Generalized closing theorems *

V.Yu. Protasov †

Abstract

We present a closing theorem for spheres in the space $\mathbb{R}^d$ that, in particular, implies the four classical theorems of Poncelet type. Then we give an independent elementary proof of the Emch theorem, which is in a sense the strongest one among the classical closing theorems. The method of the proof makes it possible to derive a generalization for pencils of circles analogous to the great Poncelet theorem.

Keywords: *Poncelet type theorems, spheres, circular series, pencils.*

AMS 2000 Mathematical Subject classification: 51M04, 51M15

I. Introduction

Closing theorems, or theorems of Poncelet type, are considered to be one of the most fascinating geometric facts. Various approaches to their proofs as well as applications to problems of elementary geometry, theory of algebraic curves, differential equations, billiards, elliptic integrals, etc., have been studied in many works (see, for example, [1]–[8] and references therein). One can spot the four best known closing theorems: Poncelet, Steiner, zigzag, and Emch theorems. We do not mention some other results, such as, for instance, the Ponzag theorem, that are actually reformulations of one of these four theorems.

This paper consists of two parts. First, in section III we derive a general closing theorem for families of Euclidean spheres in $\mathbb{R}^d$. Then, in sections IV – VI, we observe some of its corollaries. One of them, Theorem 2, gives a general closing principle for spheres in the space $\mathbb{R}^3$. The four classical theorems of Poncelet type are its direct corollaries (section V). Another one, Theorem 3, extends the Emch closing theorem to spheres in the Euclidean space $\mathbb{R}^d$. In the second part of the paper, in sections VII and VIII, we focus on the elementary proof of Emch's theorem on circular series. This theorem is the most general one among these four classical results. The Poncelet theorem in case of two circles, the planar version of the zigzag theorem, and the Steiner theorem are actually its special cases. This is

*The research is supported by the RFBR grant No 08-01-00208 and by the grant for Leading Scientific Schools No 3233.2008.1

†Dept. of Mechanics and Mathematics, Moscow State University, Vorobyovy Gory, 119992, Moscow, e-mail: vladimir_protassov@yahoo.com

shown in section VII. Therefore it would be interesting to obtain an autonomous (not relying on the Poncelet theorem) proof of the Emch theorem using only elementary geometrical tools. Such a proof is given in section VII. For the sake of simplicity we restrict ourselves to the case, when the three circles are embedded to each other. That proof is based on two auxiliary geometric results, Theorem 4 on four circles touching two concentric circles, and Proposition 1 on two chains of circles inscribed in an annulus, which may be of independent interest. Finally, in section VIII we apply this technique to derive a generalization of the Emch theorem to pencils of circles.

II. Four classical closing theorems

In this section we recall the statements of the four famous Poncelet type theorems. To formulate them in a unique way it is convenient to introduce the notion of general closing property for families of circles. This notion will be also used in the next sections, when we generalize Poncelet type theorems to the space $\mathbb{R}^d$. Suppose a circle δ and a family of circles $\mathcal{M}$ are given on the plane $\mathbb{R}^2$. Straight lines and points are also considered as circles. We call a point $z \in \mathbb{R}^d$ *singular* for the family $\mathcal{M}$ if there are more than two circles from $\mathcal{M}$ passing through z. Assume that the two conditions are satisfied:

(**a**) *the circle δ does not contain singular points for $\mathcal{M}$;*

(**b**) $\delta \notin \mathcal{M}$.

Let us now consider the following process. Take an arbitrary point D_1 on δ and draw a circle $v_1 \in \mathcal{M}$ through it (we suppose that such a circle exists; if there are two ones, then we take any of them). Let D_2 be the second point of intersection of v_1 and δ (in case of tangency we set $D_2 = D_1$). Draw a circle $v_2 \in \mathcal{M}$ through D_2 different from v_1 (if it does not exist, then we set $v_2 = v_1$). Then we denote by D_3 the second point of intersection of v_2 and δ, etc. We obtain a series of circles $\{v_k\}_{k=1}^{\infty}$. The process has period n if $v_{n+1} = v_1$ or, which is the same, $D_{n+1} = D_1$.

Definition 1 *A family of circles $\mathcal{M}$ is said to possess the closing property on a circle δ if it satisfies conditions* (**a**), (**b**) *and the following condition:*

if for some initial point D_1 the process has period $n \geq 3$ and all the points $D_1, \ldots, D_n$ are distinct, then it has the same period for any point $D_1 \in \delta$ that belongs to a circle from $\mathcal{M}$.

Now we are giving the statements of the four classical closing theorems.

Theorem A [`Poncelet [9]`] *For any quadric α and a circle δ on the plane the set of lines touching α possesses the closing property on δ.*

As usual we call quadric in $\mathbb{R}^d$ a set of points $x \in \mathbb{R}^d$ such that $(x, Ax) + (b, x) + c = 0$, where A is a self-adjoint operator, $b \in \mathbb{R}^d$, $c \in \mathbb{R}$, and $(\cdot\,,\cdot)$ denotes the standard inner product in $\mathbb{R}^d$. We deal with real nonempty quadrics only. A quadric in $\mathbb{R}^2$ is called plane. If the quadric α in Theorem A is degenerate (a pair of lines, a single line, or a point), then one replaces the tangents by parallel lines (in case of a line and of a pair of

lines) or, in case of a point, by lines passing through that point. The Poncelet theorem is usually formulated for two quadrics α and δ. Nevertheless, it can always be assumed that δ is a circle. Indeed, if the quadric δ is nondegenerate, then one can map it to a circle by a suitable stereographic projection. Therefore, in this case the Poncelet theorem for two quadrics follows from Theorem A. If δ is degenerate, then the Poncelet theorem is trivial, and the reader will easily prove it.

To illustrate the Poncelet theorem, consider the case, when α and δ are both circles, and α lies inside δ. Take a point $D_1 \in \delta$ and draw a line tangent to α, which intersects the circle δ for the second time at a point D_2. There are two tangents to the circle α passing through D_2. One of them is D_2D_1. Draw the second one D_2D_3 (the point D_3 lies on δ). Then we draw the next tangent D_3D_4 not coinciding with the previous one, etc. The Poncelet theorem says that *if this process cycles after n steps, i.e., $D_{n+1} = D_1$ (figure 1), then it will cycle for any choice of the initial point D_1 with the same number of steps n.* Thus, if there is an n-gon inscribed in the circle δ and circumscribed around the circle α, then there are infinitely many such n-gons. Moreover, any point of the circle δ is a vertex of one of those inscribed-circumscribed n-gons. A beautiful proof of this version of Theorem A using measure theory was derived by Jacobi and Bertrand (see, for instance, [7]). Proofs of Theorem A involving the theory of projective quadrics see in [5, 14]. Other proofs bases on various ideas can be found in [1] – [4]. None of them is elementary.

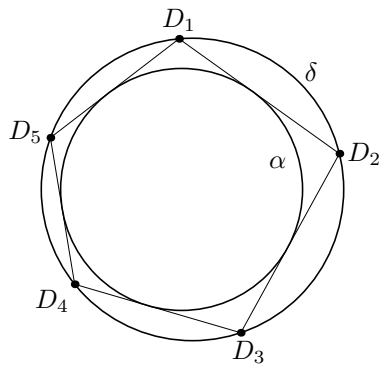

Figure 1. Poncelet theorem

To formulate the next classical closing theorem in its most general form we use the notion of index of tangency. The tangency of two circles is called interior if one of the circles lies inside the other. Suppose α_0, α_1 are circles on the plane; then for an arbitrary circle β touching both α_0 and α_1 the *index of tangency* is 0 if there is an even number of interior tangencies among the two ones: β with α_0 and β with α_1. If this number is odd, then the index is 1. This notion is naturally extended to the case, when some of the circles α_j become straight lines (the index depends on the orientation of the line). For given $i = 0, 1$ we denote by $\mathcal{M}_i$ the family of circles touching α_0 and α_1 with index i. If either α_0 or α_1 becomes a point, then $\mathcal{M}_0 = \mathcal{M}_1$. The same notation will be used in the next sections for families of spheres in $\mathbb{R}^3$ touching two given spheres.

