VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (B.Frenkin) (8) Does there exist a triangle, whose side is equal to some its altitude, another side is equal to some its bisectrix, and the third side is equal to some its median ?

   Solution. No, because the greatest side of a triangle is longer than any its median, bisector or altitude. Indeed, a segment joining a vertex of a triangle with an arbitrary point of the opposite side is shorter than one of two remaining sides. Thus each median or bisector is shorter than one of sides and so is shorter than the greatest side. This is correct also for the altitudes.

2. (D.Shvetsov) (8) Bisectors AA_1 and BB_1 of a right triangle ABC ($\angle C = 90°$) meet at a point I. Let O be the circumcenter of the triangle CA_1B_1. Prove that $OI \perp AB$.

   Solution. Let A_2, B_2, C_2 be the projections of A_1, B_1, I to AB (fig.2). Since AA_1 is a bisector, we have $AA_2 = AC$. On the other hand, AC_2 touches the incircle, thus the segment $A_2C_2 = AA_2 - AC_2$ is equal to the tangent to this circle from C. Similarly B_2C_2 is equal to the same tangent, i.e. C_2 is the midpoint of A_2B_2. By Phales theorem, C_2I meets segment A_1B_1 in its midpoint, which coincides with the circumcenter of triangle CA_1B_1 because this triangle is right-angled.

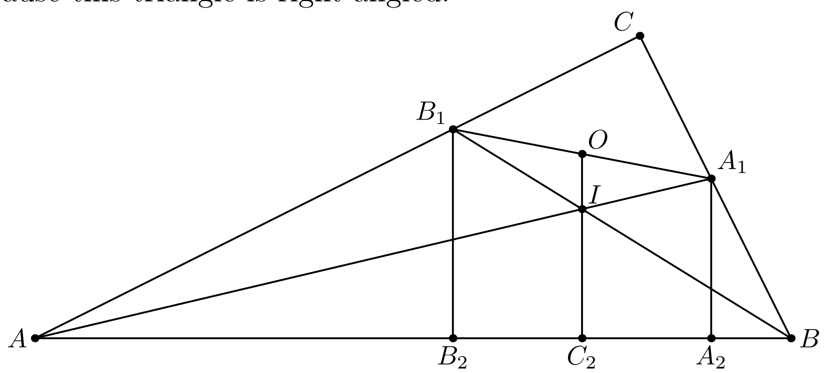

Fig. 2

3. (F.Nilov) (8) Points A', B', C' lie on sides BC, CA, AB of a triangle ABC. For a point X one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC$. Prove that the quadrilateral $XA'BC'$ is cyclic.

   Solution. Let Y be the common point of circles $AB'C'$ и $BC'A'$ distinct from C'. Then since $\angle B'YC' = \pi - \angle BAC$ and $\angle C'YA' = \pi - \angle CBA$, we obtain that $\angle A'YB' = \pi - \angle ACB$, i.e. Y lies also on circle $CA'B'$. Now note that $\angle AYB = \angle AYC' + \angle C'YB = \angle AB'C' + \angle C'A'B = 2\pi - \angle C'B'C - \angle CA'C' = \angle ACB + \angle A'C'B' = \angle AXB$ (fig.3). Similarly $\angle BYC = \angle BXC$, i.e. X and Y coincide.

