XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. First day. 8 grade

1. (D.Mukhin, D.Shiryaev) Let $ABCD$ be a cyclic quadrilateral with $AB = BC$ and $AD = CD$. A point M lies on the minor arc CD of its circumcircle. The lines BM and CD meet at point P, the lines AM and BD meet at point Q. Prove that $PQ \parallel AC$.

   Solution. The angle MPD cuts the arcs MD and BC, and the angle MQD cuts the arcs MD and AB. Therefore these angles are equal and $MPQD$ is a cyclic quadrilateral (fig.8.1). Now since $\angle DMP = \angle DMB = 90°$, we have $PQ \perp BD$, thus $PQ \parallel AC$.

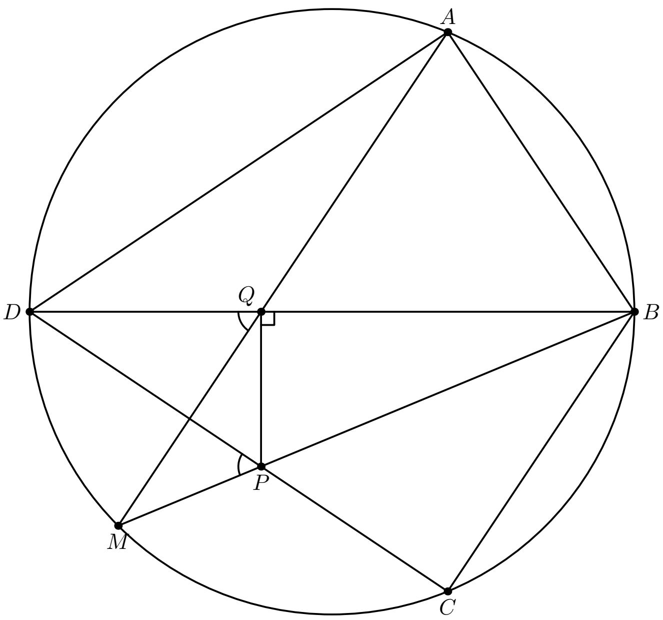

Fig. 8.1

2. (A.Sokolov) Let H and O be the orthocenter and the circumcenter of an acute-angled triangle ABC, respectively. The perpendicular bisector to segment BH meets AB and BC at points A_1 and C_1, respectively. Prove that OB bisects the angle A_1OC_1.

   First solution. Since $\angle HBC = \angle ABO = 90° - \angle C$, isosceles triangles HBC_1 and ABO are similar. Hence triangles OBC_1 and ABH are also similar, i.e., $\angle C_1OB = \angle HAB = 90° - \angle B$ (fig.8.2). Similarly $\angle A_1OB = \angle HCB = 90° - \angle B$.

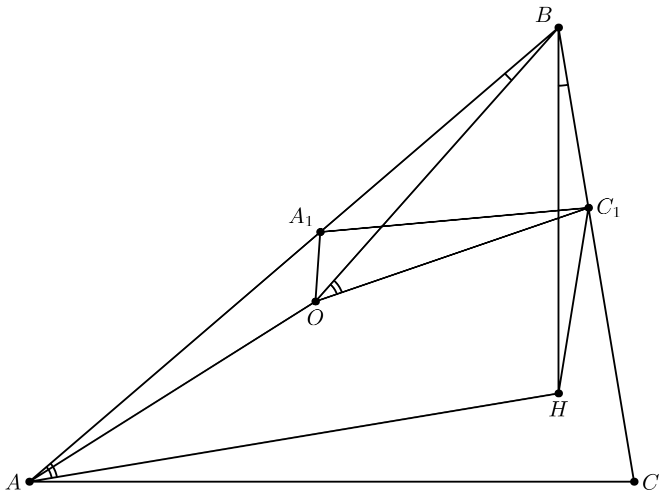

Fig. 8.2

Second solution. Use the following assertion.

Let A', B', C' be the reflections of point P about the sidelines of triangle ABC. Then the circumcenter of $A'B'C'$ is isogonally conjugated to P with respect to ABC.

Consider the triangle A_1BC_1. The reflections of H about BA_1 and BC_1 lie on the circumcircle of ABC, and the reflection of H about A_1C_1 coincide with B, thus, O and H are isogonally conjugated with respect to A_1BC_1. Then $\angle AA_1O = \angle HA_1C = \angle C_1A_1B$, i.e. C_1B is the external bisector of angle OA_1C_1. Similarly A_1B is the external bisector of angle C_1OA_1. Therefore B is the excenter of triangle A_1OC_1 and OB is the bisector of angle A_1OC_1.

3. (M.Kyranbai, Kazakhstan) Let AD, BE and CF be the medians of triangle ABC. The points X and Y are the reflections of F about AD and BE, respectively. Prove that the circumcircles of triangles BEX and ADY are concentric.

   Solution. Since $AFDE$ is a parallelogram, the midpoints of segments FE and AD coincide, therefore $EX \parallel AD$. Since FEX is a right-angled triangle, the perpendicular bisector to EX passes through the midpoint of EF, thus it coincides with the perpendicular bisector to AD (fig.8.3). Similarly we obtain that the perpendicular bisectors to DY and BE coincide.

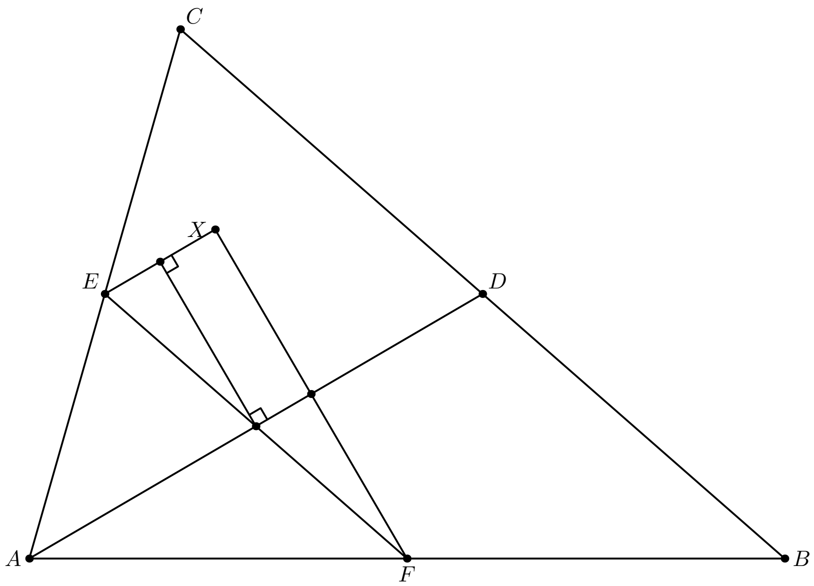

Fig. 8.3

4. (A.Shapovalov) Alex dissects a paper triangle into two triangles. Each minute after this he dissects one of obtained triangles into two triangles. After several time (at least one hour) it appeared that all obtained triangles are congruent. Find all initial triangles for which this is possible.

   Answer. Isosceles or right-angled.

   Solution. Sufficiency. Cutting an isosceles triangle by its median we obtain two congruent right-angled triangles, and cutting a right-angled triangle by the median from the right angle we obtain two isosceles triangles. Cutting each of them into two congruent triangles we obtain four congruent right-angled triangles. Similarly we can cut off each of these triangles into four congruent triangles etc.

   Necessity. After the last cutting we obtain two congruent triangles having two adjacent angles. Each of these angles is greater than a non-adjacent angle of the other triangle, thus it is equal to the adjacent angle, i.e, it is right. So the initial triangle is divided into right-angled triangles. Let their angles be α, $\beta = 90° - \alpha$ and $90°$, where $\alpha \leq \beta$. If $\alpha = 45°$ or $\alpha = 30°$, then all angles of the initial triangle are divisible by α, and direct listing of the alternatives shows that they are equal to $(45°, 45°, 90°)$, $(30°, 60°, 90°)$, $(30°, 30°, 120°)$ or $(60°, 60°, 60°)$, i.e., the triangle is right-angled or isosceles.

