI_B$.

Proof of the lemma. Let R be the intersection point of line PI_B and the line through T perpendicular to ℓ (and parallel to BI_B). Then $TQ : BI_C = TP : PB = TR : BI_B$, thus $TQ : TR = BI_C : BI_B = TC : TI_B$, i.e. triangles CTI_B and QTR are similar. It follows that triangles CTQ and I_BTR are similar as well. The angle of rotation of the similarity centered at T that maps one triangle onto the other equals $\angle CTI_B = \angle QTR = 90°$, so CQ is perpendicular to PI_B, as needed.

Return to the problem. Let the incircle touch sides AC and BC at X and Y respectively, and let Z be the midpoint of segment XY. We claim that the desired locus is the segment YZ.

To see this, observe first that the second common interior tangent to the incircles of triangles ABP and ACP passes through Y; this is well-known. Then apply the lemma to the triangle formed by the two common interior tangents to the incircles of triangles ABP and ACP and their common exterior tangent BC, and to point C on side PY of this triangle. We obtain that G lies on line XY (fig. 10.7). When P approaches B, G approaches Y; and when P approaches C, G approaches Z.

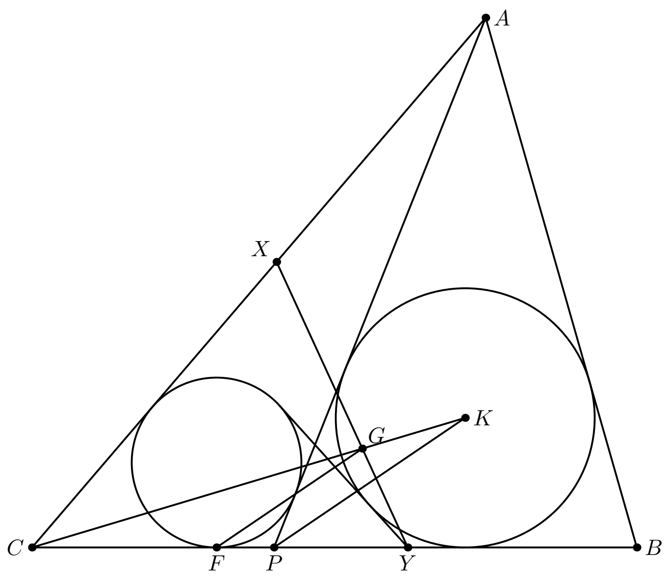


Fig. 10.7

8. (F.Nilov, Ye.Morozov) Several points and planes are given in the space. It is known that for any two of given points there exist exactly two planes

containing them, and each given plane contains at least four of given points. Is it true that all given points are collinear?

Answer. No.

Solution. Take 12 points — the midpoints of edges of cube $ABCDA'B'C'D'$, and 16 planes such that four of them pass through the center of cube and are perpendicular to its diagonals (each of these planes intersects the cube by a regular hexagon), and each of the remaining planes passes through the midpoints of four edges adjacent to the same edge of the cube (for example the midpoints of edges AB, BC, $A'B'$, and $B'C'$). It is clear that each plane contains at least four of the given points. Also for any two of the given points there exist exactly two planes containing them: the midpoints of two perpendicular edges lie on one rectangular and one hexagonal section, the midpoints of two parallel edges lying on the same face lie on two rectangular sections, and the midpoints of two opposite sections of the cube lie on two hexagonal sections.