VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. First day. 8th form. Solutions

1. (A.Blinkov) Let M be the midpoint of the base AC of an acute-angled isosceles triangle ABC. Let N be the reflection of M in BC. The line parallel to AC and passing through N meets AB at point K. Determine the value of $\angle AKC$.

   Answer. 90°.

   Solution. Let L be the common point of NK and BC (see Fig. 8.1). By means of the symmetry in BC we obtain $AM = MC = CN$ and $\angle MCB = \angle NCB$. Next, since $LN \parallel AC$, we have $\angle CNL = \angle LCM$, hence the triangle CNL is isosceles, and $LN = CN = AM$. Thus, the segments AM and LN are parallel and equal, hence the quadrilateral $ALNM$ is a parallelogram, and $AL \parallel MN \perp LC$. Finally, by the symmetry in BM we get $\angle AKC = \angle ALC = 90^\circ$.

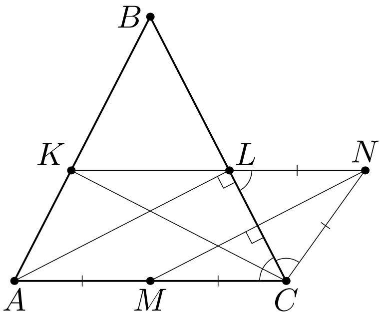

Figure 8.1

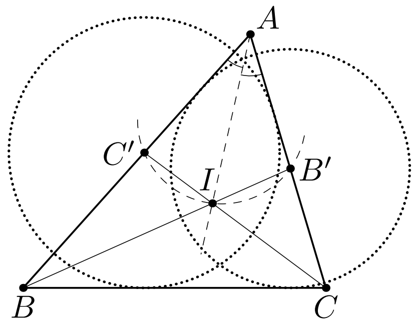

Figure 8.2

2. (A.Karlyuchenko) In a triangle ABC the bisectors BB' and CC' are drawn. After that, the whole picture except the points A, B', and C' is erased. Restore the triangle using a compass and a ruler.

   First solution. Let I be the incenter of triangle ABC. Then $\angle B'IC' = 180^\circ - (\angle IBC + \angle ICB) = 90^\circ + \frac{1}{2}\angle B'AC'$. So, denoting by O the center of the circumcircle ω of triangle BCI, we get $\angle B'OC' = 180^\circ - \angle B'AC'$. Hence one can successively reconstruct points O and I (the latter is the meeting point of the smaller arc $B'C'$ of ω with the bisector of $\angle B'AC'$, see Fig. 8.2). Finally, the points B and C can be reconstructed and the meeting points of $B'I$, AC' and $C'I$, AB' respectively.

   Second solution. Since BB' is the bisector of $\angle B$, the point B' is equidistant from the lines BC and AB. Hence the circle with center B' tangent to AC' is also tangent to BC. Analogously, line BC is tangent to the circle with center C' touching AB' (see Fig. 8.2). So, to reconstruct the points B and C, it is sufficient to draw a common outer tangent to these two circles (sharing different sides of $B'C'$ with A) and to find its common points with AB' and AC'.

3. (L.Steingarts) A paper square was bent by a line in such way that one vertex came to a side not containing this vertex. Three circles are inscribed into three obtained triangles (see Figure). Prove that one of their radii is equal to the sum of the two remaining ones.

   Solution. Assume that square $ABCD$ is bent by line XY; denote the resulting points as in Fig. 8.3.2. Recall that in any right triangle, the incenter, the points of tangency of the incircle with the legs, and the vertex of the right angle form a square; hence the inradius is equal to the segment of the tangent line from that vertex. Hence the indiameters of triangles UDX, UAP, and PVY are $d_1 = UD + DX - XU$, $d_2 = UA + AP - UP$, and $d_3 = PV + VY - PY$ respectively. Denote $a = AB$ and notice that $UX = XC$ and $VY = YB$; therefore we obtain

$$d_1+d_3-d_2 = DU+(a-CX)-CX+PV+BY-PY-(a-DU)-(a-PY-BY)+(a-PV) = \\ = 2(DU + BY - CX).$$

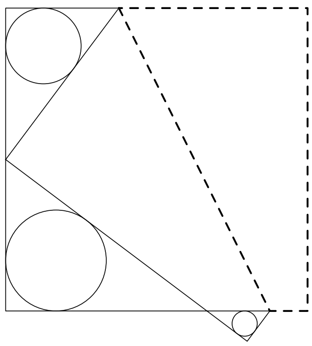

Figure 8.3.1

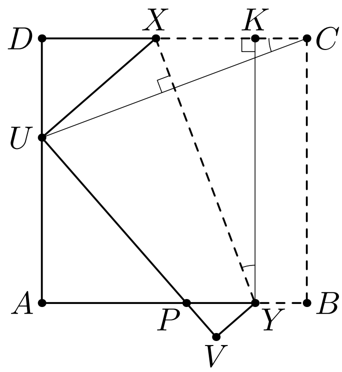

Figure 8.3.2

Let K be the projection of Y onto CD. The points C and U are symmetrical with respect to XY, hence $XY \perp CU$, and $\angle DCU = \angle KYX$. Moreover, $KY = CD = a$. Consequently, the right triangles CDU and YKX are congruent, hence $DU = KX = CX - CK = CX - BY$. This means exactly that $d_1 + d_3 - 2_d = 0$.

Remark. In the first part of the solution, one may also argue as follows. The right triangles DXY, VYP, and AUP are similar; hence the ratios of their inradii are the same as the ratios of their respective legs. Hence it suffices to prove the equality $DX + VY = AU$, or, equivalently, $DX + CK = a - DU$. The last relation follows from the relation $DU = KX$ which is proved in the second part of the Solution.

4. (A.Akopyan, D.Shvetsov) Let ABC be an isosceles triangle with $\angle B = 120^\circ$. Points P and Q are chosen on the prolongations of segments AB and CB beyond point B so that the rays AQ and CP intersect and are perpendicular to each other. Prove that $\angle PQB = 2\angle PCQ$.

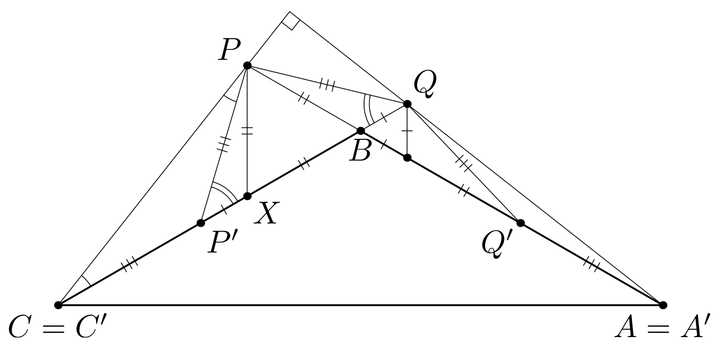

Figure 8.4

Solution. Let us choose points X and Q' on ray BC so that $BX = BP$ and $BQ' = BP + BQ$. Then triangle BPX is isosceles, and one of its angles is equal 60°; hence it is equilateral, so $PX = BP$ and $\angle PXQ' = 120°$. Then triangles PBQ and PXQ' are congruent by SAS, hence $PQ' = PQ$ and $\angle PQ'B = \angle PQB$. Analogously, choosing point P' on ray BA so that $BP' = BP + BQ$, we get $QP' = QP$ and $\angle QP'B = \angle QPB$ (see Fig. 8.4).