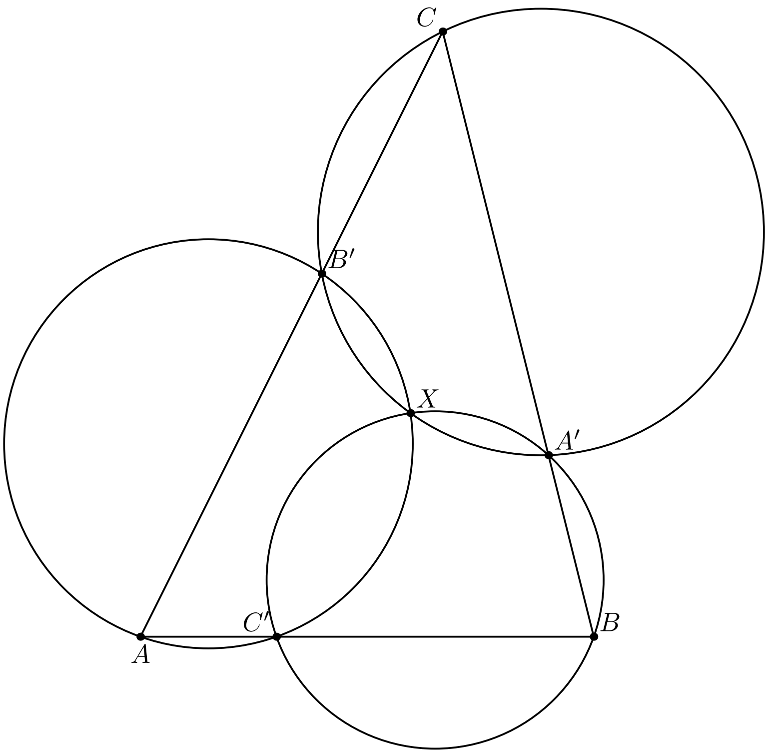

Fig. 3

4. (D.Shvetsov) (8) The diagonals of a cyclic quadrilateral $ABCD$ meet in a point N. The circumcircles of the triangles ANB and CND intersect the sidelines BC and AD for the second time in points A_1, B_1, C_1, D_1. Prove that the quadrilateral $A_1B_1C_1D_1$ is inscribed into a circle centered at N.

   Solution. Since pentagon A_1NB_1CD is cyclic, we obtain that $A_1N = B_1N$, because respective angles BDA and BCA are equal. Similarly $NC_1 = ND_1$. Also $\angle NA_1A = \angle ACD = \angle ABD = \angle DD_1N$ (fig.4). Thus $ND_1 = NA_1$, q.e.d.

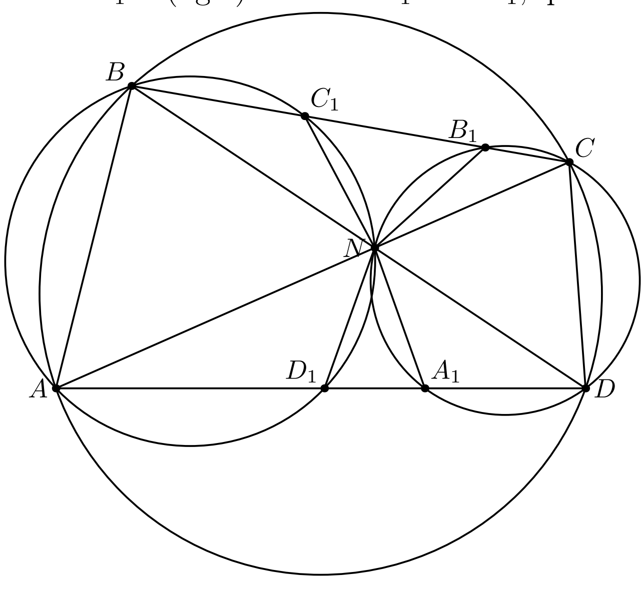

Fig. 4

5. (D.Shvetsov) (8–9) A point E lies on altitude BD of triangle ABC, and $\angle AEC = 90°$. Points O_1 and O_2 are the circumcenters of triangles AEB and CEB; points F, L are the

midpoints of segments AC and O_1O_2. Prove that points L, E, F are collinear.

Solution. Note that the medial perpendiculars to segments AE and EC are the medial lines of triangle AEC, thus they pass through F. So we must prove that FE is the median of triangle FO_1O_2. But $O_1O_2 \parallel AC$ because these two segments are perpendicular to BD. Let the line passing through E and parallel to AC meet FO_1 and FO_2 in points X and Y (fig.5). Since $FCEX$ and $FAEY$ are parallelograms, then $XE = FC = FA = EY$. Thus FE is the median of triangles FXY and FO_1O_2.

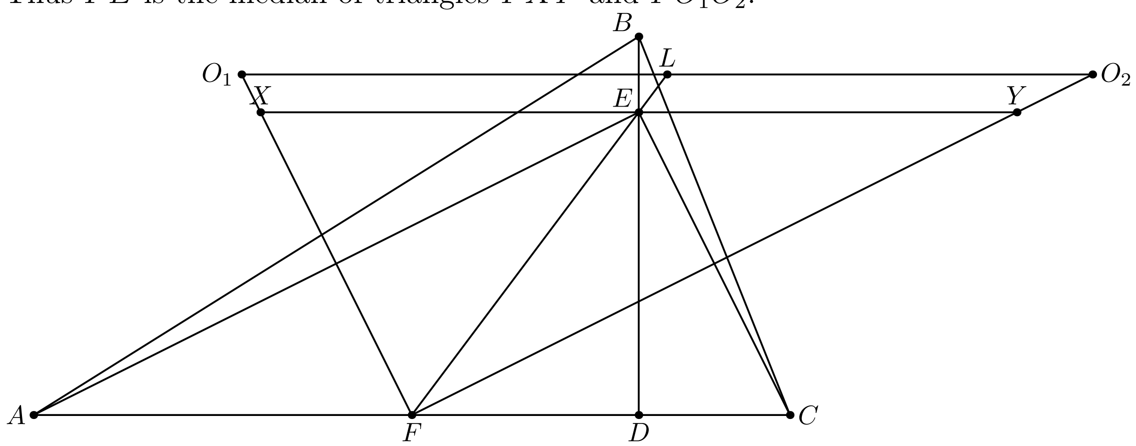

Fig. 5

6. (D.Shvetsov) (8–9) Points M and N lie on side BC of a regular triangle ABC (M is between B and N), and $\angle MAN = 30°$. The circumcircles of triangles AMC and ANB meet at a point K. Prove that line AK passes through the circumcenter of triangle AMN.

