XI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (T.Kazitzyna) Tanya cut out a convex polygon from the paper, folded it several times and obtained a two-layers quadrilateral. Can the cut polygon be a heptagon?

   Solution. Yes, for example let angle B of a quadrilateral $ABCD$ be obtuse, and three remaining angles be acute. Take a point K on side CD such that $\angle CBK < 180^\circ - \angle B$. Let points B_1, K_1 be symmetric to B, K about AD, and point K_2 be symmetric to K about BC. Then a heptagon $ABK_2CDK_1B_1$ is convex, and folding it by lines BC and AD, we obtain two-layers quadrilateral $ABCD$ (fig.1).

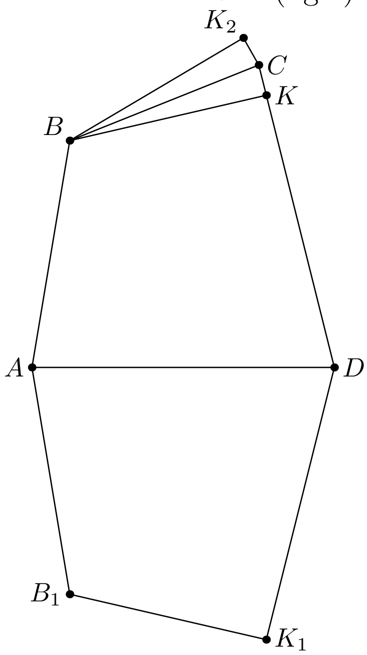

Fig.1

2. (M.Rozhkova) Let O and H be the circumcenter and the orthocenter of a triangle ABC respectively. The line passing through the midpoint of OH and parallel to BC meets AB and AC at points D and E respectively. It is known that O is the incenter of triangle ADE. Find the angles of ABC.

   Answer. $\angle A = 36^\circ$, $\angle B = \angle C = 72^\circ$.

   Solution. By the condition we obtain that AO is the bisector of angle A, i.e. $AB = AC$. Then $ODHE$ is a rhombus, $\angle ODH = 2\angle ODE = \angle B$, $\angle DOH = \angle DHO = 90^\circ - \frac{\angle B}{2} = \angle BHD$.

   Let the line passing through H and parallel to AC meet AB at point K. Since $\angle HKB = \angle A = \angle HOB$, points H, O, K, B are concyclic. Since angle KHB is right, the center of the corresponding circle lies on AB, thus it coincides with D (fig.2). Therefore, $\angle HBD = \angle BHD = 90^\circ - \frac{\angle B}{2}$. On the other hand this angle is equal to $\angle B - \frac{\angle A}{2}$, from this we obtain the answer.

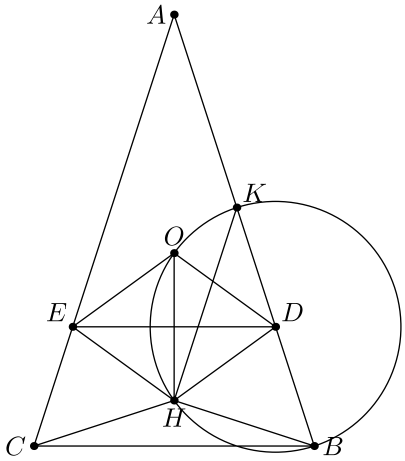

Fig.2

3. (N.Moskvitin) The side AD of a square $ABCD$ is the base of an obtuse-angled isosceles triangle AED with vertex E lying inside the square. Let AF be a diameter of the circumcircle of this triangle, and G be a point on CD such that $CG = DF$. Prove that angle BGE is less than half of angle AED.

   Solution. It is clear that F lies on sideline CD. Since $CG = DF$, we have $FG = CD = AB$, i.e. $ABGF$ is a parallelogram, and $\angle BGD = 180° - \angle AFD = \angle AED$. Thus we have to prove that $\angle BGE < \angle EGD$ or the distance from E to BG is less than its distance to CD. But the distances from E to CD and AF are equal, because FE bisects angle DFA, thus it is sufficient to prove that E is closer to BG, than to AF.

   A line through E parallel to AB meets AF at the center O of circle AED (fig.3). Therefore, $EO > AD/2 = AB/2$, which is equivalent to the desired inequality.

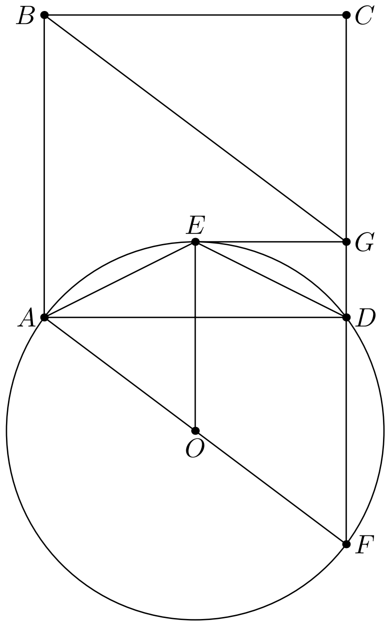

Fig.3

4. (L.Shteyngarts) In a parallelogram $ABCD$ the trisectors of angles A and B are drawn. Let O be the common points of the trisectors nearest to AB. Let AO meet the second trisector of angle B at point A_1, and let BO meet the second trisector of angle A at point

B_1. Let M be the midpoint of A_1B_1. Line MO meets AB at point N. Prove that triangle A_1B_1N is equilateral.