Now let us prolongate segments BP' and BQ' beyond the points P' and Q' by the length $Q'A' = P'C' = PQ$. Then we have $BA' = BP' + P'A' = BP + BQ + PQ = BQ' + Q'C' = BC'$. Now, triangles $QP'A'$ and $PQ'C'$ are isosceles, so $\angle P'A'Q + \angle Q'C'P = \frac{1}{2}(\angle QP'B + \angle PQ'B) = \frac{1}{2}(\angle BPQ + \angle BQP) = 30°$. Hence the angle formed by the lines QA' and PC' is equal to $180° - (\angle P'A'Q + \angle Q'C'P + \angle BA'C' + \angle BC'A') = 90°$. But, if $BA' = BC' < BA$, then this angle should be less than $90°$, and if $BA' > BA$, then it should be greater than $90°$, So we obtain $A' = A$, $C' = C$, and $\angle PQB = \angle PQ'B = 2\angle PCQ'$, QED.

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Final round. Second day. 8th form. Solutions

5. (A.Akopyan) Do there exist a convex quadrilateral and a point P inside it such that the sum of distances from P to the vertices of the quadrilateral is greater than its perimeter?

Answer. Yes.

Solution. Consider a quadrilateral $ABCD$ such that $AD = BD = CD = x$, $AB = BC = y < x/4$, and a point P on the diagonal BD such that $PD = y$ (see Fig. 8.5). Then we have $PB + PD = BD = x$ and $PA = PC > AD - PD = x - y$, hence $PA + PB + PC + PD > 3x - 2y > 2x + 2y = AB + BC + CD + DA$.

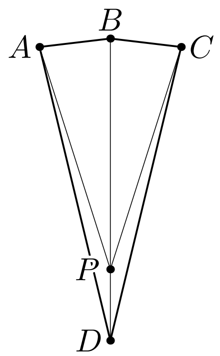

Figure 8.5

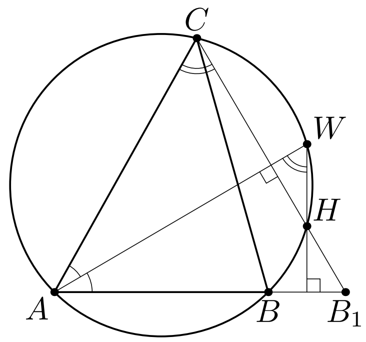

Figure 8.6

6. (A.Tumanyan) Let ω be the circumcircle of triangle ABC. A point B_1 is chosen on the prolongation of side AB beyond point B so that $AB_1 = AC$. The angle bisector of $\angle BAC$ meets ω again at point W. Prove that the orthocenter of triangle AWB_1 lies on ω.

Solution. Let H be the second point of intersection of line CB_1 with ω. Since AW is an angle bisector in the isosceles triangle AB_1C, we have $B_1H \perp AW$. If the points C and W share a common side of AH, then $\angle AWH = \angle ACH = 90° - \angle CAW = 90° - \angle WAB$, which implies $WH \perp AB_1$ (see Fig. 8.6). If they share different sides, then $\angle AWH = 180° - \angle ACH = 90° + \angle WAB$, which again follows $WH \perp AB_1$. Finally, if these points coincide, then triangle AWB_1 is right-angled, and $H = W$ is its orthocenter.

Thus, in any case point H lies on two altitudes of triangle AWB_1; hence H is its orthocenter.

7. (D.Shvetsov) The altitudes AA_1 and CC_1 of an acute-angled triangle ABC meet at point H. Point Q is the reflection of the midpoint of AC in line AA_1; point P is the midpoint of segment A_1C_1. Prove that $\angle QPH = 90°$.

First solution. Let K be the midpoint of AC. Since $KQ \parallel BC$, the line KQ bisects the altitude AA_1. So, the diagonals of quadrilateral AKA_1Q bisect each other and are perpendicular to each other, thus this quadrilateral is a rhombus. Moreover, from the symmetry we have $HQ = HK$.

Analogously, let R be the reflection of K in CC_1; then CKC_1R is a rhombus, and $HQ = HR$ (see Fig. 8.7.1). So the segments A_1Q, AK, KC, and C_1R are parallel and equal, hence QA_1RC_1 is a parallelogram, and P is a midpoint of RQ. Consequently, HP is a median, and thus an altitude, in the isosceles triangle HQR, QED.

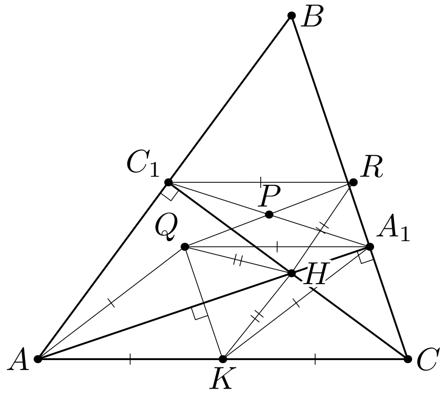

Figure 8.7.1

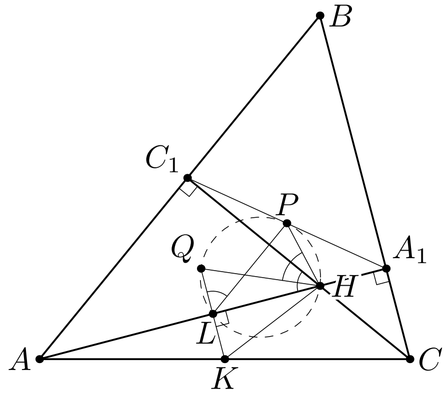

Figure 8.7.2

Second solution. Let L be the midpoint of AA_1. Then PL is a midline in the triangle AA_1C_1, so $\angle PLH = \angle BAA_1$, and hence $\angle PLQ = 90^\circ - \angle PLH = \angle C_1HA$. On the other hand, points A_1 and C_1 lie on the circle with diameter AC; therefore the triangles A_1C_1H and CAH are similar; hence the two angles $\angle PHC_1$ and $\angle KHA$ formed by their respective sides and medians are equal. Thus, $\angle QHA = \angle KHA = \angle PHC_1$, therefore $\angle PHQ = \angle C_1HA$ (see Fig. 8.7.2). So, $\angle PHQ = \angle C_1HA = \angle PLQ$, which implies that the points P, Q, L, and H are concyclic, and $\angle QPH = 180^\circ - \angle QLH = 90^\circ$.

8. (A.Zaslavsky) A square is divided into several (greater than one) convex polygons with mutually different numbers of sides. Prove that one of these polygons is a triangle.

Solution. Suppose that a square is cut into n polygons. Then each of these polygons has at most one side lying on each side of the square; next, it shares at most one side with any of the other polygons. So the total number of its edges is at most $4 + (n-1) = n+3$.