Given two circles α_0 and α_1 on the plane, α_0 is inside α_1. In this case the family $\mathcal{M}_1$ consists of circles inscribed in the annulus formed by the circles α_0 and α_1. The *process of Steiner* produces a series of circles $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{M}_1$ as follows: $v_1 \in \mathcal{M}_1$ is arbitrary, for any $k \in \mathbb{N}$ the circle v_{k+1} touches v_k and is different from v_{k-1} if $k \geq 2$. The process has period $n \geq 3$ if $v_{n+1} = v_1$.

Theorem B [`Steiner`.] *If the process of Steiner is periodic for some initial circle v_1, then it has the same period for any $v_1 \in \mathcal{M}_1$.*

Thus, if there is a closed chain of n touching circles inscribed in the annulus between α_0 and α_1 (figure 2), then there are infinitely many such chains, and any circle inscribed in the

annulus can be the first circle of a chain. This construction is sometimes called *Steiner's necklace*, or even *Steiner's telephone dialer*. In contrast to other closing theorems Theorem B has several elementary proofs. The most known one is by inversion: if one applies a suitable inversion taking α_0 and α_1 to a pair of concentric circles, then the statement becomes obvious. However, none of those elementary proofs can be extended to the other Poncelet type theorems.

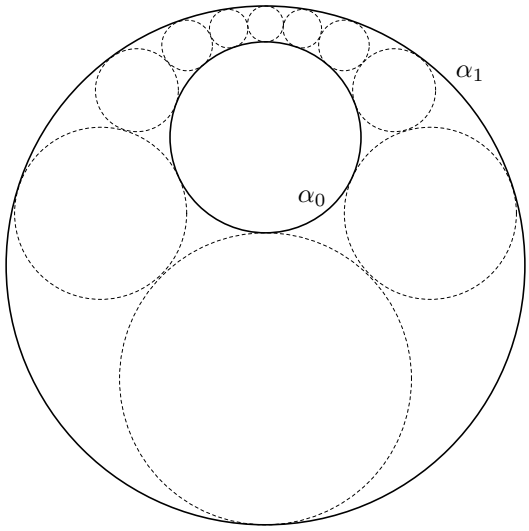

Figure 2. Steiner theorem

The third one is the zigzag theorem. It also deals with two circles, but this time the circles are not necessarily on one plane, they may have arbitrary positions in the space. Given a number $\rho > 0$ and two circles s and δ in the space $\mathbb{R}^3$. Assume this pair of circles satisfies the following condition:

(c) *the orthogonal projection of any of these circles onto the two-dimensional plane containing the other circle does not pass through its center.*

So, if one takes either of these circles and erects a perpendicular to its plane at the center of that circle, then it does not meet the other circle.

Take an arbitrary point $D_1 \in \delta$. If the sphere of radius ρ centered at D_1 intersects s, then we take any of two points of intersection and call it S_1. Then we take a point $D_2 \in \delta$ such that $D_2S_1 = \rho$ and $D_2 \neq D_1$ (if it does not exist, then $D_2 = D_1$). Further, the point $S_2 \in s$ is such that $S_2D_2 = \rho$, $S_2 \neq S_1$ (if such a point does not exist, then we set $S_2 = S_1$), and so on. The *zigzag process* produces the sequences $\{D_k\}$ and $\{S_k\}$ for a given initial segment $D_1S_1 = \rho$. Zigzag has period n if $D_{n+1} = D_1$.

Theorem C [`zigzag`.] *If the zigzag has period $n \geq 3$ for some initial point $D_1 \in \delta$ and all the intermediate points are distinct, then it has the same period for any point $D_1 \in \delta$, from which one can make the first step.*

The zigzag process can be interpreted as jumps of a flea from one circle to another with the same length of the jump ρ (figure 3). If after $2n$ jumps the flea arrives at the starting point D_1, then it will happen for any starting point on the circle δ. In other words, *if there*

is a $2n$-gon, whose even vertices lie on the circle δ, odd vertices lie on the circle s, and all sides have the same length ρ, then there are infinitely many such $2n$-gons. Moreover, any point of δ can be a vertex of such a $2n$-gon.

Theorem C originates in [10]. Its proofs based on various ideas, highly non-elementary, can be found in [3, 6]. The equivalence of the zigzag theorem and the Poncelet theorem was established in [11].

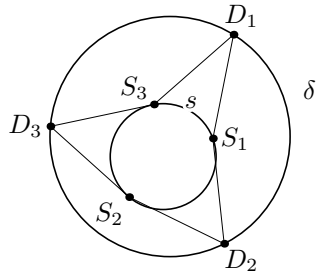

Figure 3. Zigzag theorem

Now we turn to the fourth closing theorem. We are going to see that this theorem is, in a sense, the strongest one: three others easily follow from it. In the statement we again use the families of circles $\mathcal{M}_0$ and $\mathcal{M}_1$ defined above.

Theorem D [`Emch [12].`] *There are circles α_0, α_1 and δ on the plane, each of them may become a point. Then for any $i \in \{0, 1\}$ the family $\mathcal{M}_i$ corresponding to the pair of circles α_0, α_1 possesses the closing property on δ, provided $\delta \notin \mathcal{M}_i$.*

Figure 4 illustrates Theorem D in case, when the circle δ lies between α_0 and α_1. If there is a closed chain of n circles inscribed in the annulus formed by α_0 and α_1, such that each pair on neighboring circles meets on δ, then there are infinitely many such chains. Moreover, any circle inscribed in the annulus can be the first circle of a chain.

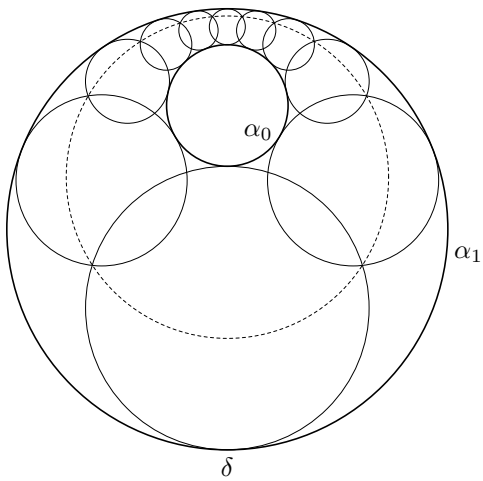

Figure 4. Emch theorem

A proof of Theorem D can be found in [3]. In [13] this theorem was derived from the Poncelet theorem by elementary geometric tools.

III. General closing principle

We are going to establish a fundamental theorem that implies not only the classical Theorems A – D, but also their multidimensional generalizations obtained in the next sections. This theorem is formulated in the space $\mathbb{R}^d$ for series of Euclidean spheres. Let us start with introducing some notation. We denote by $S(z, r) = \left\{x \in \mathbb{R}^d, |x - z| = r\right\}$ a Euclidean

sphere in $\mathbb{R}^d$ of radius r centered at z; by $P(n,c) = \{x \in \mathbb{R}^d\,,\ (n,x) = c\}$ we denote a hyperplane with a direction vector n, $|n| = 1$, and $c \in \mathbb{R}$. A sequence of spheres $S(z_k, r_k), k \in \mathbb{N}$ converges to the plane $P(n,c)$ if $r_k \to \infty$, $z_k/|z_k| \to n$ and $(|z_k|^2 - r_k^2)/2|z_k| \to c$ as $k \to \infty$. By spheres we also mean points (when $r = 0$) and planes, unless the opposite is stated (for instance, when the radius is given). In particular, $S(z,r)$ denotes a sphere or a point (when $r = 0$), but not a plane.