   Solution. Since $\angle BAM + \angle NAC = \angle MAN$ and $AB = AC$, the reflection of B in AM coincides with the reflection of C in AN. Mark this point by L. Now $\angle ALM = \angle ABM = \angle ACM$, i.e. L lies on circle ACM. Similarly L lies on circle ABN and thus coincdes with K (fig.6). So $\angle KAN = \angle NAC = 30° - \angle BAM = 90° - \angle NMA$. But the theorem on inscribed angle implies that we have the same angle between line AN and the line connecting A with the circumcenter of triangle AMN.

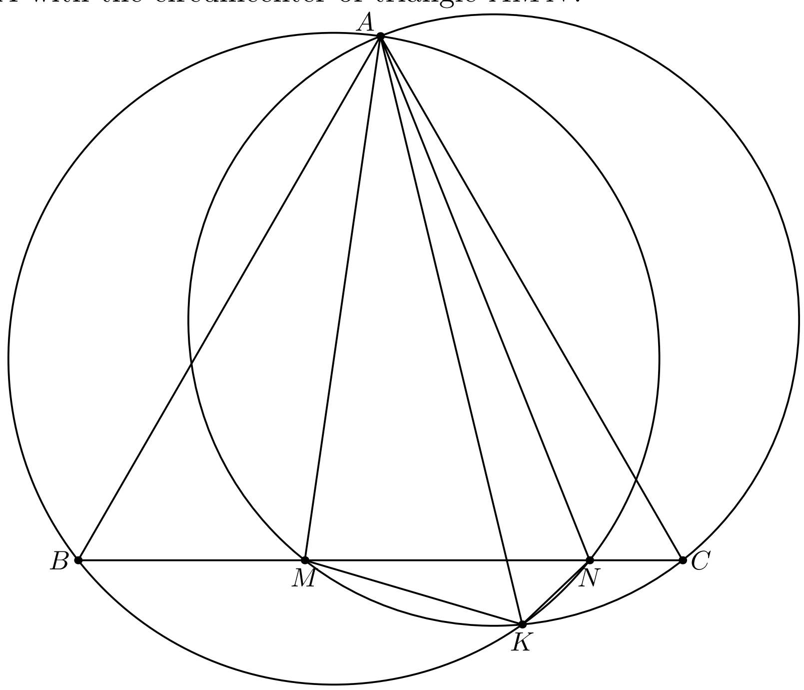

Fig. 6

7. (D.Shvetsov) (8–9) The line passing through vertex B of triangle ABC and perpendicular to its median BM intersects the altitudes dropped from A and C (or their extensions) in points K and N. Points O_1 and O_2 are the circumcenters of triangles ABK and CBN respectively. Prove that $O_1M = O_2M$.

   Solution. Consider parallelogram $ABCD$ (fig.7). Since $\angle BKA = \angle DKC = \angle BDA$, points A, B, K, D lie on a same circle and $O_1M \perp BD$. Similarly $O_2M \perp BD$. Also since triangles ABD and BCD are equal, the distances from their circumcenters to point M also are equal.