Thus, the number of sides of any polygon lies in the interval $[3, n+3]$. If none of them is a triangle, then the numbers of sides should be equal to $4, 5, \dots, n+3$. So, there exists an $n+3$-gon, and it should share a segment with every side of the square. Therefore, any other polygon can share the segments with at most two sides of the square, and the number of its sides is at most $2 + (n-1) = n+1$. Hence there is no $(n+2)$-gon; a contradiction.

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. First day. 9th form. Solutions

1. (L.Steingarts) The altitudes AA_1 and BB_1 of an acute-angled triangle ABC meet at point O. Let A_1A_2 and B_1B_2 be the altitudes of triangles OBA_1 and OAB_1 respectively. Prove that A_2B_2 is parallel to AB.

   Solution. By $\angle CAA_1 = 90° - \angle ACB = \angle CBB_1$, the right triangles OA_1B and OB_1A are similar (see Fig. 9.1). So, their altitudes A_1A_2 and B_1B_2 divide the sides OB and OA in the same ratio. This exactly means that $A_2B_2 \parallel AB$.

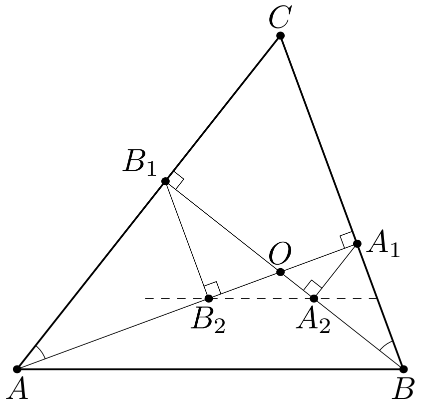

Figure 9.1

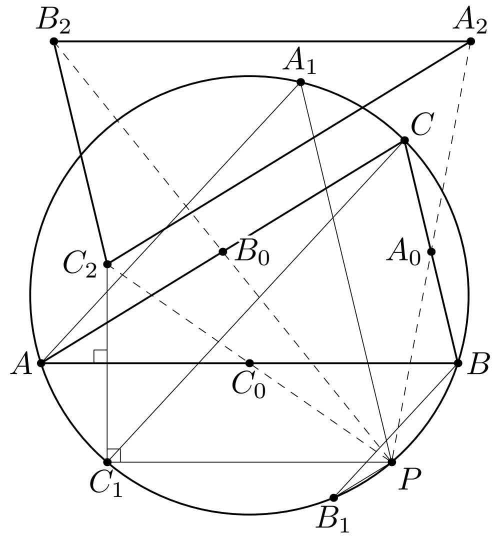

Figure 9.2

2. (D.Shvetsov, A.Zaslavsky) Three parallel lines passing through the vertices A, B, and C of triangle ABC meet its circumcircle again at points A_1, B_1, and C_1 respectively. Points A_2, B_2, and C_2 are the reflections of points A_1, B_1, and C_1 in BC, CA, and AB respectively. Prove that the lines AA_2, BB_2, CC_2 are concurrent.

   Solution. Let a, b, and c be the lines drawn through the points A_1, B_1 and C_1 and parallel to BC, CA, and AB respectively. We claim that these lines are concurrent, and their concurrency point lies on the circumcircle of ABC. Let c intersect the circumcircle at C_1 and P (if c is tangent to the circumcircle, then $P = C_1$). Then from $AB \parallel C_1P$ and $AA_1 \parallel CC_1$ we obtain $\smile BP = \smile C_1A = \smile A_1C$ (here, by $\smile XY$ we denote the measure of the arc passing from X to$Y$ clockwise). This means exactly that $A_1P \parallel BC$, hence a passes through P. Analogously, b also passes through P (see Fig. 9.2).

   Next, the points C_1 and P are symmetrical in the perpendicular bisector of AB, while the points C_1 and C_2 are symmetrical in AB; this implies that the points P and C_2 are symmetrical about the midpoint C_0 of the segment AB. Analogously, the points A_2 and B_2 are symmetrical to P about the midpoints A_0 and B_0 of the other two sides of ABC. Thus, $\overrightarrow{A_2B_2} = 2\overrightarrow{A_0B_0} = -\overrightarrow{AB}$ and analogously $\overrightarrow{A_2C_2} = -\overrightarrow{AC}$, $\overrightarrow{B_2C_2} = -\overrightarrow{BC}$. Therefore the triangles ABC and $A_2B_2C_2$ are centrally symmetric to each other, and the lines connecting their respective vertices are concurrent at the symmetry center.

3. (V.Protasov) In triangle ABC, the bisector CL was drawn. The incircles of triangles CAL and CBL touch AB at points M and N respectively. Points M and N are marked on the picture, and then the whole picture except the points A, L, M, and N is erased. Restore the triangle using a compass and a ruler.

First solution. Let K be the tangency point of the incircle of ABC with the side AB (clearly, point K lies on the segment MN). Notice that

$$MK = AK - AM = \frac{AB + AC - BC}{2} - \frac{AL + AC - LC}{2} = \frac{BL + LC - BC}{2} = LN.$$

Next, let I_a, I_b, and I be the centers of the incircles ω_a, ω_b, and ω of triangles ACL, BCL, and ABC respectively. By the angle bisector property, we get $\frac{AL}{IL} = \frac{AI_a}{I_aI} = \frac{AM}{MK}$, so $IL = \frac{AL \cdot MN}{AM}$.

Now we are ready to restore the triangle. It is easy to reconstruct successively points X, I (as the meeting point of the perpendicular to MN at K and a circle with center L), then I_a and I_b (as the meeting points of the bisectors of $\angle ALI$ and $\angle CLI$ with the perpendiculars to segment MN at its endpoints), then the circles ω_a and ω_b and, finally, the points C (the intersection of the tangents to ω_a drawn from A and L) and B (as the intersection of MN with a tangent to ω_b from C).

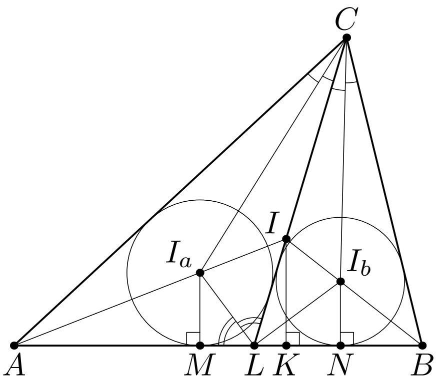

Figure 9.3

Second solution. First, let us prove the relation $1/AM + 1/ML = 1/LN + 1/NB$. Denote by $x = AC$, $y = CL$, $z = LA$ the side lengths of triangle ACL, by p, S, and r its semiperimeter, area, and inradius respectively, and by h the distance from C to line AB. Then we have $\frac{1}{AM} + \frac{1}{ML} = \frac{1}{p-y} + \frac{1}{p-x} = \frac{z}{(p-x)(p-y)}$. Next, by the Далее, $(p-x)(p-y) = \frac{S^2}{p(p-z)} = \frac{rp \cdot zh/2}{p(p-z)}$, which implies $\frac{1}{AM} + \frac{1}{ML} = \frac{2(p-z)}{rh} = \frac{2}{h\tan(\angle ACL/2)}$. Now notice that in the triangle BCL, the angle at C is the same, hence the value of $1/LN + 1/NB$ is the same.