Let us now define the closing property for families of spheres in $\mathbb{R}^d$. Suppose a circle δ and a family of spheres $\mathcal{M}$ are given in the space $\mathbb{R}^d$. We call a point $z \in \mathbb{R}^d$ *singular* for the family $\mathcal{M}$ if there are more than two spheres from $\mathcal{M}$ passing through z. The two following conditions extend conditions **(a)** and **(b)** (section II) from circles to spheres:

($\mathbf{a}'$) *circle δ does not contain singular points for $\mathcal{M}$;*

($\mathbf{b}'$) *there is no sphere in $\mathcal{M}$ containing δ.*

Consider now the same process as in section II for spheres. For an arbitrary point D_1 on the circle δ we draw a sphere $v_1 \in \mathcal{M}$ through it (we suppose that such a sphere exists; if there are two ones, then we take any of them), and denote by D_2 the second point of intersection of v_1 and δ (in case of tangency $D_2 = D_1$). Draw a sphere $v_2 \in \mathcal{M}$ through D_2 different from v_1 (if it does not exists, then we set $v_2 = v_1$), and denote by D_3 the second point of its intersection with δ, etc. We obtain a series of spheres $\{v_k\}_{k=1}^{\infty}$. The process has period n if $v_{n+1} = v_1$ or, which is the same, $D_{n+1} = D_1$.

Definition 2 *A family of spheres $\mathcal{M}$ is said to possess the closing property on a circle δ if it satisfies conditions* ($\mathbf{a}'$), ($\mathbf{b}'$) *and the following condition:*

if for some initial point D_1 the process has period $n \geq 3$ and all the points $D_1, \ldots, D_n$ are distinct, then it has the same period for any point $D_1 \in \delta$ that belongs to a sphere from $\mathcal{M}$.

The main result of this section gives sufficient conditions for a family of spheres to possess the closing property on any circle. Suppose we are given a set $\Gamma \subset \mathbb{R}^d$, which is either a plane quadric or a subset of a straight line. For arbitrary $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$ consider the set of spheres $\{\, S(z,r) \subset \mathbb{R}^d \,\}$ defined by the following relations:

$$r^2 \;=\; |z-a|^2 + b\,, \qquad z \;\in\; \Gamma\,. \tag{1}$$

So, this family consists of all spheres $S(z,r)$ such that $z \in \Gamma$ and $|z-a|^2 + b \geq 0$, in which case $r = \sqrt{|z-a|^2 + b}$. If the set Γ is unbounded, then we add to this family one or two limit planes: if Γ is a hyperbola or a pair of lines, then the two planes $P(n_k, c_k)$, $k = 0, 1$ are added, where n_k is the direction vectors of the lines or of the asymptotes of the hyperbola, $c_k = (n_k, a)$; if Γ is a subset of a line, then one plane $P(n,c)$ is added, where n is the direction vector of the line, $c = (n,a)$.

Theorem 1 *Family (1) possesses the closing property on any circle $\delta \subset \mathbb{R}^d$ that does not contain singular points for family (1) and does not lie on its spheres.*

`Proof.` First, we reduce the theorem to the planar case, i.e., to $d = 2$. Then we show that all circles of family (1) touch a suitable quadric α, after which the theorem will follow from Theorem A.

Thus, let us reduce the theorem to the case $d = 2$. We consider only spheres of family (1), the same results for planes, if they exist, will follow from the limit passage. Without loss of generality it may be assumed that the origin is located on the two-dimensional affine plane K containing the circle δ. If some sphere $S(z, r)$ of family (1) intercepts a circle on K with center z_1 and radius r_1, then $r^2 - |z|^2 = r_1^2 - |z_1|^2$. Whence z_1, r_1 satisfy (1), where a and Γ are replaced by their orthogonal projections onto K and the parameter b is also properly changed. Thus, the family of circles formed by intersections of spheres from family (1) with the plane K can be defined by similar relations on K.

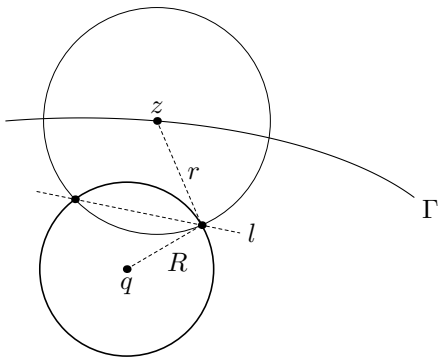

Figure 5

Let q be the center of the circle δ, R be its radius. Take an arbitrary circle γ of the family (1) intersecting δ. Let l be the line containing the common chord of δ and γ (figure 5). Any point $x \in l$ has equal powers with respect to δ and γ, hence $|x-q|^2 - R^2 = |x-z|^2 - r^2$. Expressing r^2 from (1) we obtain after simplifications $(x - a, z - q) = k$, where $k = \frac{R^2-|a|^2-|q|^2-b}{2} + (a, q)$. This linear equation on x is nontrivial ($z - q$ and k cannot vanish simultaneously), otherwise the circle γ coincides with δ, which contradicts the assumption. So, we obtain a family of lines $\mathcal{L} = \{ l(z),\ z \in \Gamma \}$, where $l(z) = l = \{ x \in \mathbb{R}^2 \mid (x - a, z - q) = k \}$. If Γ is a subset of a straight line, then all the lines of $\mathcal{L}$ concur, or they are all parallel. In this case the statement is trivial: for any point $D_1 \in \delta$ the process has period 2, whenever it can start. If Γ is a quadric, then all lines of the family $\mathcal{L}$ touch the quadric α, which is obtained from the dual quadric $(\Gamma - q)^*$ by applying the multiplication by the factor k and the translation by the vector a. Therefore, in this case the theorem follows from Theorem A.

□

In the next three sections we observe some crucial corollaries of Theorem 1. We are going to see that Theorem 1 is a quite powerful tool to prove many Poncelet type results. First, we establish a special closing theorem for spheres in the space $\mathbb{R}^3$ that imply, just as simple special cases, all four classical Theorems A–D. Another simple corollary of Theorem 1 is that the zigzag theorem holds for any pair of circles in $\mathbb{R}^d$, not necessarily in $\mathbb{R}^3$. Then we go further and derive a general closing theorem in $\mathbb{R}^d$ which is a generalization of Emch's theorem (Theorem D) for all dimensions $d \geq 2$.

IV. Closing theorem in $\mathbb{R}^3$ and the four classical theorems

In the space $\mathbb{R}^3$ given a sphere Q and spheres $S_0, S_1 \subset \mathbb{R}^3$ that are not symmetric with respect to Q (i.e. are not mapped to each other by the inversion with respect to Q). The sphere Q, and one of the spheres S_0, S_1 may become points. Let $\mathcal{M}_i$, $i = 0, 1$, be the corresponding families of spheres tangent to S_0 and S_1 (see the definition in section II). Choose $i \in \{0, 1\}$ and consider the family $\mathcal{M}$ of spheres from $\mathcal{M}_i$ that are orthogonal to Q. There are at most two singular points in $\mathbb{R}^3$ for the family $\mathcal{M}$, this will be shown in Remark 1. Orthogonality, as usual, means that two tangent planes to the spheres drawn at their common point are perpendicular to each other. Equivalently, two radii of the spheres starting at their common point form a right angle. If Q is a plane, then a sphere is orthogonal to Q iff it is centered on Q; if Q is a point, then a sphere is orthogonal to Q iff it passes through Q.

Theorem 2 *The family $\mathcal{M}$ possesses the closing property on any circle $\delta \subset \mathbb{R}^3$ that does not pass through singular points and does not lie on a sphere from $\mathcal{M}$.*

The geometrical meaning of this theorem is less obvious than for Theorems A–D because of that orthogonality condition. However, as we will see below, Theorem 2 implies all of them. In some sense, Theorem 2 is a common root for all the classical closing theorems.

The proof of Theorem 2 is by merely showing that the family $\mathcal{M}$ satisfies the assumptions of Theorem 1. We use several well-known facts of elementary geometry. Any pair of spheres S_0, S_1 of different radii has two homothety centers h_0, h_1. This means that there is a homothety centered at h_0 taking the sphere S_0 to S_1, and the same holds for the point h_1. For each $i = 0, 1$ the line joining the points of tangency of any sphere from $\mathcal{M}_i$ with S_0 and with S_1 passes through h_i. Moreover, the point h_i has the same power with respect to all spheres of the family $\mathcal{M}_i$. In the sequel we denote this power by p_i.