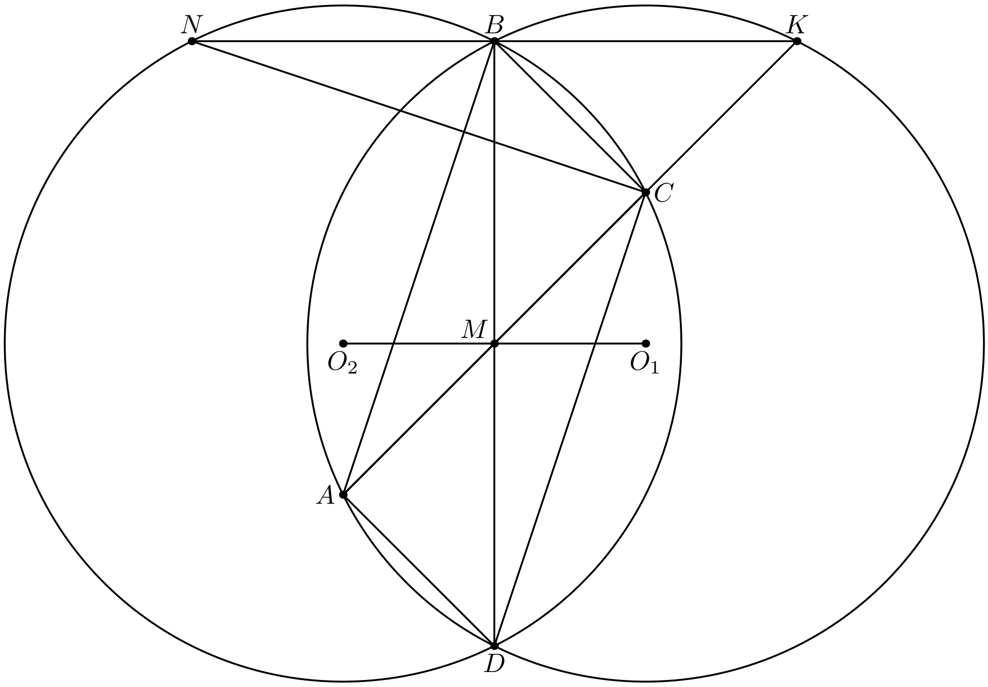

Fig. 7

8. (D.Shvetsov) (8–10) Let AH be an altitude of a given triangle ABC. Points I_b and I_c are the incenters of triangles ABH and CAH respectively; BC touches the incircle of triangle ABC at a point L. Find $\angle LI_bI_c$.

   Solution. We will prove that triangle LI_bI_c is right-angled and isosceles. Let L_b, L_c be the projections of I_b, I_c to BC, and r_b, r_c be the inradii of triangles AHB, AHC (fig.8). Since these triangles are right-angled, we have $r_b = (AH+BH-AB)/2$, $r_c = (AH+CH-AC)/2$ and $r_b-r_c = (BH-CH)/2-(AB-AC)/2 = (BH-CH)/2-(BL-CL)/2 = LH$. Thus $I_bL_b = LI_c = r_b$, $I_cL_c = LI_b = r_c$, i.e. triangles LI_bL_b and I_cL_cL are equal, $LI_b = LI_c$ and $\angle I_bLI_c = 90°$. So $\angle LI_bI_c = 45°$.

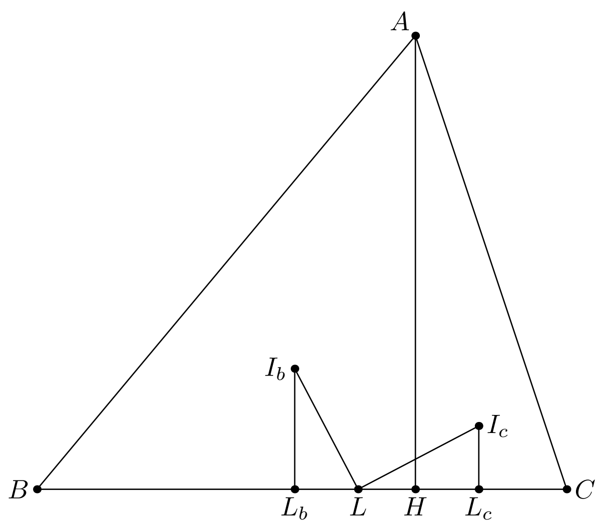

Fig. 8

9. (B.Frenkin) (8–10) A point inside a triangle is called "good" if three cevians passing through it are equal. Suppose the total number of good points is odd for an isosceles triangle ABC ($AB = BC$). Find all possible values of this number.

   Solution. Since the reflection of any good point in the altitude from B also is a good point and the total number of good points is odd, there exists a good point lying on this altitude. The cevian through this point from A is not shorter than the altitude from A. Hence the altitude from A is not shorter than the altitude from B and $AC \leq AB$. Also AC can't be longer than the altitude from B because in that case there exist two good points on this altitude. Suppose now that some good point doesn't lie on this altitude. Let AA', BB', CC' be respective cevians, and AA_1, CC_1 be the altitudes. Then $A_1A' = C_1C'$ and exactly one of points A', C' lies between the foot of respective altitude and vertex B. But this implies that respective cevians are shorter than AC. So they are shorter than BB_1 and we obtain a contradiction. Thus there exists exactly one good point.

10. (I.Bogdanov) (8–11) Let three lines forming a triangle ABC be given. Using a two-sided ruler and drawing at most eight lines, construct a point D on the side AB such that $AD/BD = BC/AC$.