Thus, knowing the lengths of segments AM, ML, and LN, we may reconstruct the length of NB and hence the point B. Further, from the relations $AC - CL = AM - LM$ and $BC - CL = BN - LN$ we get the difference $AC - BC = AM - LM - BN + LN = p$ of AC and BC, while the relation $AC/BC = AL/BL = q$ provides their ratio. Now it is easy to find the side lengths $AC = \frac{p}{q-1}$ and $BC = \frac{pq}{q-1}$, and to reconstruct the triangle.

4. (B.Frenkin) Determine all integer $n > 3$ for which a regular n-gon can be divided into equal triangles by several (possibly intersecting) diagonals.

Answer. All even n.

First solution. If $n = 2k$, then one may draw k main diagonals cutting the polygon into n congruent triangles.

Assume now that such cutting exists for some odd n. Consider the obtained triangles sharing a side with the initial n-gon P. All their angles opposite to these sides are equal; denote their value by α, and denote two other angles of an obtained triangle by β and γ. Two cases are possible.

Case 1. Assume that the angles β and γ are different, say, $\beta < \gamma$. We call a side of n-gon *β-side* or *γ-side* according to the angle of the triangle at its left endpoint (if observing from the center of P). Choose any β-side b, and consider the other side of angle β in the triangle adjacent to b. This side belongs to some diagonal of P, and the other endpoint of this diagonal also forms angle β with some side c of P (see Fig. 9.4.1). This angle β cannot be cut into some smaller angles, otherwise the angle adjacent to c is smaller than β; but it should be equal either to β or to $\gamma > \beta$, which is impossible.

Thus, this angle β belongs to a triangle with c as its side, and c is a γ-side; let us put b and c into correspondence. Conversely, considering angle β adjacent to any γ-side c we analogously find a β-side b corresponding to it. Thus all the sides are split into pairs, and their total number is even. A contradiction.

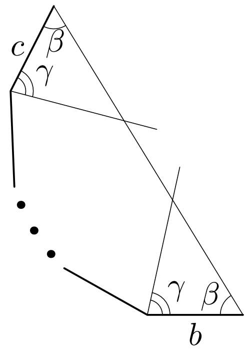

Figure 9.4.1

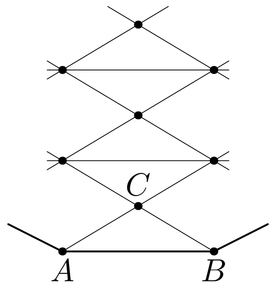

Figure 9.4.2

Case 2. Assume now that $\beta = \gamma$, and consider a triangle ABC containing the "bottom" side AB of P; its angle at C is α. The angle vertical to it is also α and belongs to some different triangle, whose side opposite to C is equal and parallel to AB. On the other side of it we find another triangle, and so on. Thus we obtain some chain of triangles (see Fig. 9.4.2); consider the last triangle UVW in this chain. If its orientation is different from that of ABC, then its top side (which is parallel to AB) is also a side of P, which is impossible for odd n (a regular n-gon has no parallel sides). Otherwise, the angle at the top vertex W is α, and W is a vertex of P. Then the sides UV and VW are (simultaneously) either sides of P or parts of its diagonals. In the first subcase we get $\alpha = \beta = \gamma = 60°$, so the angle of n-gon is $60°$, which is impossible since $n > 3$. In the second subcase, the angle of P contains (as minimum) the angle equal to α and two angles equal β (adjacent to the sides sharing the vertex W); thus $\alpha + 2\beta < 180°$. But $\alpha + 2\beta = 180°$ as the sum of three angles of a triangle; a contradiction.

Second solution. Here we present a different proof that the cutting is impossible for all odd $n > 3$.

Notice that no two diagonals are perpendicular to each other; hence for any internal meeting point of two drawn diagonals, there should exist the third diagonal passing through that point: otherwise these diagonals form two different angles which sum up to $180°$; but they should be equal to two angles of some triangle.

Next, we claim that at least two diagonals should pass through each vertex of P. Assume first that through some vertex, no diagonals are drawn. Then the triangle containing this vertex is a triangle formed by three consecutive vertices of P, and one of its angles is the angle of P. Then it is easy to see that every side of P belongs to some triangle which also contains one more side of P. Hence the sides are paired up, which is absurd.

Assume now that exactly one diagonal passes through some vertex A_i. This diagonal splits the angle at A into two different angles $\beta < \gamma$. Both these angles are adjacent to the sides of n-gon, therefore in any obtained triangle such angles are adjacent to a side equal to the side of P. Hence, the sum of all angles of triangles adjacent to the sides of P is $n(\beta + \gamma)$, which equals to the sum of all angles of P. Therefore, through each vertex passes exactly one diagonal, and the vertices are paired up, which is absurd again. The claim is proved.

Finally, assume that P is dissected into k triangles; the sum of all their angles is $180° \cdot k\pi$. The angles of P contribute $180°(n-2)$ to this sum, hence the sum of the angles at the internal points is $180°(k-n+2)$. Each such point contributes $360°$, so the number of internal points is $(k-n+2)/2$. On the other hand, each such point belongs to at least six triangles, while each vertex of P belongs to at least three of them. Hence the total number of triangles is at least $(3(k-n+2)+3n)/3 = k+2 > k$. A contradiction.

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Final round. Second day. 9th form. Solutions

5. (M.Kungozhin) Let ABC be an isosceles right-angled triangle. Point D is chosen on the prolongation of the hypothenuse AB beyond point A so that $AB = 2AD$. Points M and N on side AC satisfy the relation $AM = NC$. Point K is chosen on the prolongation of CB beyond point B so that $CN = BK$. Determine the angle between lines NK and DM.

   Answer. $45°$.

   Solution. Let L be the projection of M onto AB. Notice that $\frac{ML}{CN} = \frac{AL}{BK} = \frac{AD}{BC} = \frac{1}{\sqrt{2}}$; hence we also have $\frac{LD}{CK} = \frac{AL + AD}{BK + BC} = \frac{1}{\sqrt{2}}$. Thus, the right triangles MLD and NCK are similar, and $\angle MDL = \angle NKC$ (see Fig. 9.5). Therefore the angle between the lines NK and MD is the same as the angle between KC and LD, which is equal to $45°$.