Solution. Let K be a common point of two remote trisectors. Then in triangle ABK $\angle K = 60^\circ$, and AA_1 and BB_1 are its bisectors. Since $\angle A_1OB_1 = 120^\circ$, quadrilateral A_1KB_1O is cyclic, and since KO bisects angle K, we obtain that $OA_1 = OB_1$. Therefore, $\angle MOA_1 = 60^\circ = \angle A_1OB = \angle BON$. This yields that $ON = OA_1$ and $A_1N = A_1B_1 = B_1N$ (fig.4).

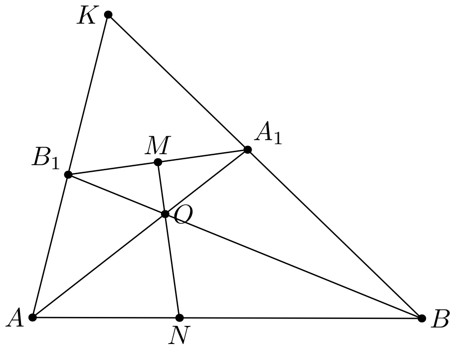

Fig.4

5. (V.Yassinsky) Let a triangle ABC be given. Two circles passing through A touch BC at points B and C respectively. Let D be the second common point of these circles (A is closer to BC than D). It is known that $BC = 2BD$. Prove that $\angle DAB = 2\angle ADB$.

   Solution. Since AD is a radical axis of two circles it meets segment BC at its midpoint M. Then $BM = BD$ and $\angle ADB = \angle DMB$. But $\angle ABM = \angle ADB$ as the angle between the chord and the tangent. By the exterior angle theorem $\angle DAB = \angle ABM + \angle AMB = 2\angle ADB$ (fig.5).

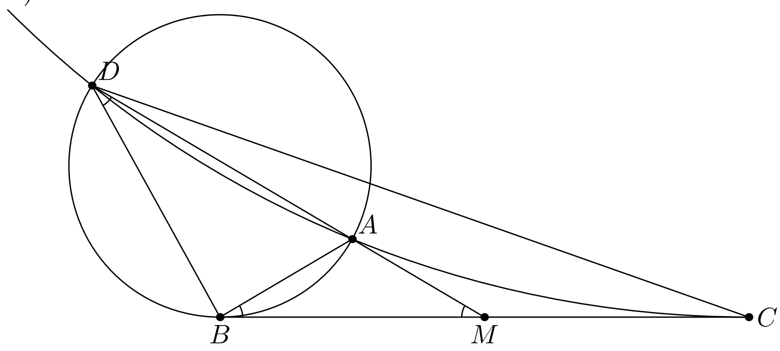

Fig.5

6. (A.Zaslavsky) Let AA', BB' and CC' be the altitudes of an acute-angled triangle ABC. Points C_a, C_b are symmetric to C' about AA' and BB' respectively. Points A_b, A_c, B_c, B_a are defined similarly. Prove that lines A_bB_a, B_cC_b and C_aA_c are parallel.

   First solution. Firstly prove next lemma.

   Let points Y', X' on sides XZ, YZ of triangle XYZ be such that $XY' = XY = X'Y$. Then $X'Y' \perp OI$, where O and I are the circumcenter and the incenter of the triangle.

   To prove the lemma it is sufficient to see that $X'O^2 - Y'O^2 = X'I^2 - Y'I^2$. Let x, y, z be the sidelengths of YZ, ZX, XY; X_0 be the the midpoint of YZ. Then $X'O^2 - OY^2 = X'X_0^2 - YX_0^2 = (z - x/2)^2 - (x/2)^2 = z(z - x)$. Similarly $Y'O^2 - OX^2 = z(z - y)$.

Also, $X'I^2 = r^2 + (z - (p - y))^2 = r^2 + (p - x)^2$, $Y'I^2 = r^2 + (p - y)^2$. Therefore, $X'O^2 - Y'O^2 = X'I^2 - Y'I^2 = z(y - x)$.

Now note that $A'A$, $B'B$, $C'C$ are the bisectors of triangle $A'B'C'$. Thus, for example, points A_b, B_a lie on $B'C'$, $A'C'$ respectively and $B'A_b = A'B_a = A'B'$. By the lemma A_bB_a is perpendicular to the line passing through the circumcenter and the incenter of triangle $A'B'C'$. Lines B_cC_b and A_cC_a are also perpendicular to this line, therefore these three lines are parallel.

Second solution. By previous solution B_a lies on $A'C'$, C_a lies on $A'B'$, A_b and A_c lie on $B'C'$. Since $A'B_a = A'B'$ and $A'C_a = A'C'$, we obtain that $B'B_a \parallel C'C_a$, thus $B'B_a/C'C_a = A'B'/A'C' = B'A_b/C'A_c$. Therefore triangles $B'A_bB_a$ and A_cC_aC' are similar, $\angle B_aA_bB' = \angle C_aA_cC'$ and $A_bB_a \parallel A_cC_a$. Similarly we prove that B_cC_b is parallel to these lines.