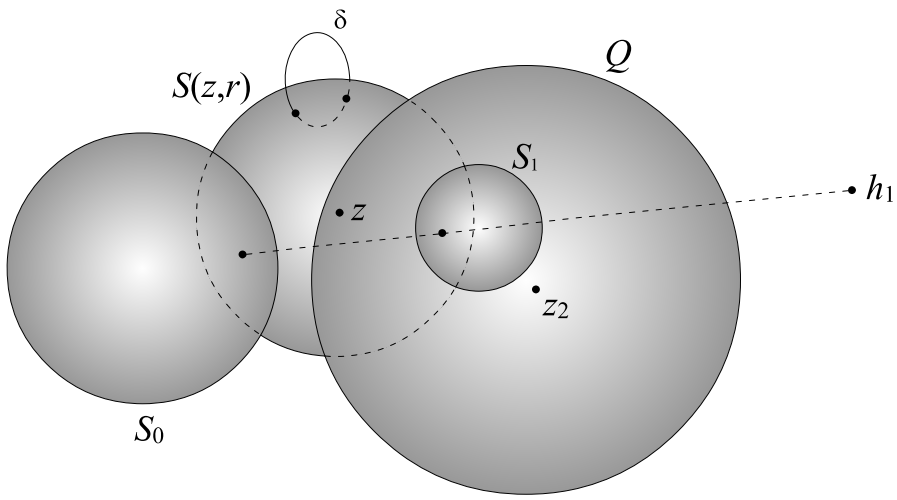

Figure 6. Closing theorem in $\mathbb{R}^3$ (Theorem 2)

`Proof of Theorem 2.` With possible inversion it may be assumed that Q, S_0 and S_1 are spheres (not planes) and that S_0, S_1 have different radii. Thus, $r_0 \neq r_1$, where $S_k =$

$S(z_k, r_k)$, $k = 0, 1$. Choose some $i \in \{0, 1\}$ take the corresponding subset $\mathcal{M}$ of the family $\mathcal{M}_i$ and consider an arbitrary sphere $S(z, r) \in \mathcal{M}$ (figure 6). The power of the point h_i with respect to this sphere is equal to p_i, so $|z-h_i|^2-r^2 = p_i$. Thus, the sphere $S(z, r)$ satisfies (1) with $a = h_i, b = -p_i$. Since this sphere is orthogonal to Q, we have $|z - z_2|^2 = r^2 + r_2^2$, where z_2 is the center of Q and r_2 is its radius. Subtracting this equality from the previous one, we obtain a linear equation in z, which defines some plane $L \subset \mathbb{R}^3$. On the other hand, centers of all spheres of $\mathcal{M}_i$ form a quadric in $\mathbb{R}^3$ with foci z_0, z_1. Therefore the centers z of spheres $S(z, r) \in \mathcal{M}$ lie on the intersection of that quadric with the plane L, i.e., on a plane quadric Γ. It now remains to apply Theorem 1.

□

Let us now derive Theorems A–D from Theorem 2.

Theorem 2 ⇒ Theorem A. S_0, S_1 are arbitrary spheres inscribed in a cone that has the quadric α as a plane section; Q is the point at infinity.

In this case all spheres of $\mathcal{M}$ are planes, because they contain the point at infinity. They intersects the plane of the quadric α by lines touching α. From this Theorem A follows.

Theorem 2 ⇒ Theorem C. In this case Q is the plane of the circle s, spheres S_i are concentric to s and have radii $|r \pm \rho|$, where r is the radius of s.

Theorem 2 ⇒ Theorem D. We set Q to be the plane of the circles $\alpha_0, \alpha_1, \delta$, the sphere S_i has its center on the plane Q and intersects it along the circle α_i, $i = 0, 1$.

In this case all spheres of $\mathcal{M}$ are centered on the plane Q and intersect that plane by circles tangent to both α_0 and α_1.

Theorem 2 ⇒ Theorem B. Theorem B is a special case of Theorem D (section VII).

Thus, the Poncelet theorem corresponds to the case of Theorem 2, when Q is the point at infinity; the zigzag theorem corresponds to the case, when S_0, S_1 are disjoint and both orthogonal to Q; finally the Emch theorem corresponds to another special case, when the circle δ lies on the sphere Q, and the spheres S_0 and S_1 are both orthogonal to Q.

Remark 1 Theorem 2 holds for general spheres Q, S_0, S_1 and a circle δ in the sense that any three spheres and a circle in general position satisfy the assumptions of Theorem 2. To see this we show that there are at most two singular points for the family $\mathcal{M}$. With possible inversion we may assume that radii of the spheres S_0, S_1 are different. If z is a singular point, then $z \notin Q$ and, moreover, the line joining z and $\tilde{z}$ (the inverse image of z with respect to the sphere Q) passes through h, and $(h_i - z, h_i - \tilde{z}) = p_i$. To show this we make an inversion with center at the point z (the images will be denoted by prime). Spheres of $\mathcal{M}$ containing z become planes passing through the center of the sphere Q' (or perpendicular to the plane Q' in case $z \in Q$), tangent to the spheres S_0', S_1'. There are at most two such planes unless the center of Q' coincides with the homothety center of the spheres S_0', S_1'. This case corresponds to the property of the point z described above. There are at most two points z with this property.

V. Zigzag theorem for two circles in $\mathbb{R}^{\mathbf{d}}$

Another immediate corollary of Theorem 1 is the extension of the zigzag theorem to spaces of all dimensions.

Corollary 1 *Theorem C holds for any pair of circles in $\mathbb{R}^d$ satisfying condition* ***(c)****.*

`Proof.` Let s and δ be arbitrary circles in $\mathbb{R}^d$ and $\rho > 0$ be the length of the jump. The family $\mathcal{M}$ of spheres of radius ρ centered on s can be defined by relations (1), where $\Gamma = s$, a is the center of the circle s, r is its radius, and $b = \rho^2 - r^2$.

□

VI. Closing theorem for spheres in $\mathbb{R}^{\mathbf{d}}$

As we already mentioned in the Introduction, the Emch theorem plays a special role among the four classical closing theorems. In some sense, it is the strongest one among them, because the Poncelet theorem for two circles, the Steiner theorem, and the zigzag theorem in case of circles on one plane follow easily from Emch's theorem. This will be shown in section VII. That is why it would be most interesting to have the Emch theorem not only in the plane, but in the space $\mathbb{R}^d$ for any $d \geq 2$. Instead of the family of circles $\mathcal{M}$ touching two given circles α_0, α_1 on the plane, as in Theorem D, we now consider a family of spheres touching d given spheres in $\mathbb{R}^d$ (figure 7). It appears that, under some general assumptions, this family possesses the closing property on any circle $\delta \subset \mathbb{R}^d$. This means that if there is a closed chain of n spheres in $\mathbb{R}^d$ touching d given spheres such that each pair on neighboring spheres intersect on the circle δ, then there are infinitely many such chains. Moreover, for any point of δ there is a chain starting in it. This is a generalization of Emch's theorem to $\mathbb{R}^d$. To formulate it one needs to overcome one difficult point. In Theorem D we deal with two families of circles $\mathcal{M}_0, \mathcal{M}_1$ touching two given circles. For d spheres in the space $\mathbb{R}^d$ there may be as many as 2^{d-1} such families of spheres. To classify them it will be convenient to use the notion of oriented sphere.