    Solution. Construct lines a, b, c, parallel to BC, CA, AB and lying at the distance from them equal to the width of the ruler. Lines a, b, BC, AC form a rhombus, and its diagonal is the bisector of angle C. Let E be the common point of this bisector with c, and F be the common point of diagonals of trapezoid formed by lines c, AB, AC and BC (fig.10). Then EF meets AB in the sought point D.

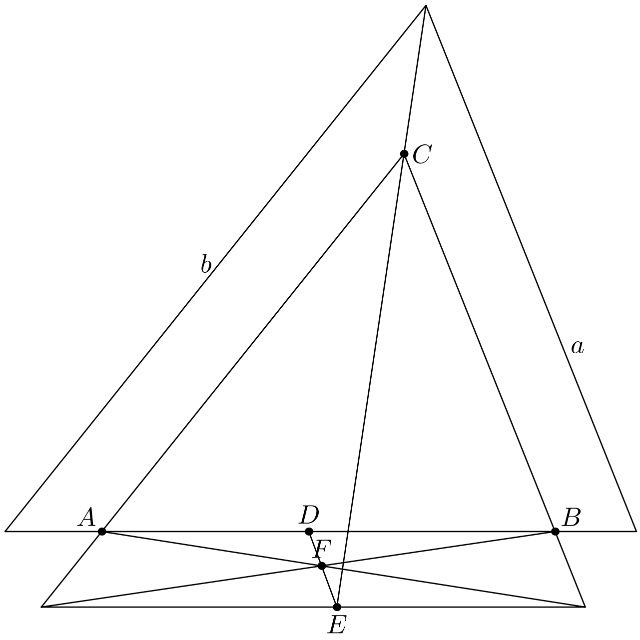

Fig. 10

11. (B.Frenkin) (8–11) A convex n-gon is split into three convex polygons. One of them has n sides, the second one has more than n sides, the third one has less than n sides. Find all possible values of n.

    Answer. $n = 4$ или $n = 5$.

    Solution. It is clear that $n > 3$. Suppose that $n > 5$. Then one of three parts of n-gon has at least $n+1$ sides, the second parts has at least n sides, the third part has at least three sides. If three pairs of the sides of these parts join inside the given polygon then at most three pairs can form its sides. If two pairs of sides join inside the polygon then at most four pairs can form its sides. In all cases the total number of sides of parts is not greater than $n+9$. If $n > 5$ this isn't possible. The examples for $n = 4, 5$ are given on fig. 11.

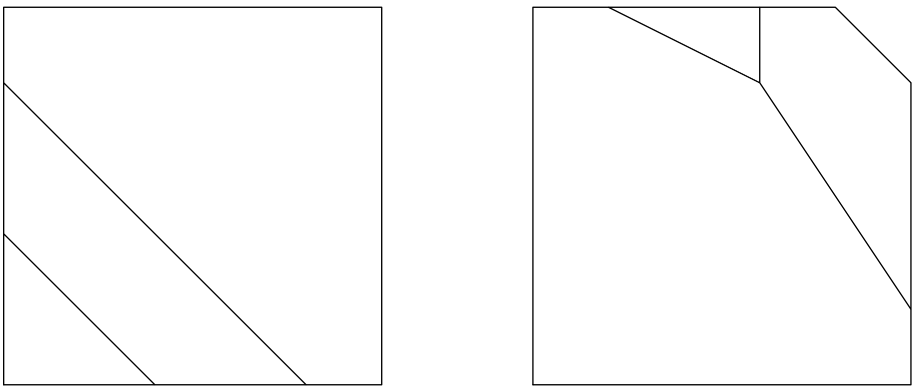

Fig. 11

12. (A.Blinkov, Y.Blinkov, M.Sandrikova) (9) Let AC be the greatest leg of a right triangle ABC, and CH be the altitude to its hypothenuse. The circle of radius CH centered at H intersects AC in point M. Let a point B' be the reflection of B with respect to the point H. The perpendicular to AB erected at B' meets the circle in a point K. Prove that:

    a) $B'M \parallel BC$;

b) AK is tangent to the circle.

Solution. a) Let N be an altitude of isosceles triangle CHM. Then $CN = NM$. Since $BH = B'H$ and $NH \parallel BC$, thus the line passing through B' and parallel to HN meets AC in point M (Phales theorem).

Second solution. Since $\angle CMH = \angle MCH = \angle CBB' = \angle CB'B = a$, points C, H, B' and M are concyclic. Thus $\angle CB'M = \angle CHM = 180^\circ - 2a$ and $\angle AB'M = a$ q.e.d.

b) From the right-angled triangle ABC we have: $CH^2 = AH \cdot BH$. Since $B'H = BH$ and $KH = CH$ then $KH^2 = AH \cdot B'H$, i.e. triangles AHK and KHB' are similar. This yields the assertion of the problem..

