ONE PROPERTY OF BICENTRAL QUADRILATERALS

ALEXEY A. ZASLAVSKY

Abstract. For a bicentral quadrilateral $ABCD$ with circumcenter O we prove that the incenters of triangles OAB, OBC, OCD, ODA are concyclic. The inverse assertion is also true: a cyclic quadrilateral having this property is circumscribed.

In this paper we will solve next problem (see [1]).

Let quadrilateral $ABCD$ be inscribed into a circle with center O and circumscribed around a circle with center I. Prove that the incenters of triangles OAB, OBC, OCD, ODA are concyclic. (see Fig. 1).

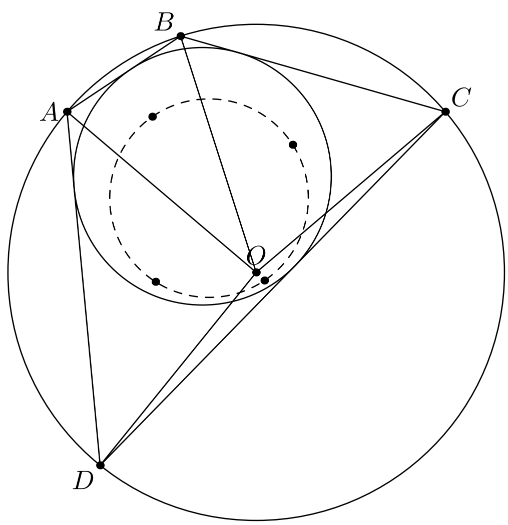

Fig. 1.

Remark 1. *Note that this circle is not the same for all quadrilaterals with given circumcircle and incircle.*

Although the formulating of the problem is elementary, it is very hard and its solution uses several nontrivial ideas. Firstly prove next

Proposition 1. *Let points A and B lie on circle ω centered at O, and let I be the incenter of triangle OAB. Then the circumcircle of triangle IAB is orthogonal to ω.*

Proof. By the trident theorem center K of circle IAB lies on the circumcircle of triangle OAB. Since $OA = OB$, we obtain that OK is a diameter of this circumcircle. Thus $KA \perp OA$ i.e. circles ω and IAB are orthogonal (see Fig. 2).

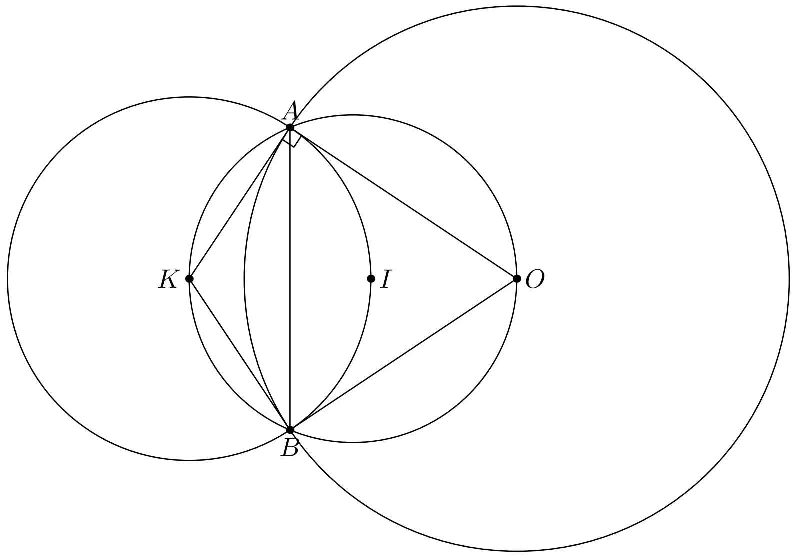

Fig. 2.

□

Now consider Poincare model of Lobachevskian geometry having ω as an absolute. From Proposition 1 we obtain that circle AIB can be considered as a line of this model. It is evident then $OI \perp AB$. Thus if we transform Poincare model to Klein model with the same absolute then I has to be the midpoint of AB. Since circles of Lobachevskian geometry correspond to Euclidean circles in Poincare model and to ellipses touching the absolute at two imaginary points in Klein model, the assertion of our problem allows next reformulating.

The midpoints of the sides of a bicentral quadrilateral lie on the ellipse touching its circumcircle at two imaginary points.

To prove this assertion use next

Proposition 2. *Let line l meet circle ω at points P and Q (real or imaginary). For an arbitrary point X denote the degree of X wrt ω as $D_\omega(X)$ and the distance from X to l as $d_l(X)$. Then the locus of such points X that the ratio $D_\omega(X) : d_l^2(X) = const$ is the conic touching ω at points P and Q.*

This proposition was proved in [2]. In [3] Nilov independently proved that for the case $D_\omega(X) = d_l^2(X)$ the locus is a parabola. Now we take an alternative proof.

Proof. Since $D_\omega(X)$ and $d_l^2(X)$ are square function of X the desired locus is a conic. It is clear that P and Q lie on it. Since D_ω maintains the sign the conic does not intersect ω. Therefore P and Q are the points of tangency. □

Inversely an arbitrary conic κ touching ω at P and Q can be obtained as a such locus. In fact let X be an arbitrary point of κ. Then the locus of points X' such that $D_\omega(X') : d_l^2(X') = D_\omega(X) : d_l^2(X)$ is a conic touching ω at P, Q and passing through X. But such conic is unique.