   For the remaining values of α the list α, β, 2α, $90°$, 2β does not contain equal angles, and all pairs of adjacent angles from this list are $(90°, 90°)$ or $(2\alpha, 2\beta)$. Let the area of all resulting triangles be 1, then the area s of the initial triangles and all areas of intermediate triangles are natural. Let us prove by induction on s that the angles of these triangles are $(\alpha, \beta, 90°)$, $(\alpha, \alpha, 2\beta)$ or $(\beta, \beta, 2\alpha)$. The base $s = 1$ is proved. A triangle T with $s > 1$ was divided into two triangles with smaller area. By induction, each of these triangles has one of three sets of angles from the list and they have two adjacent angles. If these angles are right then both triangles have a common cathetus. There are three possible cases: both opposite angles are equal to α; both angles are equal to β, or one angle is α, and the second one is β. In all these cases T belongs to one of three types. And if two adjacent angles are equal to 2α and 2β then T is right-angled with angles $(\alpha, \beta, 90°)$.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. Second day. 8 grade

5. (E.Bakaev) A square $ABCD$ is given. Two circles are inscribed into angles A and B, and the sum of their diameters is equal to the sidelength of the square. Prove that one of their common tangents passes through the midpoint of AB.

   Solution. Let O_a, O_b be the centers of the circles, T_a, T_b be their touching points with AB, and M be the midpoint of AB (fig.8.5). By the assumption of the problem, we obtain that $T_aM = MO_b$, $T_bM = MO_a$. Therefore $\angle O_aMT_a + \angle O_bMT_b = 90°$, i.e., $O_aM \perp O_bM$. Thus the line l symmetric to AB wrt O_aM is also symmetric to AB wrt O_bM. Since the distances from the centers of both circles to l are equal to their radii, we obtain that l is the common tangent.

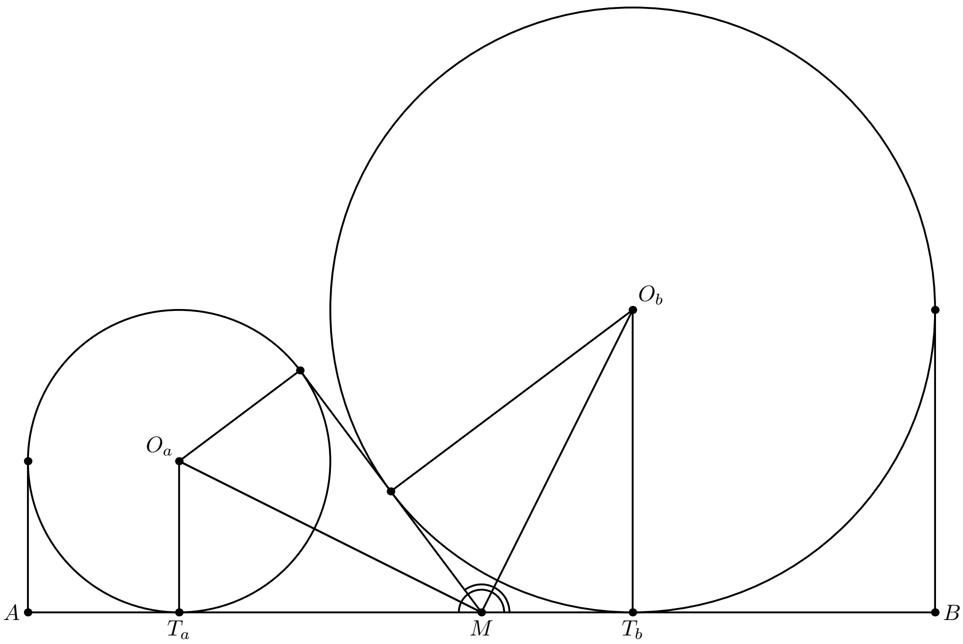

Fig. 8.5

6. (A.Shapovalov) A median of an acute-angled triangle dissects it into two triangles. Prove that each of them can be covered by a semidisc congruent to a half of the circumdisc of the initial triangle.

   Solution. Let CD be the median of triangle ABC, angle ADC be non-acute, and angle BDC be non-obtuse. Then all vertices of triangle BDC lie on the same semiplane wrt the perpendicular bisector to AB, which is the diameter of the circumcircle of ABC, therefore this triangle lies inside the corresponding semidisc. Furthermore the triangle ACD can be covered by the semidisc wuth diameter AC, hence it can be covered by a greater semidisc.

7. (E.Bakaev) Let $A_1A_2\dots A_{13}$ and $B_1B_2\dots B_{13}$ be two regular 13-gons in the plane such that the points B_1 and A_{13} coincide and lie on the segment A_1B_{13}, and both polygons lie in the same semiplane with respect to this segment. Prove that the lines A_1A_9, $B_{13}B_8$ and A_8B_9 are concurrent.

   Solution. Consider the regular polygon $C_1C_2\dots C_{13}$, where $C_1 = A_1$, $C_{13} = B_{13}$. Clearly the lines A_1A_9 and B_1B_8 coincide with C_1C_9 and $C_{13}C_8$ respectively. Furthermore $C_1C_{13} = C_8C_9$, thus C_1C_8 and $C_{13}C_9$ are the bases of an isosceles trapezoid. The points A_8 and B_9 lie on these bases and $A_1A_8 : A_8C_8 = A_1A_{13} : B_1B_{13} = C_9B_9 : B_9B_{13}$. Therefore the line A_8B_9 passes through the common point of the diagonals of the trapezoid (fig.8.7).

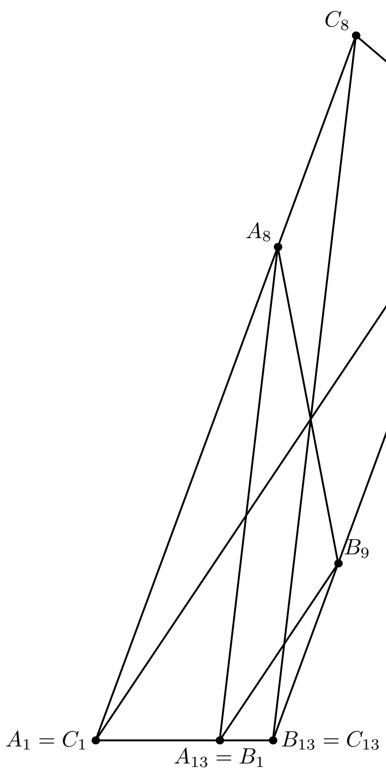

Fig. 8.7

8. (Tran Quang Hung, Vietnam) Let $ABCD$ be a square, and let P be a point on the minor arc CD of its circumcircle. The lines PA, PB meet the diagonals BD, AC at points K, L respectively. The points M, N are the

projections of K, L respectively to CD, and Q is the common point of lines KN and ML. Prove that PQ bisects the segment AB.

Solution. Firstly prove the following assertion.

Lemma. Let $AP = AC$ and $BQ = BC$ be the perpendiculars to the hypothenuse AB of a right-angled triangle ABC lying on the outside of the triangle. The lines AQ and BP meet at point R, and the lines CP and CQ meet AB at points M and N respectively. Then CR bisects the segment MN.

Proof. Since $\angle CAP = 90° + \angle CAB = 180° - \angle CBA$, we have $\angle ACP = \frac{\angle B}{2}$. Hence $BM = BC = BQ$ and similarly $AN = AC = AP$. Let the line passing through R and parallel to AB meet CP, CQ at points X, Y respectively, and let Z be the projection of R to AB (fig.8.8). Then $RX : BM = PR : PB = AR : AQ = RZ : QB$, Therefore $RX = RZ$. Similarly $RY = RZ$ (fig.8.8.1). Thus CR bisects XY, and hence it bisects MN.