7. (D.Shvetsov) The altitudes AA_1 and CC_1 of a triangle ABC meet at point H. Point H_A is symmetric to H about A. Line H_AC_1 meets BC at point C'; point A' is defined similarly. Prove that $A'C'||AC$.

   Solution. Since triangles AHC_1 and CHA_1 are similar, triangles AH_AC_1 and CH_CA_1 are also similar i.e. $\angle A'C_1B = \angle C'A_1B$. Therefore points A_1, C_1, A', C' are concyclic and lines A_1C_1 and $A'C'$ are antiparallel wrt angle B. Since A_1C_1 and AC are also antiparallel, $A'C' \parallel AC$ (fig.7).

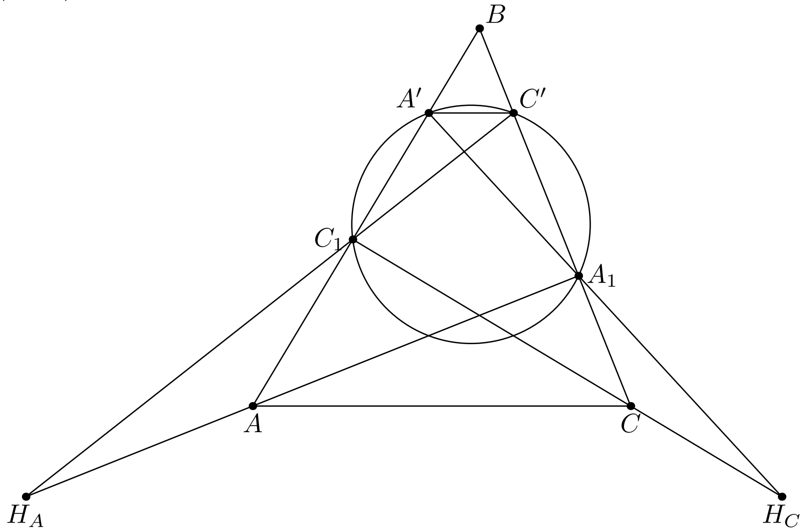

Fig.7

8. (N.Moskvitin) Diagonals of an isosceles trapezoid $ABCD$ with bases BC and AD are perpendicular. Let DE be the perpendicular from D to AB, and let CF be the perpendicular from C to DE. Prove that angle DBF is equal to half of angle FCD.

   Solution. By condition $\angle EDB = 45° - (90° - \angle A) = \angle A - 45° = \angle BDC$. Thus the distances from B to lines DE and DC are equal. Since the trapezoid is isosceles, the distance from B to DC is equal to the distance from C to AB, which is equal to the distance from B to line AB parallel to CF. Therefore, BF bisects angle CFE and $\angle BFC = 45°$. Let the perpendicular to BF from F meet BD at point K. Then $\angle CFK = \angle CBK = 45°$, thus $BFKC$ is a cyclic quadrilateral and $CK \perp BC$. Since $CF \parallel AB$, altitude CK bisects angle FCD, and from cyclic quadrilateral $BFKC$ we obtain that $\angle DBF = \angle KCF = \angle FCD/2$ (fig.8).

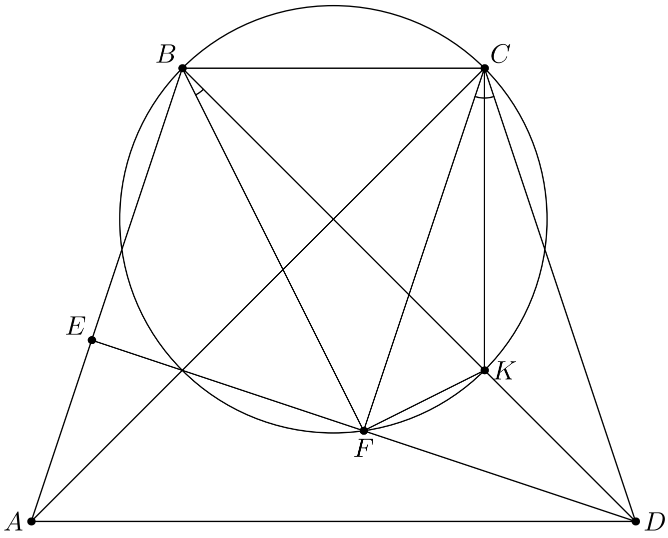

Fig.8

9. (a.Zaslavsky) Let ABC be an acute-angled triangle. Construct points A', B', C' on its sides BC, CA, AB such that:
   - $A'B' \parallel AB$;
   - $C'C$ is the bisector of angle $A'C'B'$;
   - $A'C' + B'C' = AB$.

   Solution. Let L be a common point of CC' and $A'B'$. Then $BC'/AC' = A'L/B'L = A'C'/B'C'$ and since $A'C' + B'C' = AB$ we obtain that $BC' = C'A'$, $AC' = C'B'$. Thus the reflections of C' in AC and BC lie on $A'B'$ and line CC' is symmetric to the altitude from C about the correspondent bisector i.e. CC' passes through the orthocenter of the given triangle (fig.9). The further construction is evident.