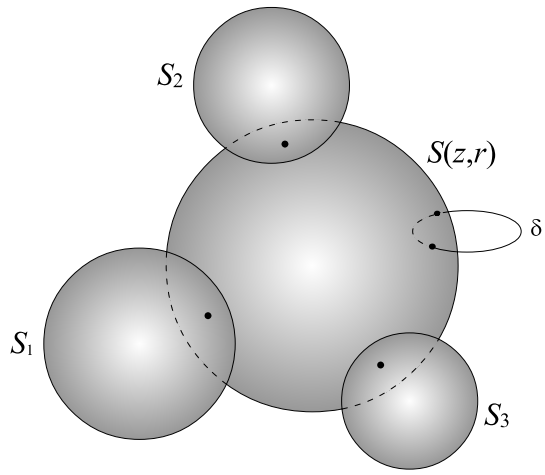

Figure 7. Generalized Emch theorem (Theorem 3)

An oriented sphere $S(z,r)$ of radius $r \in \mathbb{R}$ centered at z is the set of points $x \in \mathbb{R}^d$ such that $|x-z| = |r|$. So the radius of an oriented sphere may take any real value; $S(z,r)$ and $S(z,-r)$ are considered as two different spheres, whenever $r \neq 0$, although they correspond to the same set of points in $\mathbb{R}^d$. $P(n,c)$ denotes the oriented plane that consists of points $x \in \mathbb{R}^d$ such that $(n,x) = c$. The planes $P(n,c)$ and $P(-n,-c)$ are also considered to be different, although they correspond to the same set of points. In this section all spheres and planes are assumed to be oriented. A sphere $S(z,r)$ touches a sphere $S(z_0,r_0)$ when $|z-z_0| = |r+r_0|$; it touches a plane $P(n,c)$ when $(z,n)+r = c$. A collection of spheres $S_i = S(z_i,r_i),\ i = 1,\dots,d$ is said to be *in the general position* if the affine hull of the points $(z_i,r_i)^T \in \mathbb{R}^{d+1},\ i = 1,\dots,d$ is of dimension $d-1$. This means that the points $(z_i,r_i)^T$ are vertices of a $(d-1)$-dimensional simplex. The general position property is invariant with respect to translations and orthogonal transforms of $\mathbb{R}^d$, but not with respect to inversions. We call two collections of spheres (including planes) *equivalent* if one of them can be obtained from the other by finitely many isometries and inversions. Now we are ready to formulate the main theorem.

Let us have a collection of d spheres $S_i = S(z_i,r_i)$, $i = 1,\dots,d$. Some of them, but not all simultaneously, may become points. Consider the family $\mathcal{M}$ of spheres tangent to all the spheres S_i. Note that in case $d \geq 3$ the family $\mathcal{M}$ may be empty for some collections. For a set of d usual (non-oriented) spheres there are up to 2^{d-1} such families, depending on the orientation.

Theorem 3 *Let a collection of d spheres in $\mathbb{R}^d$ be equivalent to a collection in general position and the family $\mathcal{M}$ of spheres touching them be nonempty. If a circle $\delta \subset \mathbb{R}^d$ does not contain singular points and does not lie on a sphere from $\mathcal{M}$, then $\mathcal{M}$ possesses the closing property on δ.*

Let us see what Theorem 3 gives for small dimensions d.

$\mathbf{d = 2}$. Any pair of distinct circles on the plane is in the general position, therefore Theorem 3 becomes the Emch theorem.

$\mathbf{d = 3}$. Theorem 3 holds for *any triple* of spheres in $\mathbb{R}^3$, for which there is a sphere touching them. If a triple $S(z_i,r_i)$, $i = 1,2,3$, is not in the general position, then the points $(z_i,r_i)^T \in \mathbb{R}^4$ are collinear. Whence the centers z_i are on the same line as well. Applying the inversion with some center outside that line we get spheres of general position.

Thus, in case of small dimensions any collection of spheres are equivalent to one in general position. For $d \geq 4$ this may not be the case. Nevertheless, a "typical" set of d spheres and one circle in $\mathbb{R}^d$ does satisfy the assumptions of Theorem 3. Let us first establish this, and then we prove the theorem. We start with three auxiliary results. Observe that any sphere $S(z,r) \in \mathcal{M}$ satisfies the following system of equations:

$$r^2 \;=\; |z|^2 \;-\; 2\,(z_i,z) \;-\; 2\,r_i\,r \;+\; |z_i|^2 \;-\; r_i^2, \qquad i = 1,\dots,d; \tag{2}$$

and any plane $P(n,c) \in \mathcal{M}$ satisfies the system

$$(z_i,n) \;+\; r_i = c, \qquad i = 1,\dots,d. \tag{3}$$

Lemma 1 *For any collection of spheres in the general position the family $\mathcal{M}$ contains at most two planes.*

`Proof.` Subtracting the first equation of system (3) from the others we obtain the linear system $(z_i - z_1, n) = r_i - r_1,\ i = 2, \dots, d$ of rank $d-1$. Its solutions n form a straight line in $\mathbb{R}^d$, which contains at most two points such that $|n| = 1$.

□

Planes of the family $\mathcal{M}$, if they exist at all, are limits for the spheres of that family. Now we clarify when all spheres forming an affine plane (in the space of spheres) can touch one sphere.

Lemma 2 *Suppose an affine plane $L \subset \mathbb{R}^{d+1}$, $\dim L \geq 1$ is such that there is a sphere (or a point) $S_0 \subset \mathbb{R}^d$ tangent to every sphere of the family $\mathcal{L} = \big\{ S(z,r) \,\big|\, (z,r)^T \in L \big\}$. Then L is a line and $\mathcal{L}$ is a pencil of spheres tangent at one point. Moreover, $S_0 \in \mathcal{L}$.*

`Proof.` If $\dim L = 1$, then the center z of any sphere of $\mathcal{L}$ lies on a fixed straight line $b \subset \mathbb{R}^d$ and its radius r is a linear function of z. If all the spheres from $\mathcal{L}$ touch S_0, then the center of S_0 lies on b as well, otherwise r is not linear in z. Therefore, $\mathcal{L}$ is a pencil of tangent spheres. In particular, S_0 also belongs to $\mathcal{L}$. In case $\dim L \geq 2$ all lines on L concur at one point corresponding to the sphere S_0, which is impossible.

□

The last auxiliary result restricts the location of singular point of the family $\mathcal{M}$

Lemma 3 *For a collection of d spheres in the general position all singular points of the family $\mathcal{M}$ lie on some affine plane E of dimension $d-2$. If $\mathcal{M}$ contains two planes, then their intersection coincides with E.*

`Proof.` It suffices to show that if a point z_0 is singular, then the point $(z_0, 0)^T$ belongs to a plane $E_0 \subset \mathbb{R}^{d+1}$, which is an affine hull of the points $(z_i, r_i)^T,\ i = 1, \dots, d$. Note that $\dim E_0 = d-1$. For a sphere $S(z,r)$ that passes through the point z_0 we have $r^2 = |z|^2 - 2(z_0, z) + |z_0|^2$. Subtracting this equation from each equation of (2) we obtain the system

$$(z_i - z_0, z) + r_i r = \frac{1}{2}\big(|z_i|^2 - r_i^2 - |z_0|^2 \big), \qquad i = 1, \dots, d. \tag{4}$$

If $\big(z_0, 0\big)^T \notin E_0$, then the matrix of this system has full rank d. Therefore its solutions, i.e., points $(z,r)^T$, form a straight line $l \in \mathbb{R}^{d+1}$. Substituting the solutions in system (2) we get a quadratic equation. If all its coefficients vanish, then all the spheres corresponding to the solutions $(z,r)^T$ touch the sphere S_1. Hence by Lemma 2 they constitute a pencil of tangent spheres. Then all the spheres S_i belong to this pencil, therefore all the points $(z_i, r_i)^T$ lie on a line, which contradicts to their general position. Thus, the obtained quadratic equation is nontrivial and has at most two solutions. So, there are at most two spheres from $\mathcal{M}$ passing through z_0. Whence, if a point z_0 does not belong to a plane from $\mathcal{M}$, then it is nonsingular. If it belongs to a plane from $\mathcal{M}$, then by (3) it satisfies the equations $(z_i - z_0, n) - r_i = 0,\ i = 1, \dots, d$. This system has full rank d and so it has at most one

solution n. Comparing this system with (4) we conclude that the line l is parallel to the vector $(n,1)^T \in \mathbb{R}^{d+1}$. Substituting the solutions $(z,r) \in l$ in the first equation of (2) and taking into account that $|n| = 1$ we get a linear equation (quadratic terms disappear), which has at most one solution. Thus, there is a unique plane and at most one sphere from $\mathcal{M}$ passing through z_0.