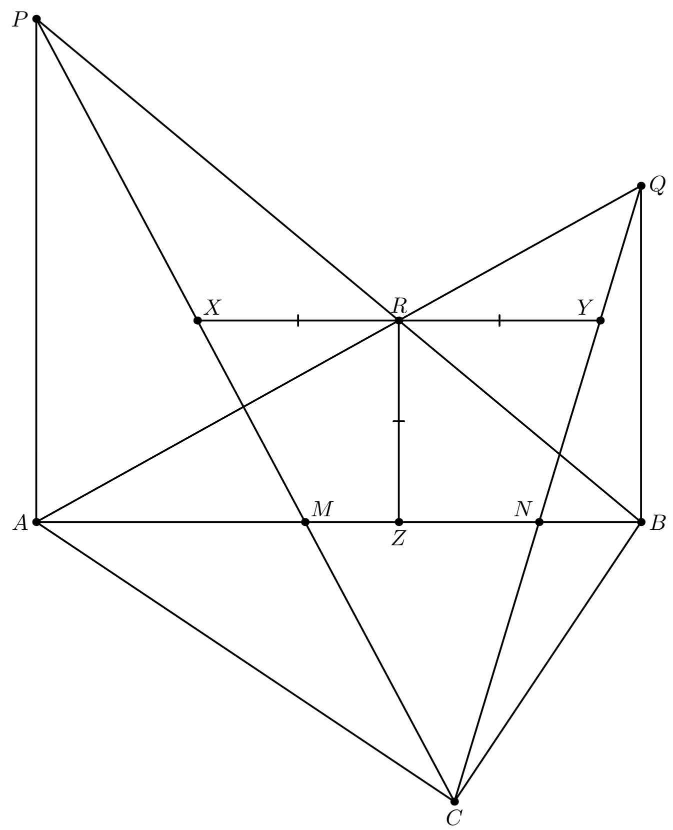

Fig. 8.8.1

Note. It it easy to see that CZ is the bisectrix of ABC and CR passes through the touching point of its incircle with the hypothenuse.

Return to the problem. Since KMD is an isosceles right-angled triangle, and $\angle KPD = 45°$, we obtain that M is the circumcenter of triangle KPD. Similarly N is the circumcenter of PCL. Furthermore $\angle MPN = 45° + (90° - \frac{\angle BDP}{2}) + (90° - \frac{\angle ACP}{2}) = 90°$. Applying the lemma to the points P, M, N, K, L we obtain the required assertion (fig.8.8.2).

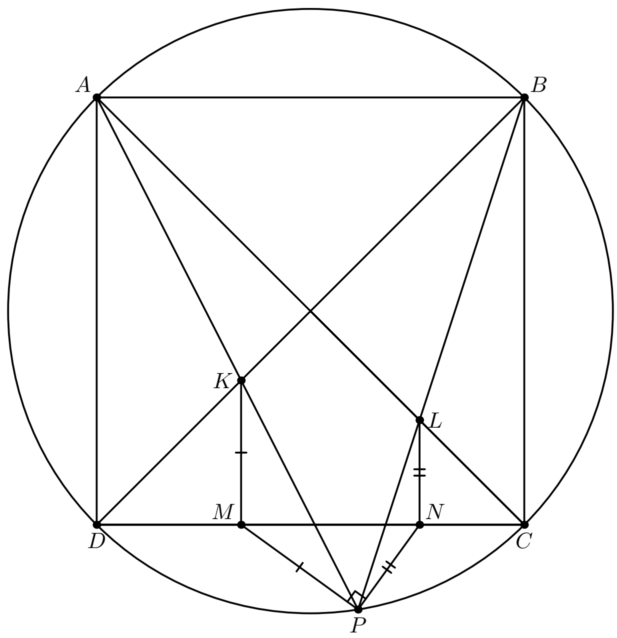

Fig. 8.8.2

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. First day. 9 grade

1. (A.Zaslavsky) Let ABC be a regular triangle. The line passing through the midpoint of AB and parallel to AC meets the minor arc AB of its circumcircle at point K. Prove that the ratio $AK : BK$ is equal to the ratio of the side and the diagonal of a regular pentagon.

   Solution. Let L be the second common point of the line and the circle (fig.9.1). Since ABC is a regular triangle we have $AL = BL + CL = BK + AK$. On the other hand KL bisects AB, thus the areas of triangles AKL and BKL are equal, i.e. $AK \cdot AL = BK \cdot BL = BK^2$. Therefore $t = AK/BK$ is the root of the equation $t^2 + t - 1 = 0$, equal to the ratio of the side and the diagonal of a regular pentagon.

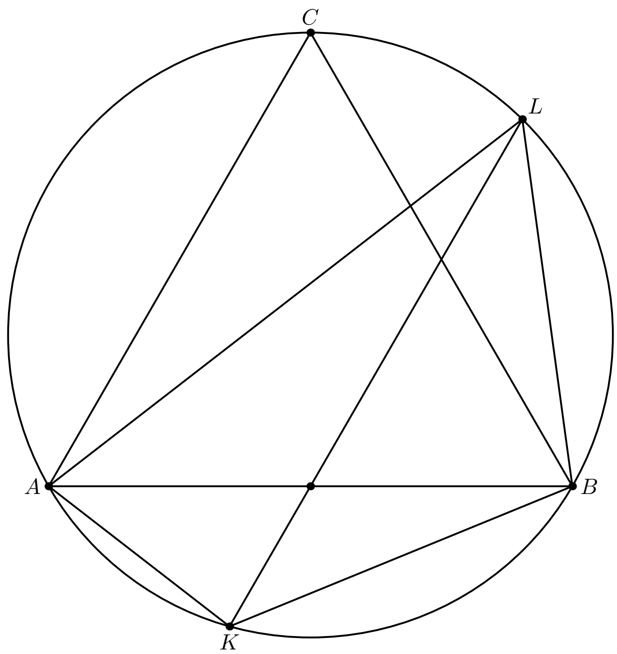

Fig. 9.1

2. (S.Berlov, A.Polyanskii) Let I be the incenter of triangle ABC, M be the midpoint of AC, and W be the midpoint of arc AB of its circumcircle not containing C. It is known that $\angle AIM = 90^\circ$. Find the ratio $CI : IW$.

   Answer. $2 : 1$.

   Solution. Let I_c be the center of the excircle touching the side AB. Since $AI_c \perp AI$ we obtain that $IM \parallel AI_c$, i.e. IM is a medial line of triangle ACI_c. Also W is the midpoint of II_c, therefore $CI = II_c = 2IW$ (fig.9.2).

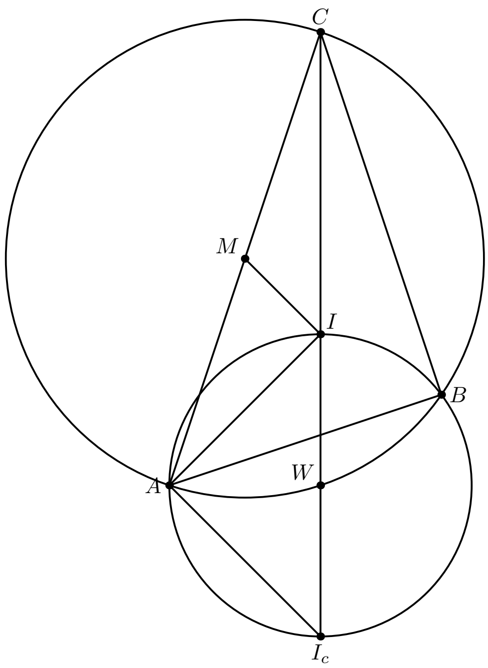

Fig. 9.2

3. (A.Mudgal, India) The angles B and C of an acute-angled triangle ABC are greater than $60°$. Points P and Q are chosen on the sides AB and AC, respectively, so that the points A, P, Q are concyclic with the orthocenter H of the triangle ABC. Point K is the midpoint of PQ. Prove that $\angle BKC > 90°$.