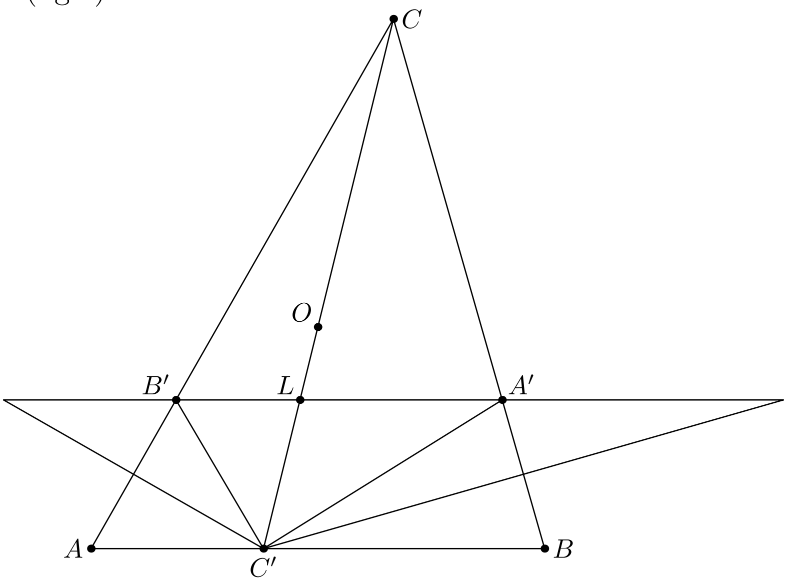

Fig.9

10. (B.Frenkin) The diagonals of a convex quadrilateral divide it into four similar triangles. Prove that it is possible to inscribe a circle into this quadrilateral.

    Solution. Let the diagonals of a quadrilateral $ABCD$ meet at point L. If for example angle ALB is obtuse, then it is greater than any angle of triangle BLC and two adjacent triangles can not be similar. Therefore the diagonals are perpendicular. Now if $\angle ABL = \angle CBL$ then BL is an altitude and a bisector of triangle ABC, thus it is also a median and $AB = BC$. Then DL is an altitude and a median of triangle ADC, therefore $AD = DC$ and the quadrilateral id circumscribed.

If angles ABL and CBL are not equal then their sum is equal to 90°. If $\angle BCL = \angle DCL$ then reason as above. Else $ABCD$ is a rectangle with perpendicular diagonals, i.e a square. Therefore a circle can be inscribed into it.

11. (A.Sokolov) Let H be the orthocenter of an acute-angled triangle ABC. The perpendicular bisector to segment BH meets BA and BC at points A_0, C_0 respectively. Prove that the perimeter of triangle A_0OC_0 (O is the circumcenter of $\triangle ABC$) is equal to AC.

    Solution. It is known that the reflections of H in the sidelines of a triangle lie on its circumcircle, i.e. the distances from them to O are equal to the circumradius R. Therefore the distances from H to points O_a, O_c, symmetric to O about BC and BA, are also equal to R. Since $BO_a = BO_c = R$, points O_a, O_c lie on A_0C_0. Also $BOCO_a$ and $BOAO_c$ are rhombus, thus $CO_a \parallel OB \parallel AO_c$, i.e. ACO_aO_c is a parallelogram and $O_aO_c = AC$. But by construction O_aO_c is equal to the perimeter of A_0OC_0 (fig.11).

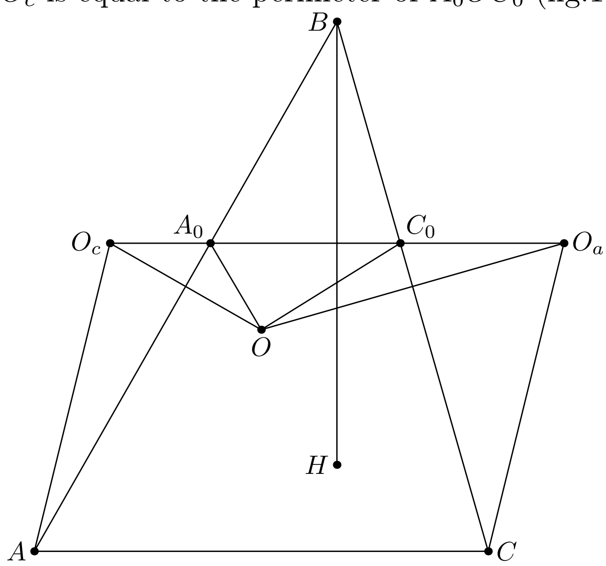

Fig.11

12. (A.Zaslavxky) Find the maximal number of discs which can be disposed on the plane so that each two of them have a common point and no three have it.

    Answer. 4.

    Solution. Consider one from n discs. Let A_iB_i be its common chords with the remaining discs. Since three discs do not intersect we obtain that for all i one of arcs A_iB_i does not contain the endpoints of the remaining chords. Cutting from each disc the segments limited by such arcs, we obtain n convex figures, each two of them have a common boundary. It is known that at most four such figures can exist on the plane. It is clear that four discs can satisfy the condition.