Remark 2. *Considering the infinite points of the desired locus it is easy to find its excentricity.*

So to solve the problem it is sufficient to find such line l that the distances from the midpoints K, L, M, N of AB, BC, CD, DA respectively to l are proportional to the correspondent sidelengths. Suppose that AB is the shortest side of $ABCD$. Let X, Y be such points that $BC \cdot \vec{XK} = AB \cdot \vec{XL}$, $AD \cdot \vec{YK} = AB \cdot \vec{YN}$. Then $d_{XY}(K) : AB = d_{XY}(L) : BC = d_{XY}(N) : AD$. Since $KLMN$ is a parallelogram and $AB + CD = AD + BC$, the ratio $d_{XY}(M) : CD$ is the same (see Fig. 3).

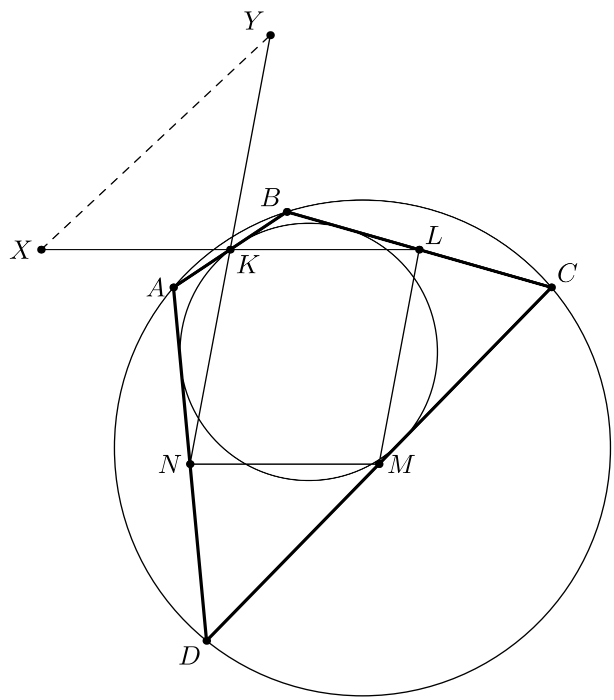

Fig. 3.

Remark 3. *As a matter of fact we proved also the inverse assertion: if $ABCD$ is a cyclic quadrilateral with circumcenter O and the incenters of triangles OAB, OBC, OCD, ODA are concyclic then $ABCD$ is circumscribed.*

Remark 4. *Using the Proposition 2 we can also prove that the midpoints of the sides of an arbitrary circumscribed quadrilateral lie on a conic touching the incircle at two points (see Fig. 4).*

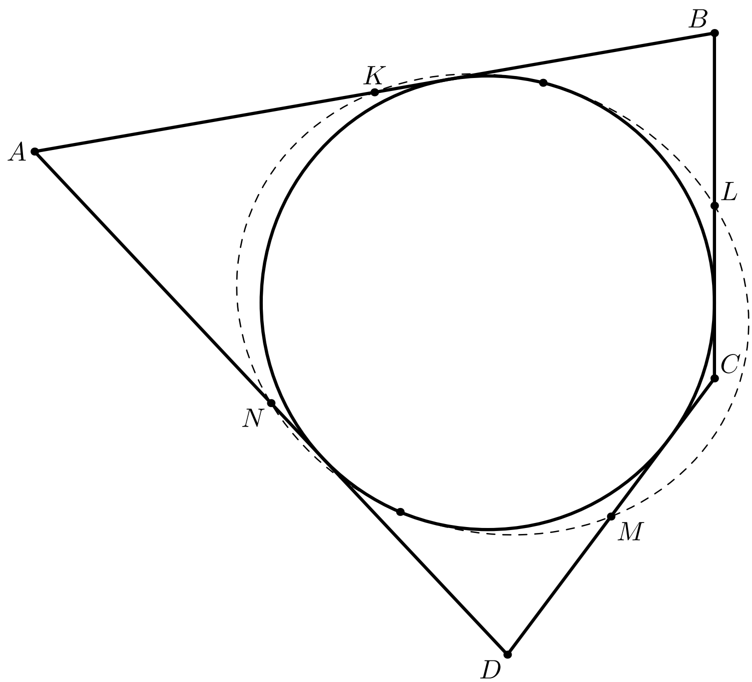

Fig. 4.

The author is grateful to F.Ivlev for the information about the considered problem.

References

[1] http://www.artofproblemsolving.com/community/c6h572848
[2] T. M. Apostol, M. A. Mnatsakanian. New Descriptions of Conics via Twisted Cylinders, Focal Disks, and Directors. Amer. Math. Monthly, 115:9 (2008).
[3] F.Nilov. Parabolic polygons. Matematicheskoe Prosveschenije, v.12, 2008 (In Russian).

Central Institute of Economy and Mathematics
E-mail address: `zaslavsky@mccme.ru`