   Solution. Let BB', CC' be the altitudes of triangle ABC. Since $\angle PHQ = 180° - \angle A = \angle B'HC'$ we obtain that the triangles $HB'Q$ and $HC'P$ are similar. Thus, when P moves uniformly along AB the point Q also moves uniformly along AC and K moves along some segment. Since $\angle AHC' > \angle CHB'$ and $\angle AHB' > \angle BHC'$, the endpoints of this segment correspond to the coincidence of P with B or Q with C, let us consider the last case (fig.9.3).

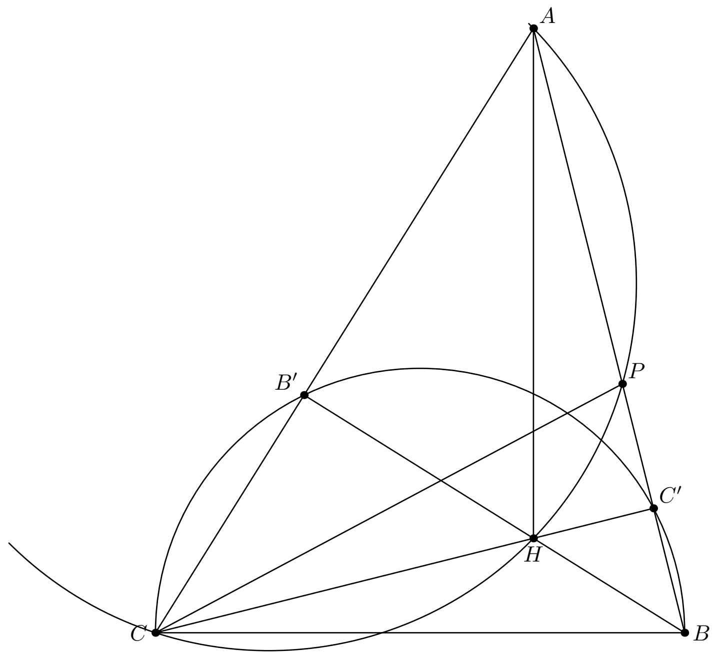

Fig. 9.3

If $Q = C$ then $\angle HCP = \angle HAP = \angle HCB$, i.e. BCP is non-isosceles triangle and the distance from K to the midpoint of BC is equal to BC'. Since $\angle B > 60^\circ$, we have $BC' < BC/2$, thus K lies inside the circle with diameter BC. Similarly the second endpoint of the segment passed by K lies inside this circle, therefore all points of the segment also lie inside it.

4. (M.Etesamifard, Iran) Points M and K are chosen on lateral sides AB and AC, respectively, of an isosceles triangle ABC, and point D is chosen on its base BC so that $AMDK$ is a parallelogram. Let the lines MK and BC meet at point L, and let X, Y be the intersection points of AB, AC respectively with the perpendicular line from D to BC. Prove that the circle with center L and radius LD and the circumcircle of triangle AXY are tangent.

   Solution. It is clear that the circumcircles of triangles ABC and AXY are perpendicular. Let E be their second common point. Since E is the center of spiral similarity mapping X and Y to B and C respectively, the triangles EXB and EYC are similar, i.e. $EB : EC = XB : YC = BD : CD$. On the other hand $LB : LD = LM : LK = LD : LC$. Hence B and C are inverse wrt the circle centered at L with radius LD i.e. this circle is also perpendicular to the circumcircle of ABC. Also the ratio of distances from B and C is the same for all points of this circle (an Apollonius circle), thus it passes through E (fig.9.4). Since both circles are perpendicular to the circumcircle of ABC and meet it at the same point we conclude that they are tangent.

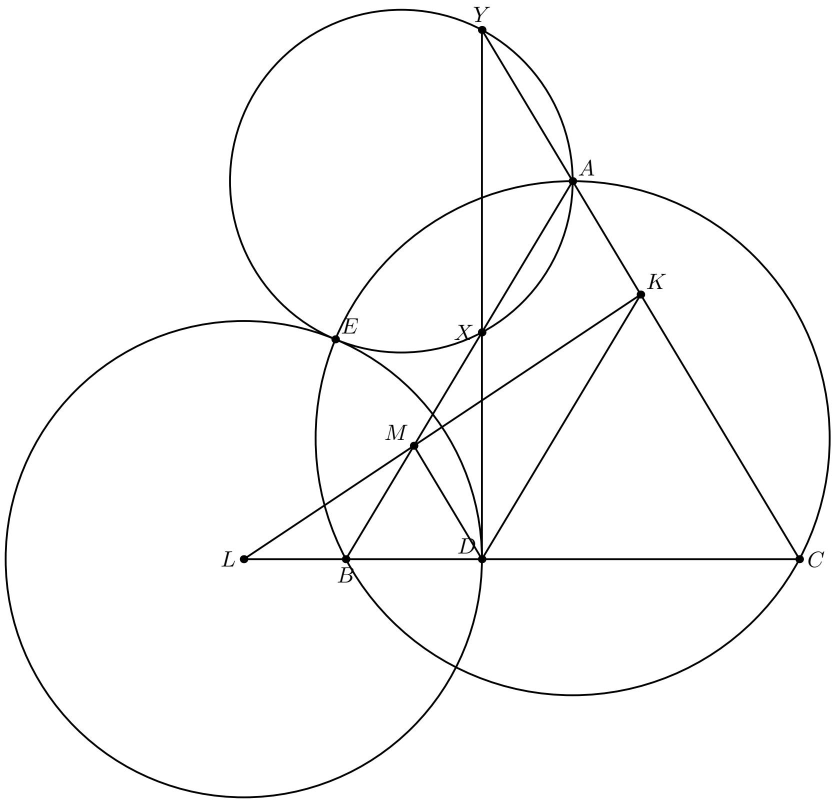

Fig. 9.4

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. Second day. 9 grade

5. (P.Kozhevnikov) Let BH_b, CH_c be altitudes of a triangle ABC. The line H_bH_c meets the circumcircle of ABC at points X and Y. Points P and Q are the reflections of X and Y about AB and AC, respectively. Prove that $PQ \parallel BC$.

   Solution. Let O be the circumcenter of triangle ABC. Since the line AO is the reflection of the altitude from A about the bisector from the same vertex, and $\angle AH_bH_c = \angle ABC$, we obtain that $AO \perp H_bH_c$, i.e. AO is the perpendicular bisector to the segment XY. Therefore, $AP = AX = AY = AQ$ and $XPQY$ is a cyclic quadrilateral (fig.9.5). Hence the lines XY and PQ are antiparallel with respect to the lines XP and YQ which are parallel to the altitudes of the triangle. But BC and H_bH_c are also antiparallel with respect to the altitudes, thus $PQ \parallel BC$.

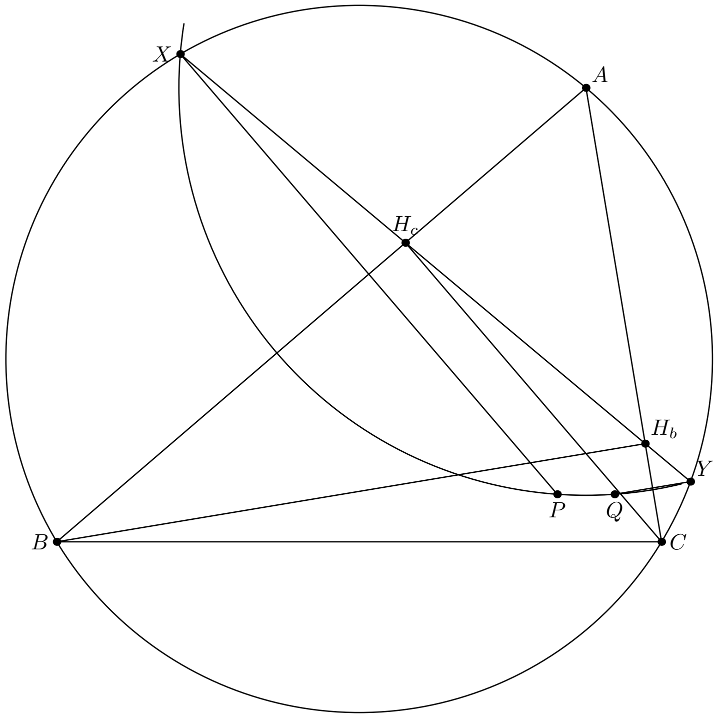

Fig. 9.5

6. (M.Etesamifard, Iran) Let ABC be a right-angled triangle ($\angle C = 90°$) and D be the midpoint of an altitude from C. The reflections of the line AB about AD and BD, respectively, meet at point F. Find the ratio $S_{ABF} : S_{ABC}$.