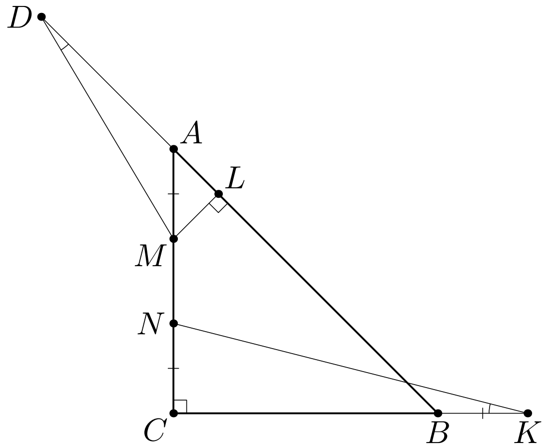

Figure 9.5

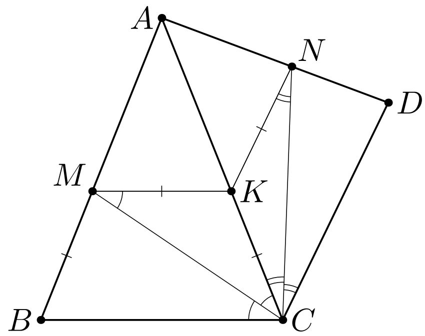

Figure 9.6

6. (M.Rozhkova) Let ABC be an isosceles triangle with $BC = a$ and $AB = AC = b$. Segment AC is the base of an isosceles triangle ADC with $AD = DC = a$ such that points D and B share the opposite sides of AC. Let CM and CN be the bisectors in triangles ABC and ADC respectively. Determine the circumradius of triangle CMN.

   Answer. $\frac{ab}{a+b}$.

   Solution. Choose a point K on segment AC so that $MK \parallel BC$. Then $\angle MCA = \angle MCB = \angle CMK$, so $MK = KC$. Moreover, by symmetry we get $KC = MB$. Next, by the bisector property we have $\frac{CK}{AK} = \frac{BM}{AM} = \frac{a}{b} = \frac{DN}{AN}$. Hence, $KN \parallel CD$. Now we may analogously obtain $KN = KC$. Thus, K is the circumcenter of triangle CMN, so its circumradius equals $KC = MB = ab/(a + b)$ (see Fig. 9.6).

7. (A.Belov) A convex pentagon P is divided by all its diagonals into ten triangles and one smaller pentagon P'. Let N be the sum of areas of five triangles adjacent to the sides of P decreased by the area of P'. The same operations are performed with the pentagon P'; let N' be the similar difference calculated for this pentagon. Prove that $N > N'$.

   Solution. Let $A_1A_2A_3A_4A_5$ be the initial pentagon, $B_1B_2B_3B_4B_5$ be the pentagon formed by its diagonals, and $C_1C_2C_3C_4C_5$ be the pentagon formed by the diagonals of $B_1B_2B_3B_4B_5$ (see Fig. 9.7). We will enumerate all the vertices cyclically, thus, for instance, $A_{i+5} = A_i$. For convenience, we will denote the area of polygon P by $[P]$.

Notice that $N' = \sum_i [B_iB_{i+1}B_{i+2}] - [B_1B_2B_3B_4B_5]$, since in the right-hand part the pentagon $C_1C_2C_3C_4C_5$ is counted with multiplicity -1, the triangles of the form $B_iB_{i+1}C_{i+3}$ — with multiplicity 1, and the triangles of the form $C_iC_{i+1}B_{i+3}$ with zero multiplicity. Thus the desired inequality is equivalent to

$$\sum_i [A_iA_{i+1}B_{i+3}] > \sum_i [B_iB_{i+1}B_{i+2}].$$

We will prove that $[A_iA_{i+1}B_{i+3}] > [B_{i+2}B_{i+3}B_{i+4}]$; adding up five such inequalities we will get the desired inequality.

Clearly, it is enough to deal with the case $i = 1$. Let us glue a triangle $A_1B_3B_4$ to each of the triangles $A_1A_2B_4$ and $B_3B_4B_5$; we get two triangles $A_1B_3A_2$ and $A_1B_3B_5$ with a common base A_1B_3. Finally, the distance from B_5 to the base is smaller than the distance from A_2; hence, $[A_1B_3A_2] > [A_1B_3B_5]$, QED.

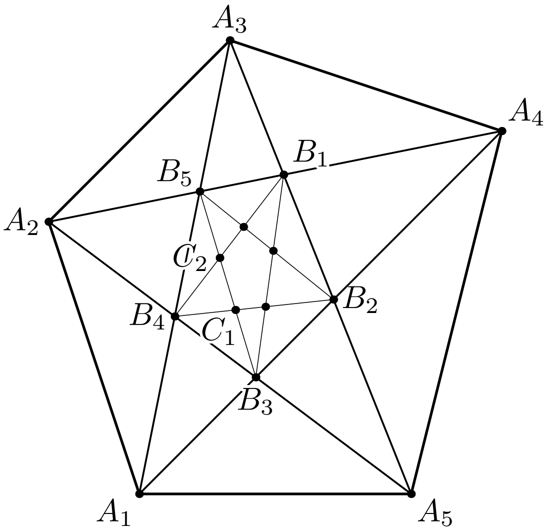

Figure 9.7

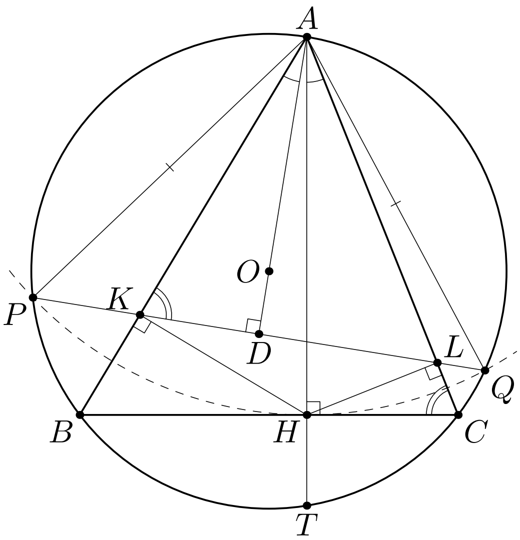

Figure 9.8

8. (M.Plotnikov) Let AH be an altitude of an acute-angled triangle ABC. Points K and L are the projections of H onto sides AB and AC. The circumcircle of ABC meets line KL at points P and Q, and meets line AH at points A and T. Prove that H is the incenter of triangle PQT.

Solution. Let O be the center of the circumcircle Ω of triangle ABC. From right triangles ABH and ACH we get $AK \cdot AB = AH^2 = AL \cdot AC$, or $AK/AL = AC/AB$. Therefore, triangles ALK and ABC are similar, and $\angle AKL = \angle ACB$. Now, since $\angle OAB = \pi/2 - \angle ACB$, we get $OA \perp KL$, which means that OA is the perpendicular bisector to the chord PQ, so $AP = AQ$. This means that TA is the bisector of $\angle PTQ$ (see Fig. 9.8).

Thus, the incenter I of triangle PQT lies on TA. Moreover, it is well-known that $AI = AP$. Thus, to prove that $I = H$ it suffices to show that $AH = AP$. Let D be the meeting point of the lines AO and KL, and let r be the radius of Ω. By the Pythagoras theorem, we have $AQ^2 - r^2 = AQ^2 - OQ^2 = (AD^2 + DQ^2) - (OD^2 + DQ^2) = AD^2 - (AD - r)^2$, which implies $AQ^2 = 2r \cdot AD$. On the other hand, notice that AH is a diameter of the circumcircle of AKL since $\angle AKH = \angle ALH = 90°$. Hence the similarity ratio of triangles AKL and ABC equals $AH/(2r)$. The segments AD and AH are the respective altitudes of these triangles, hence $AD/AH = AH/(2r)$, or $AH^2 = 2r \cdot AD = AQ^2$, QED.