□

Thus, if a circle δ does not intersect the $(d-2)$-dimensional plane E, then it does not contain singular points. For an arbitrary point $z \in \delta$ there are at most two spheres from $\mathcal{M}$ passing through z. If the circle δ is not contained in any of these two spheres, then it does not lie on any sphere of $\mathcal{M}$. Thus, we see that *in the general position a set of d spheres and one circle in $\mathbb{R}^d$ indeed satisfies the assumptions of Theorem 3.* Now we can prove the theorem.

`Proof of Theorem 3`. It suffices to consider the case of d spheres in general position. We show that the family $\mathcal{M}$ can be defined by relations (1) and then apply Theorem 1. The proof will be realized for spheres of $\mathcal{M}$, for planes (if they exist) it will follow by the limit passage. Assume first that among the spheres S_i there are two ones of different radii (regarding the sign). Subtracting the first equation of system (2) from the others we obtain $d-1$ linear equations $2(z_1-z_i,z)+2(r_1-r_i)r = \alpha_i,\ i = 2,\dots,d$, where α_i are some constants. For at least one of them the coefficient $r_1 - r_i$ does not vanish. Expressing $r = r(z)$ from that equation and substituting it in the other equations, we get a system of $d-2$ linear equations (of rank $d-2$) for the variable z. Its solutions z form a two-dimensional affine plane $L_0 \subset \mathbb{R}^d$. Since the function $r(z)$ is linear, the points $\big(z, r(z)\big)^T$ form a two-dimensional affine plane $L \subset \mathbb{R}^{d+1}$. Substituting $r(z)$ in the first equation of system (2) we obtain an equation for z, which is either quadratic or linear. If all points of L_0 satisfy it, then all the spheres associated to the points $(z,r)^T \in L$ touch the sphere S_1, which is impossible (Lemma 2). Therefore points $z \in L_0$ satisfying that equation form a plain quadric (or a line) $\Gamma \subset L_0$. Now substitute $r(z)$ in the right hand side of the first equation (2) and obtain $r^2 = |z|^2 - (a,z) + c$, where $a \in \mathbb{R}^d$ and $c \in \mathbb{R}$. Extracting the perfect square we arrive at (1). Finally, if the radii r_i are all the same, then the centers of the spheres of $\mathcal{M}$ lie on a straight line Γ, and one can easily write equation (1).

□

VII. Elementary proof of Theorem D

Now we come back to the planar Emch theorem (Theorem D). Among the four classical closing theorems it is the strongest one: three others are actually its special cases. Indeed, if the circle α_1 degenerates into a point, then an inversion with the center at this point yields Theorem A for two circles. If α_0 lies within α_1, and the circle δ is orthogonal to all circles of the family $\mathcal{M}_1$, then we obtain Theorem B. We use the fact that the center h_1 of homothety of the circles α_0 and α_1 has the same power p_1 with respect to all circles of $\mathcal{M}_1$ (see section III). Therefore the circle δ of radius $\sqrt{p_1}$ centered at h_1 is orthogonal to all circles of $\mathcal{M}_1$ and contains all their points of tangency. Hence, Theorem B follows from Theorem D. Finally, if the circles α_0 and α_1 are concentric, then we arrive at Theorem C for the case

when the circles δ and s are on one plane. Indeed, if we take as α_0, α_1 the circles of radii $|r \pm \rho|$ concentric to s (r is the radius of s), then we obtain Theorem C.

Thus, for concentric circles α_0, α_1 Theorem D becomes the "planar" version of the zigzag theorem; in case when α_0 is the point at infinity, we obtain the Poncelet theorem for two circles; finally, if α_0 is inside α_1, and δ is orthogonal to all circles of the family $\mathcal{M}_1$, then we get the Steiner theorem.

The question arises if it is possible to give a proof of Theorem D that will be elementary and autonomous (not relying on the Poncelet theorem, in contrast to the proofs of Theorems 2 and 3). In this section we give such a proof using only tools of elementary geometry, the most complicated of which are inversions and pencils of circles. To avoid technical difficulties we restrict ourselves to the following case of mutual position of circles in Theorem D:

(d) the circle δ is inside α_1, the circle α_0 is inside δ, and we consider the family $\mathcal{M}_1$ of circles touching α_1 from inside and α_0 from outside.

The idea of the proof is the following. Let a chain of circles inscribed in the annulus bounded by α_0 and α_1 intersect the circle δ in successive points $D_1, D_2, \ldots$ and another chain of circles inscribed in the annulus intersect δ in points $D'_1, D'_2, \ldots$. For any $i = 1, 2, \ldots$ draw a new circle s_i tangent to α_0 through the points D_i and D'_i. It appears that all the new circles s_i, $i = 1, 2, \ldots$ touch some circle c

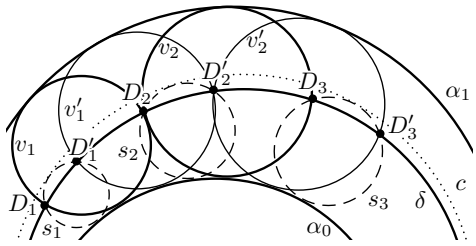

Figure 8. Proof of Theorem D

Moreover, c, δ and α_1 belong to one pencil of circles. This implies that if the first chain cycles ($D_{n+1} = D_1$), then the second also does ($D'_{n+1} = D'_1$). The idea is quite clear, but to realize it we need to establish several auxiliary results. All of them are elementary, but quite technical. Some of these results, such as Theorem 4 and Proposition 1 are, probably, of some independent interest.

We begin with a simple auxiliary fact. In the sequel we assume that an arc AB of a circle has the positive direction from A to B, $\overset{\smile}{AB}$ denotes the angle defined by that arc.

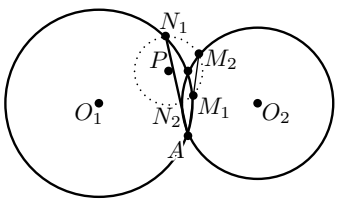

Figure 9. Lemma 4

Lemma 4 *Two circles of radii r_1, r_2 centered at O_1, O_2 intersect at points A and B. Suppose P is the fourth vertex of the parallelogram O_2AO_1P; then for any circle centered at P intersecting the first circle at some points M_1, N_1, and the second one at points M_2, N_2 (figure 9) the following hold:*

a) the lines M_1M_2 and N_1N_2 pass through A;

b) $\frac{M_1N_1}{M_2N_2} = \frac{AN_1}{AM_2} = \frac{AM_1}{AN_2} = \frac{BM_1}{BM_2} = \frac{BN_1}{BN_2} = \frac{r_1}{r_2}$.

`Proof.` a) The triangles PO_1M_1 and M_2O_2P are equal by three equal sides, hence $\angle M_1O_1P = \angle M_2O_2P$. Furthermore, $\angle PO_1B = \angle PO_2B$, since O_2O_1PB is an equilateral trapezium. Subtracting the second equality from the first one, we obtain $\angle M_1O_1B = \angle M_2O_2B$, therefore $\overset{\smile}{BM_1}=\overset{\smile}{BM_2}$. Consequently, $\angle M_1AB = \angle M_2AB$, and thus the line M_1M_2 passes through A. The proof for N_1N_2 is the same.

b) It follows from (a) that the chords M_1N_1 and M_2N_2 define equal angles on the two circles: $\angle M_1AN_1 = \angle M_2AN_2$. Whence $\frac{M_1N_1}{M_2N_2} = \frac{r_1}{r_2}$. Similarly, $\frac{BM_1}{BM_2} = \frac{BN_1}{BN_2} = \frac{r_1}{r_2}$. Finally, since the quadrangle $M_1M_2N_1N_2$ is inscribed in a circle, it follows that the triangles M_1AN_1 and N_2AM_2 are similar with the factor $\frac{M_1N_1}{M_2N_2} = \frac{r_1}{r_2}$. Therefore $\frac{AN_1}{AM_2} = \frac{AM_1}{AN_2} = \frac{r_1}{r_2}$.