   Answer. 4/3.

Solution. Let CH be the altitude and K, L be the common points of the line passing through C and parallel to AB with AF and BF respectively (fig.9.6). Since the trapezoid $AKLB$ is circumscribed around the circle with diameter CH, we obtain that KD and LD are the bisectors of angles AKL and BLK respectively. Hence $\angle CKD = 90^\circ - \angle HAD$, i.e. the triangles KCD and DHA are similar, and $KC = CD^2/AH = CH^2/(4AH) = BH/4$. Similarly $CL = AH/4$. Therefore the ratio of the altitudes of similar triangles FKL and FAB is equal to $1/4$, and the ratio of the altitudes of triangles AFB and ABC is $4/3$.

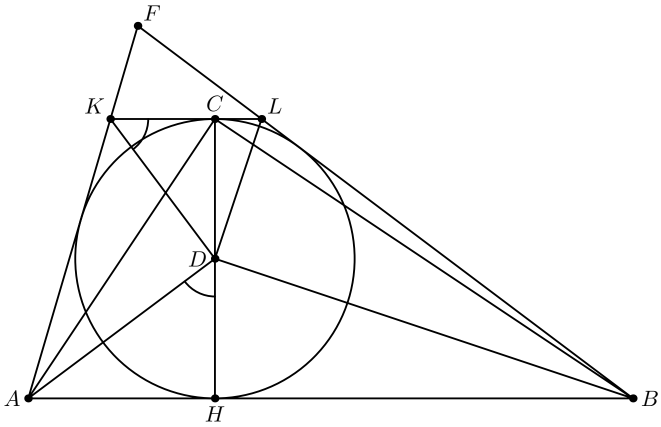

Fig. 9.6

7. (P.Kozhevnikov) Let a and b be parallel lines with 50 distinct points marked on a and 50 distinct points marked on b. Find the greatest possible number of acute-angled triangles all whose vertices are marked.

Answer. 41650.

First solution. Let $n = 50$.

Introduce coordinates so that lines a and b are given by equations $y = 0$ and $y = 1$ respectively. Denote by $A_1, A_2, \ldots, A_n$ marked points on a so that their x-coordinates $a_1, a_2, \ldots, a_n$ are ordered: $a_1 < \ldots < a_n$. Similarly define $B_1, \ldots, B_n$ with x-coordinates $b_1 < \ldots < b_n$. Let A^- and A^+ be points on a such that their x-coordinates satisfy conditions $a^- < a_1$ and $a^+ > a_n$. Similarly define B^- and B^+.

Estimate. The total number of triangles with all vertices marked is $T = 2\binom{n}{2}n = n^2(n-1)$. The number of non-acute triangles is not less than the number N of non-acute angles among angles $A_iA_jB_k$ and $B_iB_jA_k$. Let us estimate N.

Fix $t \in \{1, 2, \ldots, n\}$ and $s \in \{1, 2, \ldots, n\}$, then WLOG $t \le s$, and consider the segment A_tB_s (similarly consider B_tA_s). The segment A_tB_s forms two pairs of equal angles with lines a and b. Note that either $\angle A^-A_tB_s$ and $\angle A_tB_sB^+$ both are non-acute, or $\angle A^+A_tB_s$ and $\angle A_tB_sB^-$ both are non-acute. In the first case all angles $\angle A_iA_tB_s$ and $\angle A_tB_sB_j$ with $i < t$ and $j > s$ are non-acute; the number of such angles is $(t-1)+(n-s) = n-1-(s-t)$. In the second case all angles $\angle A_iA_tB_s$ and $\angle A_tB_sB_j$ with $i > t$ and $j < s$ are non-acute; the number of such angles is $(n-t)+(s-1) = n-1+(s-t)$. Anyway the total number of non-acute angles of the form $\angle A_iA_tB_s$ or $\angle A_tB_sB_j$ is not less than $n-1-(s-t)$. Thus we have the estimate $N \ge n(n-1) + 2\sum\limits_{1\le t<s\le n}(n-1-(s-t)) = \frac{(n-1)(2n^2-n)}{3}$ (here the summand $n(n-1)$ corresponds to n segments A_sB_t with $t = s$). A direct calculation shows that the number of acute triangles with all vertices marked is not greater than $T - N \le \frac{(n-1)n(n+1)}{3} = 41650$.

Example. Let us mark points so that $0 < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < 1/10$. Note that in this example all angles $\angle A_iB_kA_j$ and $\angle B_iA_kB_j$ are acute. Moreover, for each segment A_tB_s (or A_sB_t) with $t \le s$, among the angles of the form $\angle A_iA_tB_s$ and $\angle A_tB_sB_j$ exactly $n-(s-t)+1$ are non-acute. This means that this example is optimal, since our estimate in this case is sharp.

Second solution. Clearly, the maximum is achieved when the points of both sides are located sufficiently closely, in such a way that all angles having the vertex on one line and the sides passing through two points of the second one are acute. Let the points on one line be colored blue and the projections to this line of the points on the second one be colored red. Then we obtain the following reformulation of the problem.

Suppose 50 blue and 50 red points are marked on a line. Find the maximal number of triples having the medial point of one color and two extreme points of the other color.

Let $A_1, \ldots, A_{50}$, $B_1, \ldots, B_{50}$ be the red and the blue points respectively, ordered from left to right. Consider two adjacent points A_i and B_j. If A_i lies on the left side from B_j, these two points form a good triple with $B_1, \ldots, B_{j-1}$ and $A_{i+1}, \ldots, A_{50}$, i.e. we have $n-1+(j-i)$ good triples. When $i > j$ we can transpose A_i and B_j and increase the number of such good triples, and this operation retains all remaining good triples. Hence in the optimal disposition any point A_i lies to the right from B_{i-1}, but to the left from B_{i+1} (the order of A_i and B_i can be arbitrary). In particular, the alternating disposition is optimal. The number of acute-angled triangles in this disposition is calculated in the previous solution.

8. (I.Frolov) Let AK and BL be the altitudes of an acute-angled triangle ABC, and let ω be the excircle of ABC touching the side AB. The common internal tangents to circles CKL and ω meet AB at points P and Q. Prove that $AP = BQ$.

   First solution. Let R be the internal center of similitude of CKL and ω, the incircle of ABC touch AB at C_1, the incircle of triangle PQR touch AB at C_1', ω touch AB at C_2, and C_2C_3 be a diameter in ω. Then C, R, C_3 are collinear. Furthermore C, C_1, C_3 are collinear because C is the homothety center of the incircle and the excircle. Analogously, R, C_1', C_3 are collinear. So C_1' coincides with C_1. Thus the midpoints of AB, C_1C_2, $C_1'C_2$, and PQ coincide, as needed.

   Second solution. Let us prove that the assertion of the problem is correct when we replace CKL by an arbitrary circle centered on the altitude and passing through C. As in the first solution we obtain that the common point R of internal common tangents lies on CC_1. So we have the following reformulation of the problem.

   The tangents from an arbitrary point R of line CC_1 to the excircle meet AB at points P and Q. Prove that these points are symmetric wrt the midpoint of AB.