13. (S.Berlov) (9) Given a convex quadrilateral $ABCD$ such that $AB = BC$. A point K lies on the diagonal BD, and $\angle AKB + \angle BKC = \angle A + \angle C$. Prove that $AK \cdot CD = KC \cdot AD$.

    Solution. Let L be a point on BD such that $\angle ALB = \angle A$. Since triangles ABL and DBA are similar we have $BL \cdot BD = AB^2 = BC^2$. Thus triangles CBL and DBC are also similar, i.e. $\angle BLC = \angle C$ and L coincides with K. The sought equality clearly follows from these two similarities.

14. (S.Berlov) (9–10) Given a convex quadrilateral $ABCD$ and a point M on its side AD such that CM and BM are parallel to AB and CD respectively. Prove that $S_{ABCD} \geq 3S_{BCM}$.

    Solution. Since $\angle ABM = \angle BMC = \angle MCD$ we have $S_{ABM}/S_{BMC} = AB/MC$ and $S_{BMC}/S_{CMD} = BM/CD$. But triangles ABM and MCD are similar, so these two ratios are equal and $S_{BMC}^2 = S_{ABM} \cdot S_{MCD}$. By Cauchi inequality $S_{BMC} \leq (S_{ABM} + S_{MCD})/2$ which is equivalent to the assertion of the problem.

15. (D.Prokopenko, A.Blinkov) (9–11) Suppose AA_1, BB_1 and CC_1 are the altitudes of an acute-angled triangle ABC, AA_1 meets B_1C_1 in a point K. The circumcircles of triangles A_1KC_1 and A_1KB_1 intersect the lines AB and AC for the second time at points N and L respectively. Prove that

    a) the sum of diameters of these two circles is equal to BC;

    b) $A_1N/BB_1 + A_1L/CC_1 = 1$.

    Solution. a) Triangles AB_1C_1, A_1BC_1 and A_1B_1C are similar to triangle ABC with coefficients $\cos A$, $\cos B$, $\cos C$ respectively. Thus $\angle KA_1C_1 = \angle KA_1B_1 = 90^\circ - \angle A$, and by sinuses theorem the diameters of circumcircles of triangles AKB_1 and A_1KC_1 are equal to $B_1K/\cos A$ and $C_1K/\cos A$ respectively. So their sum is $B_1C_1/\cos A = BC$.

    b) An equality proved in p.a) can be written as

    $$\frac{A_1N}{\sin B} + \frac{A_1L}{\sin C} = BC.$$

    Dividing by BC we obtain sought relation.

16. (F.Nilov) (9–11) A circle touches the sides of an angle with vertex A at points B and C. A line passing through A intersects this circle in points D and E. A chord BX is parallel to DE. Prove that XC passes through the midpoint of segment DE.

    Solution. Note that $\angle BCD = \angle ECX$ because the respective arcs lie between parallel chords. Furthermore since $\angle ABD = \angle AEB$, triangles ABD and AEB are similar and so

$BD/BE = AD/AB$. Similarly $CD/CE = AD/AB$, i.e. $BD{\cdot}CE = CD{\cdot}BE = BC{\cdot}DE/2$ (the last equality follows from Ptolomeus theorem).

Now let CX meet DE at point M (fig. 16). Then triangles CBD and CME are similar, thus $BD \cdot CE = CB \cdot EM$. From this and previous equalities we obtain $EM = ED/2$.