Remark. The proof of the relation $AQ = AH$ may be shortened by means of the inversion with center A and radius AQ. Under this inversion, the line PQ and the circle Ω interchange, hence the points B and K also interchange, and $AQ^2 = AB \cdot AK = AH^2$.

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. First day. 10th form. Solutions

1. (A.Shapovalov) Determine all integer n such that a surface of an $n \times n \times n$ grid cube can be pasted in one layer by paper 1×2 rectangles so that each rectangle has exactly five neighbors (by a line segment).

   Answer. All even n.

   Solution. Consider any even n. Divide each face into 2×2 squares, and paste each such square with two rectangles in such a way that the long sides of the rectangles in one square are adjacent to the short ones in a neighboring square. Let us show that such pasting is possible. It is easy to see that one may cover four side faces of the cube, leaving top and bottom faces uncovered. Next, one may paste one bordering row on the top face (the arrangement of the rectangles around the corner looks as on Fig. 10.1, or symmetrically to it). This row determines the arrangement of rectangles on the top face uniquely, and it is easy to see that all four bordering rows of squares will satisfy the conditions. The covering of the bottom face is analogous.

   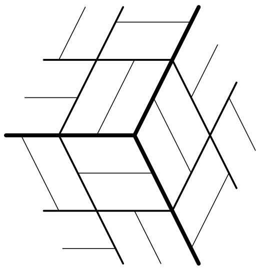

   Figure10.1

   Now assume that a pasting is possible for some odd n. The total number of rectangles is $6n^2/2 = 3n^2$; if each of them has five neighbors, then the total number of pairs of neighboring rectangles is $3n^2 \cdot 5/2$; but this number is not integer, which is absurd.

2. (A.Zaslavsky, B.Frenkin) We say that a point inside a triangle is *good* if the lengths of the cevians passing through this point are inversely proportional to the respective side lengths. Find all the triangles for which the number of good points is maximal.

   Answer. All acute-angled triangles.

   Solution. Let AA_1, BB_1, and CC_1 be the altitudes of a triangle ABC, and H be its orthocenter. Consider any good point P; let AA_P, BB_P, CC_P be the cevians passing through P. Then we have $AA_P/AA_1 = BB_P/BB_1 = CC_P/CC_1$; hence the right triangles AA_1A_P, BB_1B_P, and CC_1C_P are similar, so $\angle A_1AA_P = \angle B_1BB_P = \angle C_1CC_P$. There are two ways how these angles may be oriented. (Recall that an *oriented angle* $\angle(\ell, m)$ is the angle at which one needs to rotate ℓ clockwise to obtain a line parallel to m.)

   Case 1. Suppose that $\angle(A_1A, AA_P) = \angle(B_1B, BB_P) = \angle(C_1C, CC_P)$ (in particular, the triangle ABC is acute-angled; if, for instance, $\angle A \geq \pi/2$, then the angles $\angle B_1BB_P$ and $\angle C_1CC_P$ are acute, and their orientation s are opposite). The first equality yields that the points P, H. A, B are concyclic; analogously, P lies on the circumcircles of triangles ACH and BCH. But these three circles have exactly one common point H; hence $P = H$.

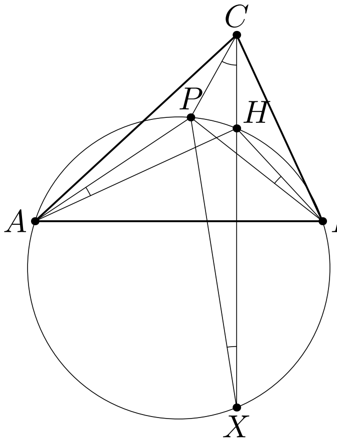

Figure 10.2.1

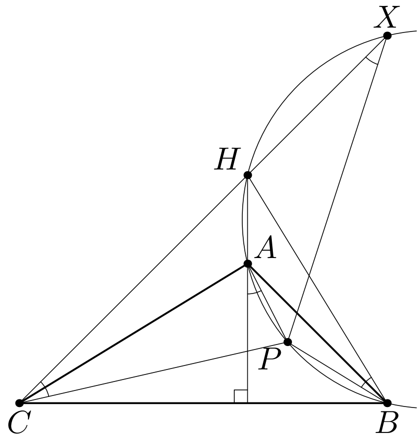

Figure 10.2.2

Case 2. Now suppose that two of the oriented angles are equal, while the third (say, $\angle(C_1C, CC_P)$) is opposite to them. Then, as in Case 1, the point P lies on the circumcircle Ω_C of triangle ABH (recall that $\angle(AH, HB) = -\angle(AC, CB)$, so Ω_C is symmetrical to the circumcircle Ω of ABC with respect to the line AB). Let X be the second meeting point of Ω_C with the line CH (then the points X and C are also symmetrical in AB; on Figs. 10.2.1 and 10.2.2 two possible configurations are shown). Then $\angle(PX, XC) = \angle(PB, BH) = -\angle(PC, CX)$; if these angles are nonzero, then this relation shows that the triangle PCX is isosceles, $PC = PX$. But then the point P lies on the perpendicular bisector AB of segment CX, which is impossible. Thus, $\angle(PB, BH) = 0$, and $P = H$.

Consequently, a point inside the triangle is good only if it is the orthocenter (and, obviously, the orthocenter of an acute-angled triangle is good). So, in an acute-angled triangle there exists exactly one good point, and there are no good point i other triangles.

Remark. In Case 2, one may apply a shorter (but less elementary) argument. The locus of points P satisfying the relation $\angle(B_1B, BB_P) = -\angle(C_1C, CC_P)$ is an equilateral hyperbola circumscribed about triangle ABC. Two such hyperbolas may have at most four common points, and these points are A, B, C, and H.

3. (A.Karlyuchenko) Let M and I be the centroid and the incenter of a scalene triangle ABC, and let r be its inradius. Prove that $MI = r/3$ if and only if MI is perpendicular to one of the sides of the triangle.

First solution. Let C_1 and C_2 be respectively the tangency points of side AB with the incircle ω and excircle ω_C of triangle ABC. Denote by C' the midpoint of AB. It is well known that $C_1C' = C_2C'$. Next, consider a homothety with center C mapping ω_C to *omega*; under this homothety, point C_2 maps to a point C_3 on ω opposite to C_1 (since the tangents in C_1 and C_3 to ω are parallel; see Fig. 10.3.1). Then IC' is a midline of triangle $C_1C_2C_3$, hence $C'I \parallel CC_2$. Thus, under a homothety with center M and coefficient -2, the point I maps to a point N lying on CC_2 (analogously, N lies on the segments connecting other vertices with the corresponding point of tangency of other excircles; N is called *the Nagel point* of triangle ABC). Therefore, N is obtained from M' by a homothety with center I and coefficient 3.