   It can be proved that the correspondence between P and Q preserves the cross-ratios. Hence it is sufficient to find two pairs P, Q symmetric wrt AB. For $R = C$ the points P, Q coincide with A, B, and for $R = C_1$ they coincide with C_1, C_2. In both cases they are symmetric.

   Note. In the formulation of the problem, we can replace the excircle by the incircle and the internal common tangents by the external ones. Moreover, arguing as in the second solution we obtain that the common points of tangents with AB are the same in both cases.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. First day. 10 grade

1. (D.Shvetsov) Let A and B be the common points of two circles, and CD be their common tangent (C and D are the tangency points). Let O_a, O_b be the circumcenters of triangles CAD, CBD respectively. Prove that the midpoint of segment O_aO_b lies on the line AB.

   Solution. Let C', D' be the touching points of the circles with the second common tangent. The angles ACD and ADC are equal to the halves of arcs AC and AD of the corresponidng circles, and the angles BCD and BDC are equal to the halves of arcs BC and BD which are equal to $C'A$ and $D'A$. Therefore, the sum of four angles is equal to the half-sum of arcs $C'AC$ and $D'AD$. Since the last arc is homothetic to $C'C$, this half-sum is equal to π. Thus the circumcenters of triangles CAD and CBD are symmetric with respect to CD, i.e., the midpoint of O_aO_b coincides with the midpoint of CD which lies on the radical axis AB of the circles (fig.10.1).

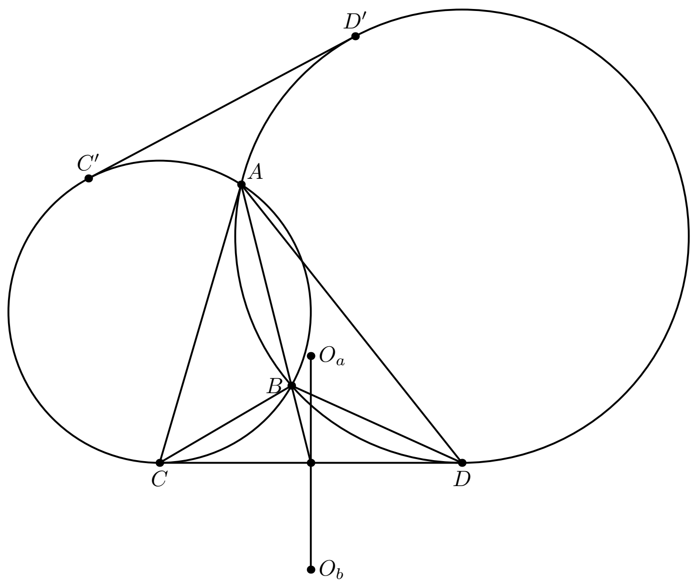

Fig. 10.1

2. (A.Peshnin) Prove that the distance from any vertex of an acute-angled triangle to the corresponding excenter is less than the sum of two greatest sidelengths.

   Solution. Let in a triangle ABC $\angle A = 2\alpha$, $\angle B = 2\beta$, $\angle C = 2\gamma$ and $\alpha \geq \beta \geq \gamma$. Let I_a, I_b, I_c be the excenters and p be the semiperimeter of the triangle .Then the inequalities $2\alpha < 90° < 2\beta + 2\gamma$ and $\beta \geq \gamma$ yield that $2\beta > \alpha$. Also since $AI_a \cos\alpha = BI_b \cos\beta = CI_c \cos\gamma = p$, we have $AI_a \geq BI_b \geq CI_c$, and it is sufficient to prove that $AI_a < AC + BC$. We can obtain this in several ways.

First way. Note that a point K symmetric to B with respect to the external bisector CI_a lies on AC, and $CK = CB$. Since BI_a is the external bisector of angle B, we have $\angle I_aKA = \alpha + \gamma$, and since $\angle I_aAK = \alpha$, we obtain that $\angle AI_aK = 2\beta + \gamma$ (fig.10.2.1). Since $2\beta > \alpha$ we have $\angle AKI_a < \angle AI_aK$, i.e. $AI_a < AK = AC + BC$.

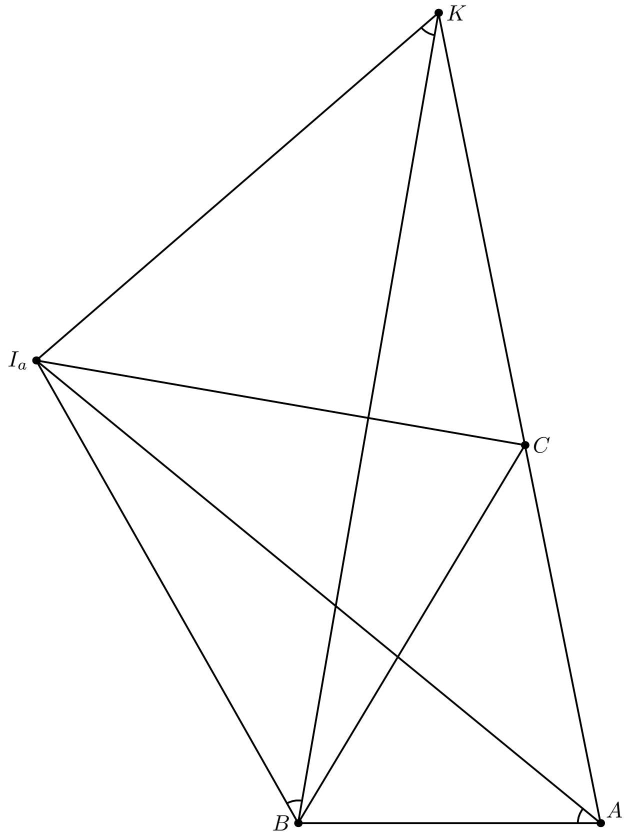

Fig. 10.2.1

Second way. Let the excircle touches AB at point T, and let U be the reflection of B about T. Since $AT = p$, we have $AU = 2p - AB = AC + BC$. Also, in the triangle AUI_a we have $\angle UAI_a = \alpha$, $\angle AUI_a = \angle I_aBT = \pi/2 - \beta$, thus $\angle AI_aU = \pi/2 - \alpha + \beta$. Since $2\beta > \alpha$ we obtain that $\angle AI_aU > \angle AUI_a$ and $AU > AI_a$ (fig.10.2.2).

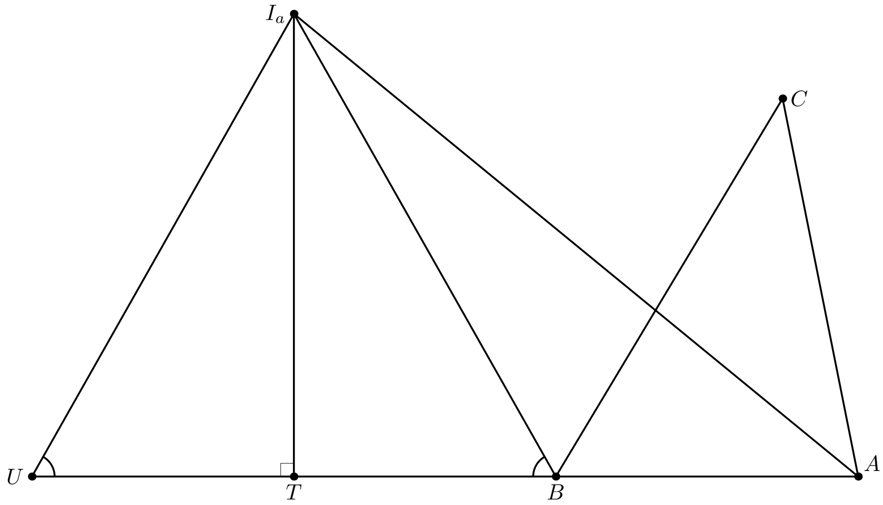

Fig. 10.2.2

Third way. We will use the following facts.