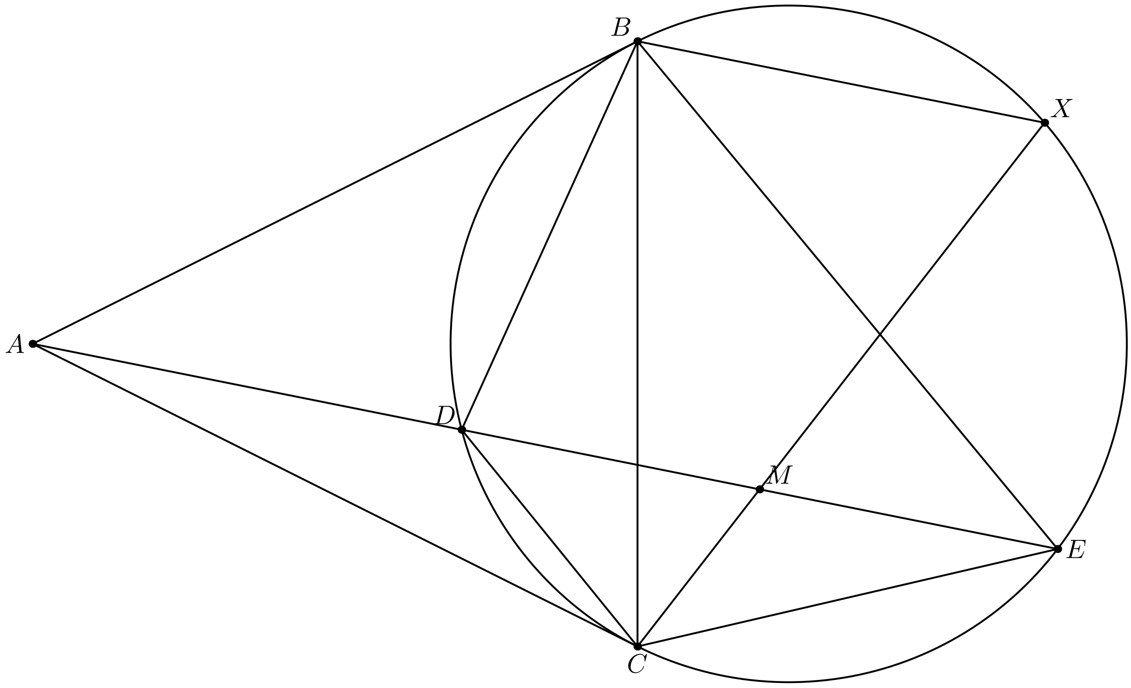

Fig. 16

17. (S.Tokarev) (9–11) Construct a triangle, if the lengths of the bisectrix and of the altitude from one vertex, and of the median from another vertex are given.

First solution. Let $l = CL$, $h = CH$ be the bisector and the altitude from vertex C, $m = BM$ be the median from vertex B and ϕ be the angle of right-angled triangle with hypothenuse l opposite to cathetus with length h. Let p be the line passing through C and parallel to AB, and B' be the reflection of point B in p.

Suppose that triangle ABC is constructed. Then $\angle CLB = \phi$ or $\angle CLB = 180° - \phi$, and in both cases $\angle B'CM = 2\phi$. In fact, if $\angle CLB = \phi$, then $\angle B'CM = 360° - 2\angle CBA - \angle BCA = 2(180° - \angle CBA - \angle BCL) = 2\phi$ (Fig.17.1). The second case is similar.

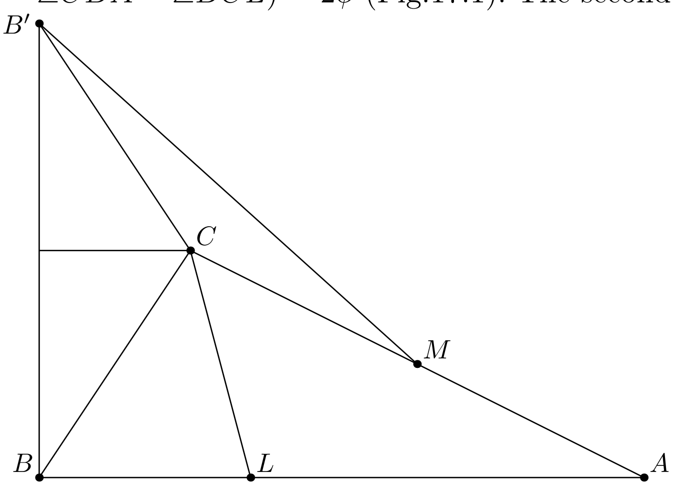

Fig. 17.1

Since $\angle B'CM = 2\phi$ we obtain the following construction.

Construct two parallel lines with distance h between them. Let B be a point lying on one of these lines and p be the second line. Now construct point B' and point M equidistant from two lines and such that $BM = m$. Construct angle ϕ and two arcs with endpoints B' and M equal to $360° - 4\phi$.

If C_1 and C_2 are the common points of these arcs with line p, and A_i $(i = 1, 2)$ is the reflection of C_i in M, then each of triangles A_1BC_1 and A_2BC_2 is sought.

Indeed, the altitudes of these triangles from C_1, C_2 are equal to h, and segment $BM = m$ is their common median. Furthermore if L_1, L_2 are the feet of respective bisectors then our construction yields that one of angles C_1L_1B and C_2L_2B is equal to ϕ, and the second one is equal to $180° - \phi$. Thus by definition of ϕ we obtain $C_1L_1 = C_2L_2 = l$.