13. (A.Rudenko, D.Khilko) Let AH_1, BH_2 and CH_3 be the altitudes of a triangle ABC. Point M is the midpoint of H_2H_3. Line AM meets H_2H_1 at point K. Prove that K lies on the medial line of ABC parallel to AC.

Solution. Let P be the projection of H_3 to AC. Triangle H_3PH_2 is right-angled, and M is the midpoint of its hypothenuse, thus $MP = MH_2$ and $\angle MPH_2 = \angle MH_2A$. It is known that $\angle ABC = \angle H_1H_2P = \angle H_3H_2A$, therefore $MP \parallel KH_2$. From this we obtain that $\frac{AM}{AK} = \frac{AP}{AH_2}$. Triangles AH_2H_3 and ABC are similar, thus $\frac{AP}{AH_2} = \frac{AH_3}{AB}$. Then $\frac{AM}{AK} = \frac{AP}{AH_2} = \frac{AH_3}{AB}$, and $H_3M \parallel BK$ (fig.13). Also $\angle H_3H_2B = 90° - H_3H_2A = 90° - H_1H_2C = \angle BH_2K$. Therefore $\angle H_2BK = \angle H_3H_2B = \angle BH_2K$, and triangle BH_2K is isosceles. It is clear that the medial line parallel to AC is the perpendicular bisector to BH_2. Thus it passes through K.

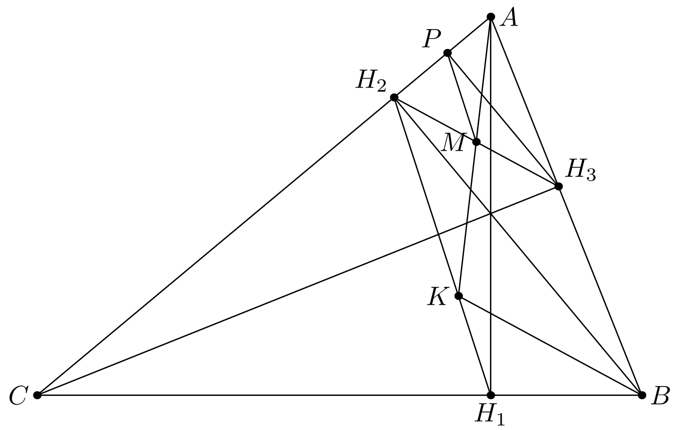

Fig.13

14. (A.Myakishev) Let ABC be an acute-angled, nonisosceles triangle. Point A_1, A_2 are symmetric to the feet of the internal and the external bisectors of angle A wrt the midpoint of BC. Segment A_1A_2 is a diameter of a circle α. Circles β and γ are defined similarly. Prove that these three circles have two common points.

    Solution. It is known that the circles having the feet of internal and external bisectors as opposite points are perpendicular to the circumcircle. Thus circles α, β, γ symmetric to them about the diameters of the circumcircles are also perpendicular to it, i.e. the degrees of the circumcenter O wrt these three circles are equal. Since the midpoints of the segments between the feet of the bisectors are concurrent, the centers of three circles are also concurrent by the Menelaos theorem. The perpendicular from O to the correspondent line is the common radical axis of three circles, therefore they have two common points.

15. (V.Yassinsky) The sidelengths of a triangle ABC are not greater than 1. Prove that $p(1-2Rr)$ is not greater than 1, where p is the semiperimeter, R and r are the circumradius and the inradius of ABC.

    Solution. Since the area of a triangle with sidelengths a, b, c is equal to $abc/4R = pr$, the desired inequality is equivalent to $a + b + c - abc \leq 2$. But

    $$a+b+c-abc = a+b+c(1-ab) \leq a+b+1-ab = 1+a+b(1-a) \leq 1+a+1-a = 2.$$

16. (B.Frenkin) The diagonals of a convex quadrilateral divide it into four triangles. Restore the quadrilateral by the circumcenters of two adjacent triangles and the incenters of two mutually opposite triangles.

    First solution. Let L be a common point of the diagonals of quadrilateral $ABCD$; O, I be the circumcenter and the incenter of triangle LAB; O' be the circumcenter of triangle LAD; I' be the incenter of triangle LCD. Then OO' is the perpendicular bisector to LA, and II' contains the bisector of angle LAB. Thus we can define the directions of lines LA, LB and construct the perpendicular bisector to LB.

Let X, Y, Z be the midpoints of arcs LA, LB, AB of circle LAB. Then I is the orthocenter of triangle XYZ and since we know angle ALB we can find angle XIY. Denote this angle as φ. Now we have to solve next problem.

An angle with vertex O and a point I are given. Construct on the sides of the angle such points X, Y that $OX = OY$ and $\angle XIY = \varphi$.

Take on the sides of the angles two arbitrary points X_1, Y_1 such that $OX_1 = OY_1$ and find such point I_1 on ray OI that $\angle X_1I_1Y_1 = \varphi$. The homothety with center O, transforming I_1 to I, transforms X_1, Y_1 to the desired points. The further construction is evident.