1. **The trident theorem.** The circumcenter of triangle BCI_a coincide with the midpoint W of the arc BC of circle ABC.

2. Let a circle be given, its chord AB be fixed, and point X move on an arc AB. Then the sum $AX + BX$ increases while X comes closer to the midpoint of the arc.

By the trident theorem we have $AI_a = AW+WB$. Since $\angle WBA-\angle WAB = 2\beta > 2(\alpha-\beta) = \angle CAB-\angle CBA$ we obtain that C is closer to the midpoint of arc ACB than W. Therefore, $AW + BW < AC + BC$.

Note. In fact, we proved that the segment joining any vertex to the corresponding excenter is less than the sum of the opposite and the greatest of two adjacent sidelengths.

3. (A.Sokolov) Let $ABCD$ be a convex quadrilateral, and let $\omega_A, \omega_B, \omega_C, \omega_D$ be the circumcircles of triangles BCD, ACD, ABD, ABC, respectively. Denote by X_A the product of the power of A with respect to ω_A and the area of triangle BCD. Define X_B, X_C, X_D similarly. Prove that $X_A + X_B + X_C + X_D = 0$.

Solution. If the quadrilateral is cyclic then the assertion is evident. Now note that D lies outside ω_D iff $\angle A + \angle C > \angle B + \angle D$, i.e. iff C lies inside ω_C. So the signs of X_C and X_D are opposite.

Let CD meet AB at point P and meet ω_C, ω_D for the second time at C', D' respectively. Then the ratio of areas of triangles ABC and ABD is equal to the ratio of their altitudes, which is equal to $\frac{PC}{PD}$. Since $PC\cdot PD' = PA\cdot PB =$

$PC' \cdot PD$, this ratio is equal to $\frac{PC}{PD} = \frac{PC'}{PD'} = \frac{PC-PC'}{PD-PD'} = \frac{CC'}{DD'}$. On the other hand, the ratio of the absolute values of the powers of C and D with respect to the corresponding circles is $\frac{CD \cdot CC'}{CD \cdot DD'} = \frac{CC'}{DD'}$ (fig.10.3). Therefore, $|X_C| = |X_D|$ and $X_C + X_D = 0$. Similarly $X_A + X_B = 0$.

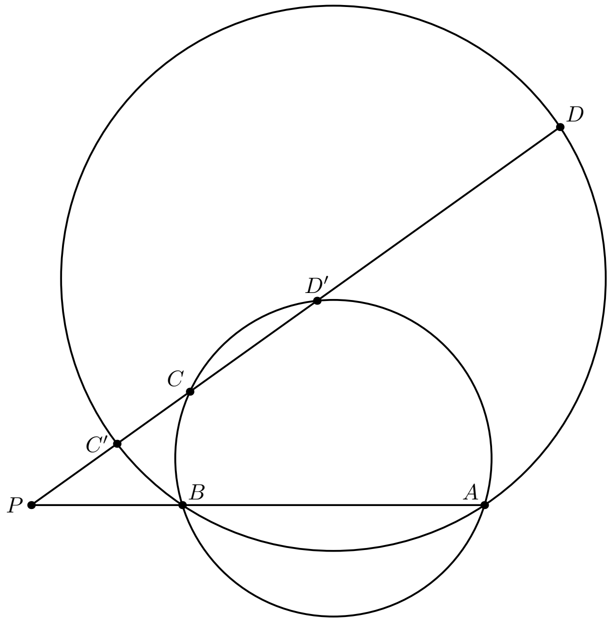

Fig. 10.3

If $AB \parallel CD$ then $S_{ABC} = S_{ABD}$, $CC' = DD'$ and we also obtain that $X_C + X_D = 0$.

Note. The equality $|X_A| = |X_B| = |X_C| = |X_D|$ is also valid when the four points do not form a convex quadrilateral.

4. (A.Zaslavsky) A scalene triangle ABC and its incircle ω are given. Using only a ruler and drawing at most eight lines, rays or segments, construct points A', B', C' on ω such that the rays $B'C'$, $C'A'$, $A'B'$ pass through A, B, C, respectively.

Solution. Let A_0, B_0, C_0 be the touching points of the incircle with BC, CA, AB respectively. Then the required points A', B', C' are such that $A'A_0C'C_0$, $B'B_0A'A_0$ and $C'C_0B'B_0$ are harmonic quadrilaterals. Consider a projective transform preserving ω and mapping the common point of AA_0, BB_0, CC_0 to its center. This transform maps ABC to a regular triangle. Then the triangles $A_0B_0C_0$ and $A'B'C'$ are also regular and $A'A_0C'C_0$ is an isosceles trapezoid. Let K be a midpoint of A_0C_0. The harmonicity condition $\angle C_0A'C' = \angle KA'A_0$ now reads $\angle KA'A_0 = \angle A_0C_0A'$, i.e., $\triangle KA'A_0 \sim \triangle A'C_0A_0$, whence $\angle A'KA_0 = 2\pi/3$ and $A'K \parallel BC \parallel B_0C_0$ (fig. 10.4).

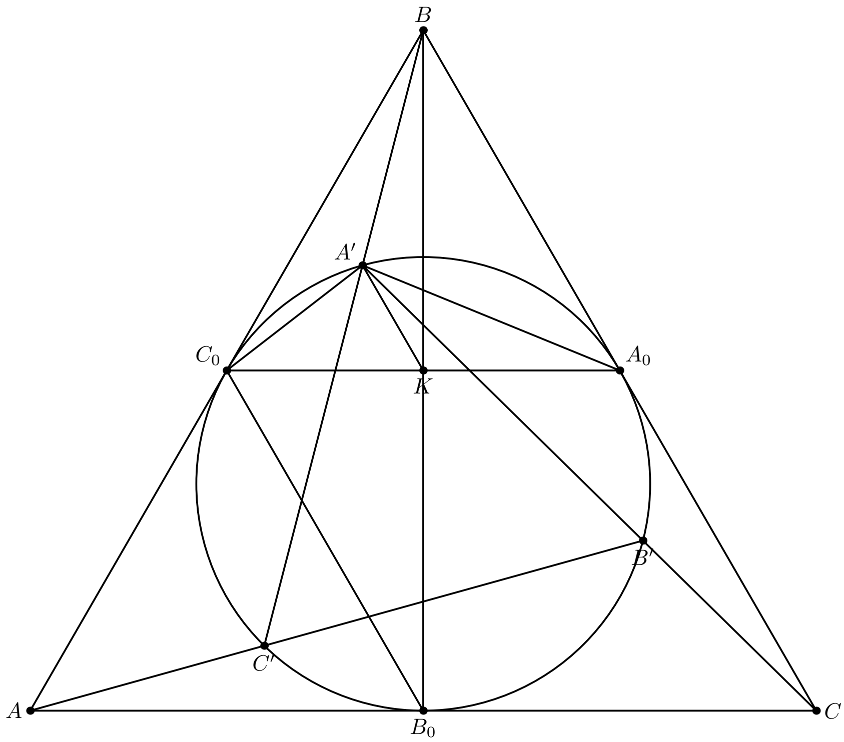

Fig. 10.4

Now, after the inverse transform we obtain the following construction.

1–2. Draw A_0C_0, BB_0 and find their common point K.

3–4. Draw BC, B_0C_0 and find their common point L.

5. Draw KL and find its common point A' with the arc A_0C_0.

6. Draw CA' and find its second common point B' with ω.

7. Draw AB' and find its second common point C' with ω.

XIII Geometrical Olympiad in honour of I.F.Sharygin
Solutions. Final round. Second day. 10 grade

5. (A.Garkavy) Let BB', CC' be the altitudes of an acute-angled triangle ABC. Two circles passing through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.

   First solution. Since $BP^2 = BQ^2 = BA \cdot BC'$ and the quadrilaterals $AC'A'C$, $AB'A'B$ are cyclic (AA' is the altitude) we have $CP \cdot CQ = CB^2 - BP^2 = CB^2 - BA \cdot BC' = BC^2 - BC \cdot BA' = BC \cdot CA' = CA \cdot CB'$. Clearly this is equivalent to the required assertion.