Now we turn to the problem. Suppose that $MI = r/3$. Then point N lies on ω. Without loss of generality we may assume that the tangent at N to ω intersects sides AC and BC; then ω and C share different sides of this tangent, and hence $N = C_3$. Since $IC_3 \perp AB$, we obtain $MI \perp AB$ (see Fig. 10.3.2).

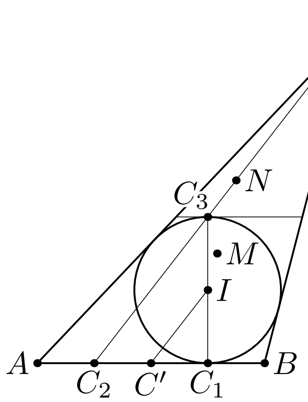

Figure 10.3.1

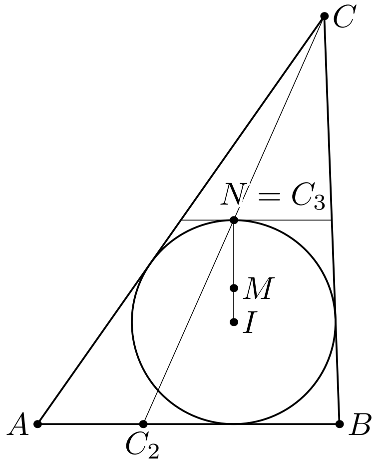

Figure 10.3.2

Conversely, if $AB \perp IM$, then N lies on the line IC_2; moreover, it also lies on the line CC_2. Since the triangle ABC is scalene, these lines are distinct, hence $N = C_2$, and thus $r = IN = 3IM$.

Second solution. Suppose that $AB \perp IM$. By the Pythagoras theorem, $AM^2 - BM^2 = (AC_1^2 + C_1M^2) - (BC_1^2 + C_1M^2) = (p-a)^2 - (p-b)^2 = c(b-a)$ (here, again, C_1 is the tangency point of AB with the incircle). Using the standard formula for the median length, we get $AM^2 = \frac{1}{9}(2b^2 + 2c^2 - a^2)$ and $BM^2 = \frac{1}{9}(2a^2 + 2c^2 - b^2)$, whence $c(b-a) = \frac{1}{3}(b-a)(a+b)$, or $a+b = 3c$, that is, $p = 2c$. It is easy to show that the converse is also true: namely, if $p = 2c$, then $AB \perp IM$. Finally, from $c(IM + r)/2 = S_{ABM} = S_{ABC}/3 = pr/3$ we obtain $IM + r = 4r/3$, or $IM = r/3$.

Conversely, assume that $MI = r/3$. Notice that $IA^2 + IB^2 + IC^2 = (\overrightarrow{IM} + \overrightarrow{MA})^2 + (\overrightarrow{IM} + \overrightarrow{MB})^2 + (\overrightarrow{IM} + \overrightarrow{MC})^2 = MA^2 + MB^2 + MC^2 + 2\overrightarrow{IM} \cdot (\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}) + 3MI^2 = MA^2 + MB^2 + MC^2 + 3MI^2$. So, if $MI = r/3$, then $IA^2 + IB^2 + IC^2 = MA^2 + MB^2 + MC^2 + \frac{1}{3}r^2 = \frac{1}{3}(a^2 + b^2 + c^2 + r^2)$. Next, by the Pythagoras theorem $IA^2 = r^2 + (p-a)^2$. Finally, using the relation $r^2 = S^2/p^2 = (p-a)(p-b)(p-c)/p$, we obtain

$$\frac{a^2 + b^2 + c^2 + r^2}{3} = (p-a)^2 + (p-b)^2 + (p-c)^2 + 3r^2,$$

or

$$\frac{a^2 + b^2 + c^2}{3} - (p-a)^2 - (p-b)^2 - (p-c)^2 = \frac{8r^2}{3} = \frac{8(p-a)(p-b)(p-c)}{3p},$$

which rewrites as $(p-2a)(p-2b)(p-2c) = 0$. It was mentioned above that if some expression in the brackets vanishes then IM is perpendicular to the corresponding side.

4. (B.Frenkin) Consider a square. Find the locus of midpoints of the hypothenuses of right-angled triangles with the vertices lying on three different sides of the square and not coinciding with its vertices.

Answer. All the points of a curvilinear octagon bounded by the arcs of eight parabolas with foci at the vertices of the square and directrices containing a side (non-adjacent to the focus); the midpoints of the sides of the square should be excluded (see Fig. 10.4.1).

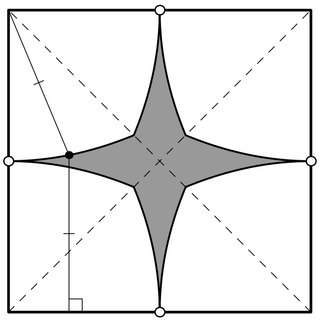

Figure 10.4.1

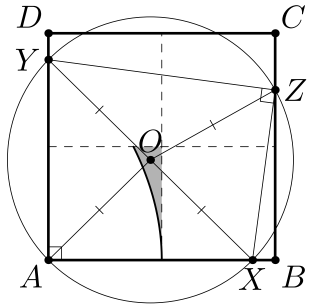

Figure 10.4.2

Solution. Notice first that the midpoint of the hypothenuse lies inside a square. If the endpoints of a hypothenuse lie on opposite sides of the square, then its midpoint lies on the midline of the square. Now assume that the endpoints X and Y of the hypothenuse of a triangle XYZ lie on sides AB and AD of a square $ABCD$ respectively, while the vertex Z lies on the side BC (see Fig. 10.4.2). Denote by O the midpoint of XY. The points A and Z belong to the circle with diameter XY; hence $OA = OX = OY = OZ$, and the distance from O to A is less that the distances to the other vertices of the square, but is not less than the distance from O to the line BC.

The locus of points equidistant from A and BC is the parabola with focus A and directrix BC (the vertex of this parabola is the midpoint of AB). So, the point O lies between this parabola and BC in the quarter of the square closest to A. Point O may lie on the parabola, but it cannot lie on the midline of the square (otherwise $Y = B$).

Analogously, one may consider other arrangements of points; taking the union of the obtained sets, we obtain the curvilinear octagon P bounded by the arcs of eight parabolas. The vertices of P are the midpoints of the sides of the square (they do not belong to the locus) and the points of intersection of parabolas with the diagonals of the square. Since the midlines of the square also lie in P, we obtain that the total locus lies in P. It remains to show that each point O in P (distinct from the midpoints of the sides) belongs to the locus.

If O lies on the midline parallel to AB, and it is not farther from AD than from BC, then one may take its projections onto AB and VD as the midpoints X and Y of a hypothenuse, and find Z as a meeting point of AD with the circle with diameter XY. If O lies in the quarter closest to A, between the parabola with focus A and directrix BC and the corresponding midline, then one chooses X and Y as the second intersection points of sides AB and AD with the circle with center O and radius OA, while Z may be chosen as any point of intersection of the same circle with side BC (such a point exists since the distance from O to BC is less than OA, but $OB > OA$).