Second solution. In the notations of previous solution it is sufficient to find point L. In fact OO' is the perpendicular bisector to AL, and II' is the bisector of angle ALB. Constructing the perpendicular from L to OO' we find line AL. Reflecting it about II' we obtain line BL. Constructing a circle passing through L with center O we find A and B as its common points with AL and BL. The circle through L with center O' meets BL at D. Now construct the circle with center I', touching AL and BL, the tangent to this circle from D meets AL at C.

To find L use the trident theorem: a common point of the perpendicular bisector to a side of a triangle with its circumcircle lies on equal distances from the incenter and the endpoints of the side. Take an arbitrary circle ω_1 with center O. Let it meet OO' at point K. Constructing the perpendicular from K to II' and reflecting ω_1 about it, we obtain circle ω_2. Let OI meet ω_2 at point I_1. Reflecting I_1 about this perpendicular, we obtain point L' on ω_1. The homothety with center O, transforming I_1 to I, transforms L' to L.

17. (F.Nilov) Let O be the circumcenter of a triangle ABC. The projections of points D and X to the sidelines of the triangle lie on lines l and L such that $l \parallel XO$. Prove that the angles formed by L and by the diagonals of quadrilateral $ABCD$ are equal.

    Solution. By condition D and X lie on the circumcircle of ABC, and l and L are its Simson lines. Let chords CC', DD' and XX' be parallel to AB. By Simson lines properties l and OX are perpendicular to CD', and $L \perp CX'$. Thus we have to prove that arcs $X'D$ and $X'C'$ are equal. But these arcs are equal to $D'X$ and CX respectively, and the equality of these two arcs ic evident (fig.17).

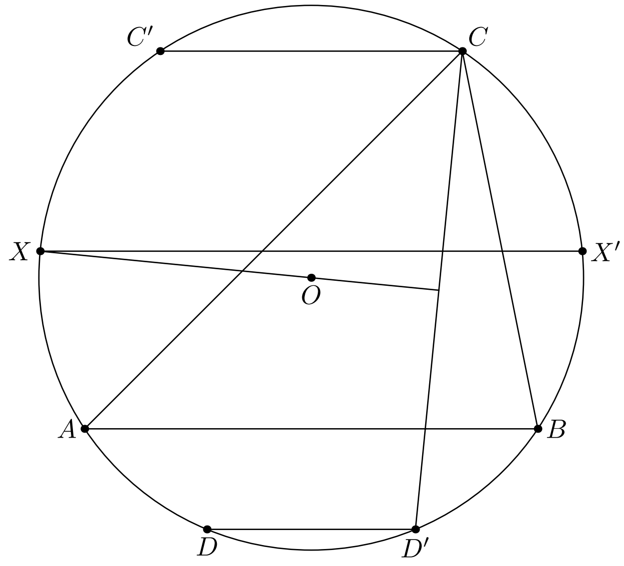

Fig.17

18. (V.Yassinsky) Let $ABCDEF$ be a cyclic hexagon, points K, L, M, N be the common points of lines AB and CD, AC and BD, AF and DE, AE and DF respectively. Prove that if three of these points are collinear then the fourth point lies on the same line.

    Solution. Consider a projective map saving the circumcircle and transforming L to its center. It transforms $ABCD$ and KL to a rectangle and its symmetry axis respectively. If one of points M, N lies on this axis then E and F are symmetric about it, therefore the remaining point also lies on KL.

19. (F.Ivlev) Let L and K be the feet of the internal and the external bisector of angle A of a triangle ABC. Let P be the common point of the tangents to the circumcircle of the triangle at B and C. The perpendicular from L to BC meets AP at point Q. Prove that Q lies on the medial line of triangle LKP.

    Solution. Since BC is the polar of P wrt the circumcircle ω of triangle ABC we obtain that P lies on the polar of L. Since the quadruple B, C, L, K is harmonic, K also lies on the polar of L. Therefore KP is the polar of L wrt ω, and the medial line of triangle KLP is the radical axis of ω and L. Prove that Q also lies on this axis.

    Let M be the midpoint of KL. Since M is the center of circle AKL perpendicular to ω, M lies on the polar of A. But M also lies on the polar of P, thus AP is the polar of M wrt ω and the common chord of ω and circle AKL. But LQ is the radical axis of circle AKL and L, therefore, Q is the common point of three radical axes.

20. (A.Zaslavsky) A circle and an ellipse lying inside it with a focus C are given. Find the locus of the circumcenters of triangles ABC, where AB is a chord of the circle touching the ellipse.

    Solution. Let CH be an altitude of triangle ABC. Then H lies on the circle having the greatest axis of the ellipse as diameter. Let O and R be the center and the radius of the given circle, and O' be the circumcenter of ABC. Using the cosine law to triangles $AO'O$ and $AO'C$, we have $R^2 = O'A^2 + O'O^2 - 2O'A \cdot O'O \cos\angle AO'O$, $OC^2 = O'C^2 + O'O^2 - 2O'C \cdot O'O \cos\angle CO'O$. Since $O'O \parallel CH$ and $O'A = O'C$, we obtain subtracting the second equality from the first one that $R^2 - OC^2 = 2O'O \cdot CH$.