   Second solution. Let C_0 be the reflection of C' about B. Then $BC_0 \cdot BA = BC' \cdot BA = BP^2 = BP \cdot BQ$, so the points A, P, C_0, Q lie on some circle ω. Let H_0 be the point on ω opposite to A. Then $H_0C_0 \perp BC$. Hence, the reflection of $H - 0$ about B (whic is the midpoint of PQ), lies on the altitude CC'; on the other hand, this reflection also lies on the altitude AA' of the triangle APQ. Thus, the point H_0 is symmetric to the orthocenter H of ABC about B. Therefore, $BH_0 \cdot BB' = BH \cdot BB' = BC' \cdot BA = BC_0 \cdot BA$, which shows that B' also lies on ω (fig.10.5).

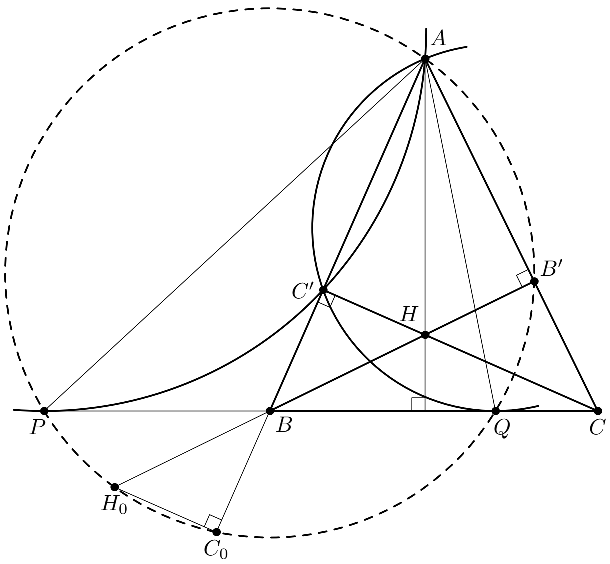

Fig. 10.5

Note. In fact, we implemented a well-known fact that C' is the projection of the orthocenter of the triangle APQ onto its median AB. This yields, in particular, that the triangles ABC and APQ have a common orthocenter H.

6. (I.I.Bogdanov) Let the insphere of a pyramid $SABC$ touch the faces SAB, SBC, SCA at points D, E, F respectively. Find all possible values of the sum of angles SDA, SEB and SFC.

   Answer. 2π.

   Solution. Since the triangles SCE and SCF are congruent we have $\angle SFC = \angle SEC$. Similarly $\angle SEB = \angle SDB$ and $\angle SDA = \angle SFA$. Hence $\angle SDA + \angle SEB + \angle SFC = \angle SFA + \angle SDB + \angle SEC = \frac{6\pi-(\angle ADB+\angle BEC+\angle CFA)}{2}$. But the angles ADB, BEC, CFA are equal to the angles AGB, BGC, CGA, where G is the tangency point of the insphere with the face ABC. Therefore their sum is equal to 2π.

   Note. One can show that, in fact, each of the triples of angles $(\angle SDA, \angle SDB, \angle ADB)$, $(\angle SEB, \angle SEC, \angle BEC)$, $(\angle SFC, \angle SFA, \angle AFC)$, and $(\angle AGB, \angle BGC, \angle CGA)$ contains the same three angles, perhaps permuted.

7. (I.Frolov) A quadrilateral $ABCD$ is circumscribed around circle ω centered at I and inscribed into circle Γ. The lines AB and CD meet at point P, the lines BC and AD meet at point Q. Prove that the circles PIQ and Γ are orthogonal.

   Solution. Since $ABCD$ is cyclic, the bisectors of the angles formed by its opposite sidelines are perpendicular. Thus $\angle PIQ = 90^\circ$, and PQ is a diameter of circle PIQ. Let R be the common point of the diagonals. Then the circle PIQ meets PR in a point S such that $PR \perp QS$. Since PR is the polar of Q with respect to Γ, we obtain that Q and S are inverse with respect to this circle, thus any circle passing through these two points is orthogonal to Γ (fig.10.7).

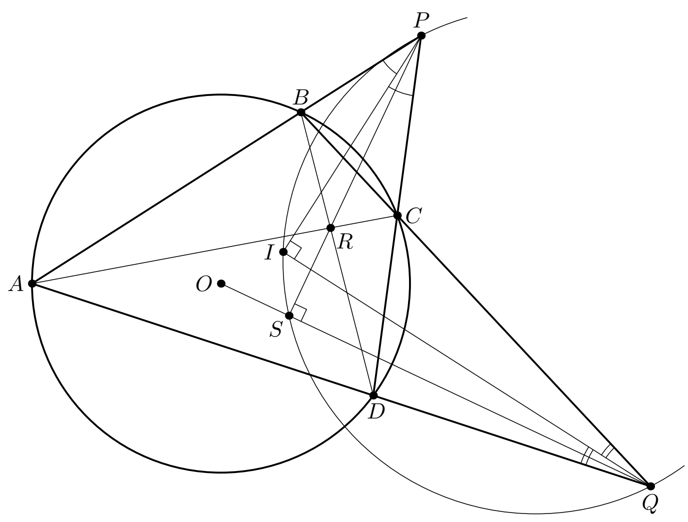

Fig. 10.7

Note. The assertion of the problem is valid for an arbitrary cyclic quadrilateral if we define I as the common point of the bisectors of angles APC and AQC.

8. (M.Saghafian,Iran; I.I.Bogdanov) Let S be a set of points in the plane, $|S|$ is even; no three points of S are collinear. Prove that S can be partitioned into two sets S_1 and S_2 so that their convex hulls have equal number of vertices.

Solution

Denote by $k(X)$ the number of vertices in the convex hull $\operatorname{conv} X$ of X.

Let $A = A_1A_2 \dots A_n = \operatorname{conv} S$, and let T be the set of all points in S lying (strictly) inside A. Set $X_i = \{A_1, \dots, A_i\} \cup T$, $Y_i = \{A_{i+1}, \dots, A_n\}$.

Let i be the minimal index such that $k(X_i) \geq k(Y_i)$. Clearly, $i < n$. If $i = 0$, then we may find a subset $T' \subseteq T$ such that $k(T') = n$ (removing the points from T one by one). Then $T' \sqcup (S \setminus T')$ is a required partition.

Assume now that $1 \leq i \leq n-1$. By the minimality of i, we get

$$k(X_i) - 1 \leq k(X_{i-1}) \leq k(Y_{i-1}) - 1 \leq k(Y_i).$$

So, either $k(X_i) = k(Y_i)$ (and they form a required partition), or

$$k(X_i) - 1 = k(X_{i-1}) = k(Y_{i-1}) - 1 = k(Y_i).$$

Let us consider the latter case.

Set $X = X_i$, $Y = Y_i$. Since $k(X) + k(Y)$ is odd, there exists at least one *extra* point $M \in X$ not on the boundary of $\operatorname{conv} X$ and $\operatorname{conv} Y$. If M is outside $\operatorname{conv} Y$, simply move it from X to Y to obtain the required partition. Otherwise all such extra points lie in $\operatorname{conv} X \cap \operatorname{conv} Y$. In particular, this intersection is nonempty.

Now let $X' = X \setminus \operatorname{conv} Y$. Then all points of X' lie on the boundary of $\operatorname{conv} X$ (all inner points of $\operatorname{conv} X$ lie also inside $\operatorname{conv} Y$), hence $k(X') < k(X)$ and so $k(X') \leq k(Y)$. If $k(X') = k(Y)$ then X' and $S \setminus X'$ form the required partition. Otherwise add to X' points from $X \cap \operatorname{conv} Y$ one by one until we get the set X'' with $k(X'') = k(Y)$. Then X'' and $S \setminus X''$ form the required partition.