VIII Geometrical Olympiad in honour of I.F.Sharygin

Final round. Second day. 10th form. Solutions

5. (F.Nilov) A quadrilateral $ABCD$ with perpendicular diagonals is inscribed into a circle ω. Two arcs α and β with diameters AB and CD lie outside ω. Consider two crescents formed by the circle ω and the arcs α and β (see Figure). Prove that the maximal radii of the circles inscribed into these crescents are equal.

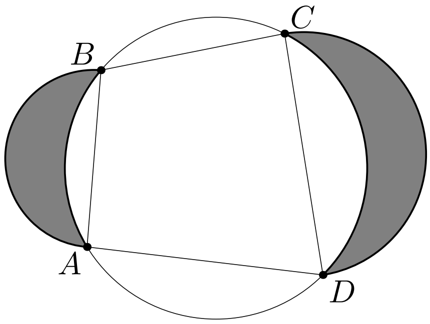

Figure 10.5.1

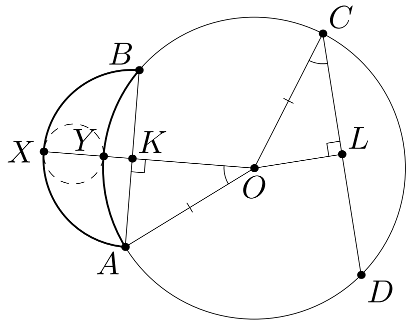

Figure 10.5.2

Solution. Let X and Y be respectively the midpoints of the arc α and the arc AB of the circle ω. Denote by O the center of ω. Then the crescent with vertices A and B is situated between the concentric circles centered at O with radii OY and OX (see Fig. 10.5.2). Hence the diameter of any circle inscribed into this crescent is at most XY; on the other hand, the circle with diameter XY lies inside the crescent. Thus the maximal diameter of a circle inside this crescent equals XY.

Since $AC \perp BD$, the sum of arcs AB and CD of the circle ω equals $180°$. Let K and L be the midpoints of segments AB and CD respectively; then $\angle AOK = 90° - \angle COL = \angle OCL$, hence the right triangles AOK and OCL are equal by hypothenuse and acute angle. So $OX = OK + KX = OK + KA = (AB + CD)/2$, and therefore $XY = (AB + CD)/2 - r$, where r is the radius of ω. Analogously we obtain that the maximal radius of a circle inscribed into the second crescent also equals $(AB+CD)/2-r$.

6. (V.Yassinsky) Consider a tetrahedron $ABCD$. A point X is chosen outside the tetrahedron so that segment XD intersects face ABC in its interior point. Let A', B', and C' be the projections of D onto the planes XBC, XCA, and XAB respectively. Prove that $A'B' + B'C' + C'A' \le DA + DB + DC$.

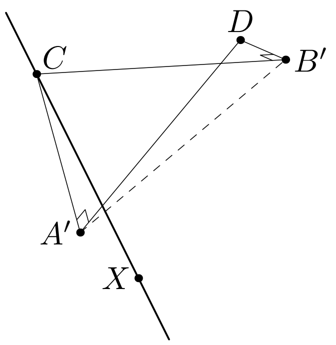

Figure 10.6

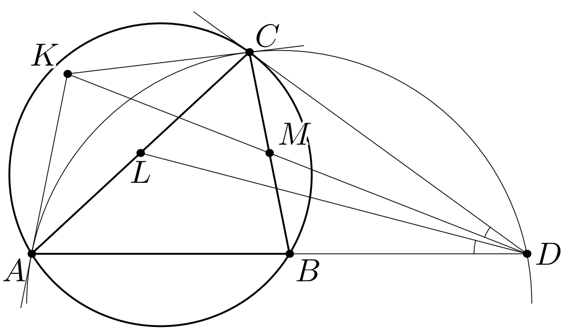

Figure 10.7

Solution. Since $DA' \perp (XBC)$, we get $\angle DA'C = 90°$; analogously, $\angle DB'C = 90°$ (see Fig. 10.6). Hence the points A' and B' lie on the sphere with diameter DC, and the distance between them is does not exceed the diameter: $A'B' \le DK$. Analogously, we get $A'C' \le DB$ and $B'C' \le DA$. Adding up these inequalities we get the desired one.

7. (F.Ivlev) Consider a triangle ABC. The tangent line to its circumcircle at point C meets line AB at point D. The tangent lines to the circumcircle of triangle ACD at points A and C meet at point K. Prove that line DK bisects segment BC.

Solution. It is well known that in any triangle XYZ, the symmedian XX' (that is, the line symmetrical to the median from the vertex X with respect to the angle bisector of angle X) passes through the common point of the tangents to the circumcircle of XYZ at Y and Z. Thus, the line DK is a symmedian in triangle ACD. Next, the triangles ACD and CBD are similar. So, denoting by DL and DM respectively their medians from D, we get $\angle CDK = \angle ADL = \angle CDM$, which implies that M lies on DK.

8. (D.Shvetsov) A point M lies on the side BC of square $ABCD$. Let X, Y, and Z be the incenters of triangles ABM, CMD, and AMD respectively. Let H_x, H_y, and H_z be the orthocenters of triangles AXB, CYD, and AZD. Prove that H_x, H_y, and H_z are collinear.

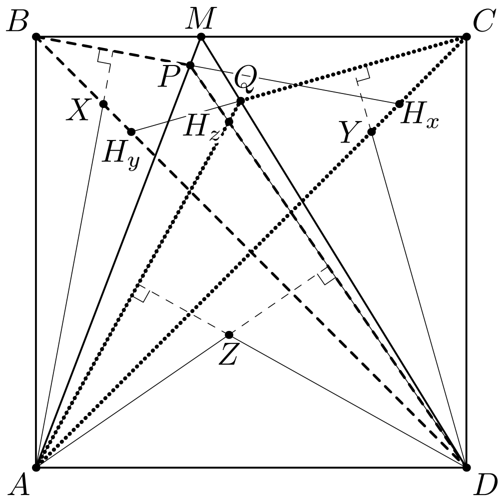

Figure 10.8

Solution. Clearly, the points X and Y lie on the diagonals BD and AC respectively. Hence the lines AC and BD contain some altitudes of triangles AXB and CYD respectively. Let us choose points P and Q on the segments AM and DM respectively so that $AP = DQ = AD$. Then AX is an angle bisector (and hence an altitude) of an isosceles triangle ABP. Thus, the orthocenter H_x is the common point of the lines BP and AC. Analogously, H_y is the common point of the lines CQ and BD. Finally, from similar arguments we get that $AZ \perp DP$ and $BZ \perp AQ$, so H_z is the common point of the lines AQ and BP (see Fig. 10.8).

Now let us apply the Desargues' theorem to the triangles BPD and CAQ. Since the lines BC, PA, and DQ which connect the corresponding points of these triangles are concurrent at M, we get that the common points of the lines containing the corresponding sides of these triangles are collinear. But these points are exactly H_x, H_y, and H_z.