□

Before formulating the crucial auxiliary fact, Theorem 4, let us recall that a *pencil of circles* is a set of circles on the plane orthogonal to two different circles (all circles may degenerate into lines and points, unless the opposite is stated). Circular pencils are straight lines in the three-dimensional space of circles. Any pair of circles $\mathbf{b}, \mathbf{c} \subset \mathbb{R}^2$ is contained in a unique pencil that will be denoted by $\mathcal{P}\{\mathbf{b}, \mathbf{c}\}$. For every $t \in \mathbb{R} \cup \{\infty\}$ the set of points on the plane, for which the ratio of powers with respect to given circles $\mathbf{b}$ and $\mathbf{c}$ equals to t is either empty or a circle of the pencil $\mathcal{P}\{\mathbf{b}, \mathbf{c}\}$.

Now we are going to establish the main theorem. Take two circles with a common center P, we call them the bigger circle and the smaller one. Consider the families of circles $\mathcal{M}_j$, $j = 0, 1$ touching them. Take an arbitrary pair $\beta_0, \beta_1 \in \mathcal{M}_0$ and a pair $\gamma_0, \gamma_1 \in \mathcal{M}_1$. The points of intersection of β_i and γ_k will be denoted by A^0_{ik} and A^1_{ik} (the first point is farther from the center P then the second one). Finally, draw one more circle centered at P. Let it meet each of the circles β_i and γ_k at two points b^s_i, $s = 0, 1$ (respectively c^s_k). If a point goes around the circle β_i counterclockwise from its point of tangency with the bigger circle, then it meets the point b^0_i first and b^1_i second. The same with the points c^s_k (figure 10). In the notation of Theorem 4 all superscripts are taken modulo 2, for example, $A^2_{10} = A^0_{10}$, $c^3_1 = c^1_1$.

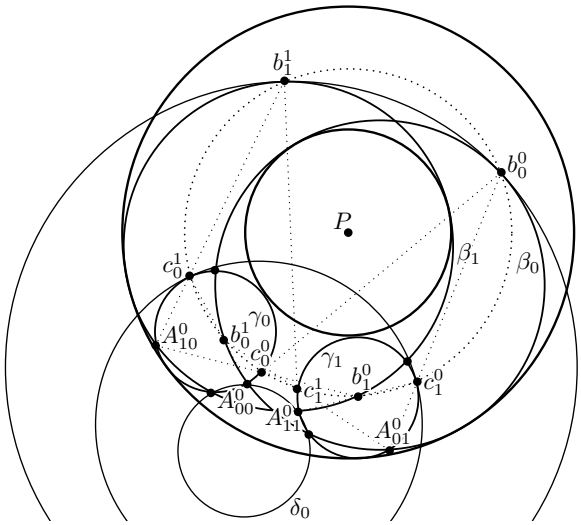

Figure 10. Theorem 4

Theorem 4 *Given two circles with a common center P and arbitrary circles $\beta_i \in \mathcal{M}_0, \gamma_k \in \mathcal{M}_1,\ i,k=0,1$. Suppose A_{ik}^s are the corresponding 8 points of their intersection; then*

a) 4 points $A_{ik}^{i+k},\ i,k \in \{0,1\}$ lie on one circle (denote it by δ_0).The same holds for 4 points $A_{ik}^{i+k+1},\ i,k \in \{0,1\}$ (a circle δ_1).

b) Let an arbitrary circle centered at P meet β_i, γ_k at points b_i^s, c_k^s respectively. Then for every $i,k \in \{0,1\}$ the lines $b_i^0 c_k^0$ and $b_i^1 c_k^1$ pass through the point A_{ik}^0.

c) Choose arbitrary $q,j \in \{0,1\}$ and draw a circle touching the circles β_i at the points $b_i^{i+q},\ i=0,1$ and a circle touching the circles γ_k at the points $c_k^{k+q+j+1},\ k=0,1$. Then these two circles belong to one pencil with δ_j.

Thus, if four circles $\beta_i, \gamma_k,\ i,k \in \{0,1\}$ touch two concentric circles, then the eight points of their intersection A_{ik}^s are naturally split into two quadruples, each of them lies on one circle. Drawing a third concentric circle we get the 8 additional points of intersection b_i^s, c_k^s that form 8 triples of collinear points (figure 10). Finally, if we consider two pairs of circles touching β_i, γ_k at the points of their intersection with the third circle, then we obtain 4 triples of circles, each of them is in one pencil. So this configuration of seven circles produces two circles, 8 lines and 4 pencils.

`Proof of Theorem 4`. Let R and r be radii of the circles β_0 and γ_0 respectively, and O_1, O_2 be their centers. Then $O_2 A_{00}^0 O_1 P$ is a parallelogram with sides of lengths r and R. By Lemma 4 the lines $b_0^0 c_0^0$ and $b_0^1 c_0^1$ concur at A_{00}^0. This proves item b) for $i=k=0$, the proof for other indices is the same.

Now draw a circle $\mathbf{b}$ touching β_0 and β_1 at the points b_0^0 and b_1^1 respectively, and a circle $\mathbf{c}$ touching γ_0 and γ_1 at the points c_0^1 and c_1^0. We assume that both $\mathbf{b}$ and $\mathbf{c}$ do not degenerate into lines. Denote by x the radius of the circle $\mathbf{b}$ taken with the sign: it is positive if this circle touches β_0 from outside, and negative otherwise. Similarly y is the radius of $\mathbf{c}$ with the sign. Denote also by B the second point of intersection of the line $b_0^0 A_{00}^0$ with the circle $\mathbf{b}$, and by C the second point of intersection of the line $c_0^1 A_{00}^0$ with the circle $\mathbf{c}$. The similarity

of circles implies that $b_0^0 B = \frac{x}{R} b_0^0 A_{00}^0$. Therefore the power of the point A_{00}^0 with respect to the circle $\mathbf{b}$ is $A_{00}^0 b_0^0 \cdot A_{00}^0 B = \big(1+\frac{x}{R}\big)\big(A_{00}^0 b_0^0\big)^2$. Similarly, the power of the point A_{00}^0 with respect to $\mathbf{c}$ is $\big(1+\frac{y}{r}\big)\big(A_{00}^0 c_0^1\big)^2$. Lemma 4 yields $A_{00}^0 b_0^0 / A_{00}^0 c_0^1 = R/r$. Therefore, the ratio of the powers of the point A_{00}^0 with respect to the circles $\mathbf{b}$ and $\mathbf{c}$ is equal to $\frac{(R+x)R}{(r+y)r}$. In the same way we obtain that for each of the points $A_{11}^0, A_{10}^1, A_{01}^1$ the ratio of powers with respect to $\mathbf{b}$ and $\mathbf{c}$ also equals to $\frac{(R+x)R}{(r+y)r}$. Whence, these 4 points are on one circle (δ_0) that belongs to the pencil $\mathcal{P}\{\mathbf{b},\mathbf{c}\}$. This proves items a) and c) for $j\,,q=0$. The proof for other $j\,,q$ is the same.

□

Corollary 2 *Under the assumptions of Theorem 4 for any $j=0,1$ the following hold: for an arbitrary circle $\mathbf{b}$ touching the circles β_0 and β_1 in the same way (both from inside or both from outside) there exists a circle $\mathbf{c}\in\mathcal{P}\{\delta_j,\mathbf{b}\}$ touching γ_0 and γ_1 in the same way.*

Remark 2 Through any point A of the circle γ_0 (different from points of its tangency with the bigger circle and with the smaller one) one can draw two circles of the family $\mathcal{M}_0$. Precisely for one of them A is the closest to P point of intersection with γ_0.

Theorem 4 b) implies, in particular, that *for any $i,k\in\{0,1\}$ the line joining the points of tangency of β_i,γ_k with the bigger circle passes through the point A_{ik}^0.* The same is true for the smaller circle. We need the converse, whose proof will be a simple exercise for the reader.