Let the translation to vector CO transform H to H'. Then O, H' and O' are collinear and $OH' \cdot OO' = (R^2 - OC^2)/2$ do not depend on AB. Therefore O' and H' are symmetric about some circle concentric with the given one. Since the locus of points H' is a circle, The desired locus is also a circle.

21. (A.Yakubov) A quadrilateral $ABCD$ is inscribed into a circle ω with center O. Let M_1 and M_2 be the midpoints of segments AB and CD respectively. Let Ω be the circumcircle of triangle OM_1M_2. Let X_1 and X_2 be the common points of ω and Ω, and Y_1 and Y_2 the second common points of Ω with the circumcircles of triangles CDM_1 and ABM_2. Prove that $X_1X_2||Y_1Y_2$.

    Solution. Let K be a common point of AB and CD. Since angles OM_1K and OM_2K are right, OK is a diameter of Ω. Since arcs OX_1 and OX_2 of this circle are equal it is sufficient to prove that arcs KY_1 and KY_2 are also equal, or $\angle KM_1Y_1 = \angle KM_2Y_2$.

    Let N_1, N_2 be the second common points of circles CDM_1 and ABM_2 with AB and CD respectively. Then $KM_1 \cdot KN_1 = KC \cdot KD = KA \cdot KB$, therefore, $N_1K \cdot N_1M_1 = N_1A \cdot N_1B$. Thus the powers of N_1 wrt circles Ω and ABM_2 are equal, i.e. N_1 lies on M_2Y_2. Similarly N_2 lies on M_1Y_1 (fig.21). But it is clear that quadrilateral $M_1M_2N_2N_1$ is cyclic, which yields the desired equality.

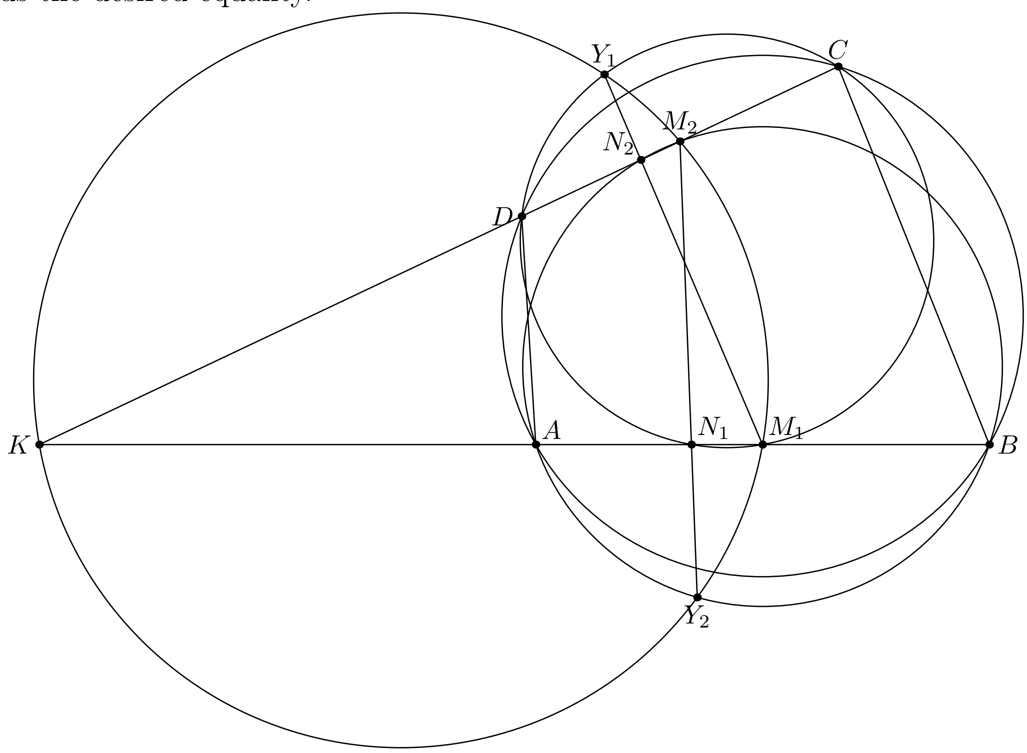

Fig.21

22. (A.Belov-Kanel) The faces of an icosahedron are painted into 5 colors in such a way that two faces painted into the same color have no common points, even vertices. Prove that for any point lying inside the icosahedron the sums of the distances from this point to the red faces and to the blue faces are equal.

    Solution. Prove that there exists a unique coloring satisfying the condition. Call the distance between two faces the minimal number of edges intersecting in the path from one face to the second one. Then the distance between two opposite faces is equal to

5. Also there exist 3 faces with distances 1 and 4 from any fixed face, and 6 faces with distances 2 and 3 from it.

Consider one of red faces. The faces with distances 1 or 2 from it can not be red. If the opposite face is red, then all remaining faces can not be red. If there exists a red face with distance 4 from the initial one, then there are only two faces without common vertices with two red faces. Since these two faces are adjacent only one from them can be red. Finally only three faces with distance 3 from the considered one can be red simultaneously. Thus there exists at most four red faces. This is also correct for all remaining colors, therefore there are exactly four faces of each color. The planes of four monochromatic faces form a regular tetrahedron. But for any point inside a tetrahedron the sum of the distances from it to the faces is equal to the altitude of the tetrahedron. This evidently yields the assertion of the problem.