Lemma 5 *Suppose an arbitrary straight line passing through the point of tangency of given circles α and β meet these circles at points A and B respectively. The circle s passes through B and touches α at the point A. Then all such circles s touch a fixed circle different from α. This circle touches β and is concentric to α.*

Now we are formulating the assertion, from which Theorem D follows immediately. Consider arbitrary circles α_0,α_1 and δ satisfying condition **(d)**. Take two series of circles $\{v_k\},\{v_k'\}\subset\mathcal{M}_1$ and the corresponding points $\{D_k\}$ and $\{D_k'\}$ on the circle δ. We assume that these sequences both go around δ in positive direction and that D_1' belongs to the arc D_1D_2. Denote by s_k the circle passing through the points D_k, D_k' and tangent to α_0 from outside.

Proposition 1 *All the circles $s_k, k\in\mathbb{N}$ touch a fixed circle of the pencil $\mathcal{P}\{\delta,\alpha_1\}$.*

`Proof` consists of the consecutive application of Theorem 4 to the pairs of circles v_k, v_k' and s_k, s_{k+1} for all $k\geq 1$. A suitable inversion maps the pairs v_1, v_1' and s_1, s_2 to the pairs β_0,β_1 and γ_0,γ_1 from Theorem 4. To see this we make an inversion with center at the second point of intersection of the circles MBK and MCL (the first point of their intersection is M), where K,M,L are the points of tangency of the circle α_0 with s_1,s_2,v_1 respectively, and B and C are the second points of intersection of the circle v_1 with the circles s_1 and s_2 respectively. By Lemma 5 the images of the circles s_1,s_2,v_1 touche two concentric circles, one of them is the image of α_0. In order to be defined we assume that the image of α_0 is the smaller concentric circles and that the images of s_1,s_2 are situated between these concentric circles.

Thus, s_1, s_2, v_1 are mapped to the circles γ_0, γ_1 β_0, and the points D_1 and D_2 are mapped to A^1_{00} and A^0_{01} respectively. Let X_1, X_2 be images of the points D'_1, D'_2. Draw a circle β_1 through X_2 that touches both these concentric circles from within so that X_2 is the closest (to the center) point of intersection of γ_1 and β_1 (Remark 2). Thus, $X_2 = A^1_{11}$. By Theorem 4 a) the points $A^1_{00}, A^0_{01}, A^1_{11}, A^0_{10}$ lie on a circle. By the assumptions $A^1_{00}, A^0_{01}, A^1_{11}, X_1$ are on a circle as well. Whence $A^0_{10} = X_1$, and β_1 is the image of v'_1.

Thus, an inversion takes circles v_1, v'_1, s_1, s_2 to the circles $\beta_0, \beta_1, \gamma_0, \gamma_1$ respectively. According to Corollary 2 the pencil $\mathcal{P}\{\delta, \alpha_1\}$ contains some circle $\mathbf{c}$ touching the circles s_1, s_2 in the same way. It is located between the circles δ and α_1. Indeed, if we fix the points D_1, D_2 and move the point D'_1 along the arc D_1D_2, then at the extreme positions we have: $\mathbf{c} = \delta$ in case $D'_1 = D_1$, and $\mathbf{c} = \alpha_1$ in case $D'_1 = D_2$. The circle $\mathbf{c}$ changes continuously in the same pencil $\mathcal{P}\{\delta, \alpha_1\}$ when D'_1 moves. Hence if for some D'_1 the circle $\mathbf{c}$ is not between δ and α_1, then for some interior point D'_1 of the arc D_1D_2 one has either $\mathbf{c} = \delta$ or $\mathbf{c} = \alpha_1$. Neither of these cases is possible, because for any interior point of the arc D_1D_2 the circle s_1 touches neither δ nor α_1.

Taking the next pairs of circles v_2, v'_2 and s_2, s_3, we obtain a circle that belongs to the same pencil $\mathcal{P}\{\delta, \alpha_1\}$, lies between δ and α_1, and touches both s_2 and s_3. This circle coincides with $\mathbf{c}$ because the pencil $\mathcal{P}\{\delta, \alpha_1\}$ has at most one circle between δ and α_1 that touches s_2. Hence $\mathbf{c}$ touches s_3. Thus we consecutively prove that $\mathbf{c}$ is tangent to all the circles s_k.

□

`Proof of Theorem D in case (d).` Let $v_1, \ldots, v_n$ be a periodic series of circles ($v_{n+1} = v_1$), $D_1, \ldots, D_n$ be the corresponding points on δ. Take an arbitrary series $v'_1, v'_2, \ldots$ and the corresponding sequence of points $D'_1, D'_2, \ldots$. With possible renumbering it may be assumed that these sequences both go around the circle δ in positive direction and that the point D'_1 is located on the arc D_1D_2. Consider the circles s_k, $k \geq 1$ from Proposition 1 and conclude that they are all tangent to a circle $\mathbf{c} \in \mathcal{P}\{\delta, \alpha_1\}$. The arc D_1D_2 has only one point D'_1, for which the circle passing through the points D'_1 and $D_{n+1} = D_1$ and touching α_0 from outside is tangent to $\mathbf{c}$. Therefore $D'_{n+1} = D'_1$ and $s_{n+1} = s_1$, which completes the proof.

□

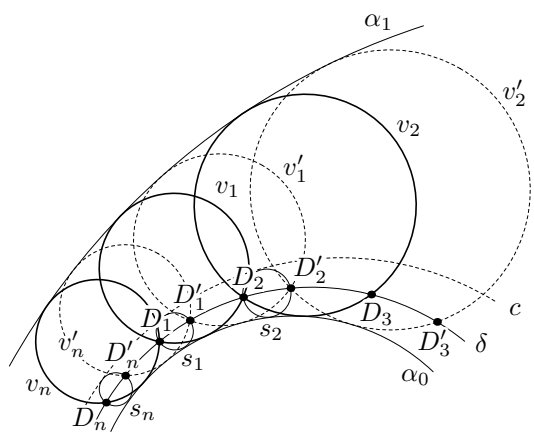

Figure 11. Proof of Theorem D

VIII. Generalization of Theorem D

The method developed in the previous section makes it possible to go a bit further and to obtain a generalization of the Emch theorem for several pencils of circles analogous to the great Poncelet theorem [14, theorem 16.6.7]. We formulate it only for one case of mutual position of circles. Let us have a circle δ and two sequences of circles $\{\alpha_0^k\}$ and $\{\alpha_1^k\}$. Each sequence $\{\alpha_i^k\}$ is contained in a pencil $\mathcal{A}_i$, $i = 0, 1$ that also contains the circle δ. We assume that the circles $\{\alpha_0^k\}$ are all inside δ and that δ is inside all the circles $\{\alpha_1^k\}$. By $\mathcal{M}_1^k$ we denote the families of circles tangent to α_1^k and α_0^k with index 1 (touching α_0^k from without and α_1^k from within).

We choose an arbitrary point $D_1 \in \delta$ and draw a circle $v_1 \in \mathcal{M}_1^1$ through it. Then take the second point D_2 of intersection of v_1 with the circle δ, draw a circle $v_2 \in \mathcal{M}_1^2$ through it, etc. The circles v_k are chosen in each iteration so that the sequence $\{D_k\}$ goes along δ in positive direction.

Theorem 5 *If the process starting at some point D_1 has period $n \geq 3$ and all the points $D_1, \ldots, D_n$ are different, then it has the same period for any initial point.*

The proof is literally the same as the proof of Theorem D and is based on the following analog of Proposition 1. Given two series of circles $\{v_k\}$, $\{v'_k\}$, where $v_k, v'_k \in \mathcal{M}_1^k$ $k \in \mathbb{N}$ and the corresponding sequences of points $\{D_k\}$, $\{D'_k\}$ on δ. Assume that D'_1 is located on the arc D_1D_2. Choose an arbitrary circle $\alpha_0 \in \mathcal{A}_0$ lying inside δ and denote by s_k the circle passing through the points D_k, D'_k and touching α_0 from outside. Then all the circles s_k, $k \in \mathbb{N}$ are tangent to one circle of the pencil $\mathcal{A}_0$. The proof of this fact is realized in the same way as for Proposition 1 and we leave it to the reader.

Acknowledgement. The author is grateful to the anonymous Referee for valuable comments and remarks, and to A.Akopyan and E.Gorskaya for their help in making illustrations.

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