23. (M.Yagudin) A tetrahedron $ABCD$ is given. The incircles of triangles ABC and ABD with centers O_1, O_2, touch AB at points T_1, T_2. The plane π_{AB} passing through the midpoint of T_1T_2 is perpendicular to O_1O_2. The planes π_{AC}, π_{BC}, π_{AD}, π_{BD}, π_{CD} are defined similarly. Prove that these six planes have a common point.

Solution. Consider four spheres having those circles as diametral sections. Then for example π_{AB} is the radical plane of two spheres touching AB, therefore it contains the radical center of four spheres. The remaining planes also pass through this point.

24. (N.Beluhov) The insphere of a tetrahedron $ABCD$ with center O touches its faces at points A_1, B_1, C_1 and D_1.

a) Let P_a be a point such that its reflections in lines OB, OC and OD lie on plane BCD. Points P_b, P_c and P_d are defined similarly. Prove that lines A_1P_a, B_1P_b, C_1P_c and D_1P_d concur at some point P.

b) Let I be the incenter of $A_1B_1C_1D_1$ and A_2 the common point of line A_1I with plane $B_1C_1D_1$. Points B_2, C_2, D_2 are defined similarly. Prove that P lies inside $A_2B_2C_2D_2$.

Solution. a) Let B_a be such a point that A_1B_a is a diameter in the circumcircle of $\triangle A_1C_1D_1$ with center O_b and radius R_B. Define C_a, D_a, O_b... and so on similarly. Let also the inscribed sphere of $ABCD$ be ω, and its inradius be r. Finally, denote by $d_a(X)$ the distance from a point X to the plane $(B_1C_1D_1)$, and similarly for $d_b(X)$ and so on.

By symmetry, B_a is the reflection of A_1 in BO. So, since the plane (BCD) touches ω, P_aB_a also touches ω. Let Q be the projection of P_a in the plane $(A_1C_1D_1)$. We see that $\angle P_aB_aO = 90 \Rightarrow \triangle P_aQB_a \sim \triangle B_aO_aO \Rightarrow d_b(P_a) : R_B = P_aB_a : r$. Analogously, $d_c(P_a) : R_C = P_aC_a : r$ and $d_d(P_a) : R_D = P_aD_a : r$. Since $P_aB_a = P_aC_a = P_aD_a$ (as tangents to a sphere), this means that the distances from P_a to the faces of the tetrahedron $A_1B_1C_1D_1$ are in ratios $d_b(P_a) : d_c(P_a) : d_d(P_a) = R_B : R_C : R_D$. Analogous reasoning shows that the distances from P_b to the corresponding faces of the same tetrahedron are in ratios $R_A : R_C : R_D$, and so on for P_c and P_d.

But the locus of the points whose distances to three given planes are in given ratios is a line trough the intersection of these planes, and the locus of the points whose distances to two given planes are in given ratio is a plane trough the intersection of these planes. Thus, the lines A_1P_a and B_1P_b lie in the same plane and intersect in some point P. By the loci argument, this point also lies in the lines C_1P_c and D_1P_d.

b) Notice that the interior of the tetrahedron $A_2B_2C_2D_2$ is the locus of the points X such that the four inequalities hold: $d_a(X)+d_b(X)+d_c(X) \geq 2d_d(X)$, $d_b(X)+d_c(X)+d_d(X) \geq 2d_a(X)$, and so on. This is easy to see using baricentric coordinates with respect to $A_1B_1C_1D_1$. Indeed, if α, β, γ and δ are the coordinates of some point X, and d_A and so on denote the equal distances from A_2 to the three corresponding faces of $A_1B_1C_1D_1$, then $d_a(X) = \beta d_B + \gamma d_C + \delta d_D$ and so on, yielding $3\alpha d_A = d_b(X)+d_c(X)+d_d(X)-2d_a(X)$ and so on. Thus, the inequalities hold exactly when α, β, γ and δ are positive, and this happens exactly when X lies inside $A_2B_2C_2D_2$ (more elementary, but not as simple arguments can also be applied).

Thus, it suffices to show that $R_A + R_B + R_C > 2R_D$ (and so on, symmetrically).

Notice that all faces of the tetrahedron $A_1B_1C_1D_1$ are acute-angled triangles, and the points O_a, O_b and so on are interior to them (this follows easily from the fact that its vertices are the tangency points of the inshere with the faces of $ABCD$). Obviously, $2R_A + 2R_B + 2R_C \geq B_1C_1 + C_1A_1 + A_1B_1$ (as diameters are greater than chords). Let K, L and M be the midpoints of the sides of $\triangle A_1B_1C_1$. The point O_d lies inside the quadrilateral, say, A_1LKB_1 (as it lies inside $\triangle KLM$), thus $A1L + LK + KB_1 > A_1O_d + O_dB_1$. But $A_1L + LK + KB_1 = \frac{1}{2}B_1C_1 + \frac{1}{2}C_1A_1 + \frac{1}{2}A_1B_1$, and the inequality desired follows.