Note. It is evident that if $l < h$ or $m < h/2$, then the solution doesn't exist. If $l = h$ and $m \geq h/2$, then the sought triangle is unique and isosceles (if $m = h/2$ it degenerates to a segment). When $l > h$ and $m = h/2$, we obtain two equal triangles symmetric wrt line BB'.

If $l > h$ and $m = l/2$ then one of two triangles is degenerated. In all other cases the problem has two solutions.

Second solution. Having an altitude and a bisector from vertex C, we can construct this vertex and line AB. Consider now the following map of this line to itself. For an arbitrary point X find a point Y such that its distances from X and AB are equal to the given median from B and to the half of the given altitude (fig. 17.2). Now find a common point X' for AB and for the reflection of CY in the bisector. Obviously this map is projective and B is its fixed point. So we obtain the well-known problem of constructing the fixed point of a projective map.

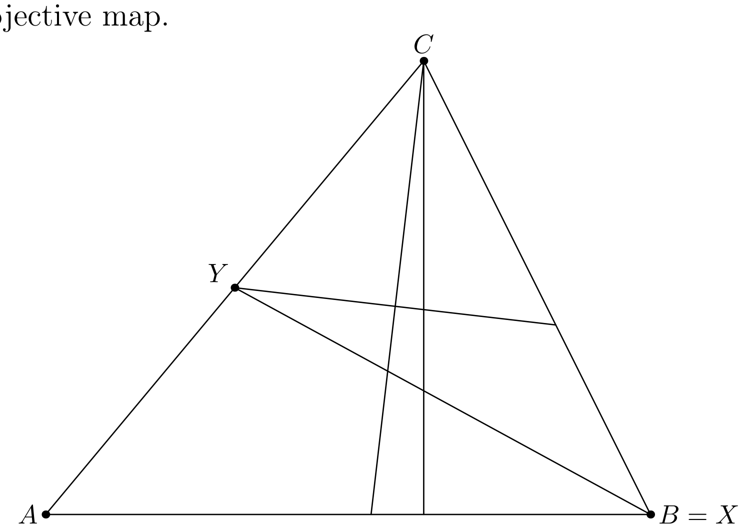

Fig. 17.2

18. (D.Prokopenko) (9–11) A point B lies on a chord AC of a circle ω. Segments AB and BC are diameters of circles ω_1 and ω_2 centered at O_1 and O_2 respectively. These circles intersect ω for the second time in points D and E respectively. The rays O_1D and O_2E meet in a point F, and the rays AD and CE meet in a point G. Prove that line FG passes through the midpoint of segment AC.

Solution. Since $\angle ADB = \angle BEC = 90^\circ$, points D and E lie on the circle with diameter BG. Also $\angle FDG = \angle ADO_1 = \angle DAC = \angle GED$. Thus FD (and similarly FE) touches this circle (fig. 18). So GF is the symmedian of triangle GED. Since triangle GDE is similar to triangle GCA, this line is the median of the last triangle.

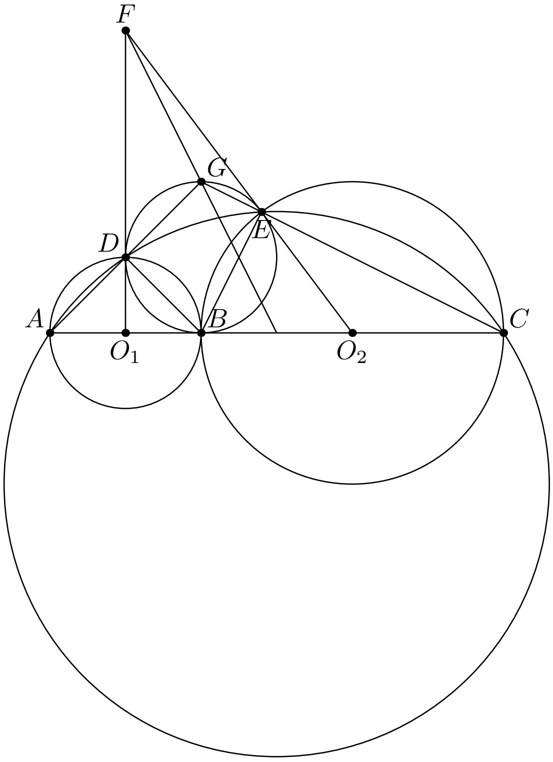

Fig. 18

19. (V.Yasinsky, Ukraine) (9–11) A quadrilateral $ABCD$ is inscribed into a circle with center O. Points P and Q are opposite to C and D respectively. Two tangents drawn to that circle at these points meet the line AB in points E and F (A is between E and B, B is between A and F). Line EO meets AC and BC in points X and Y respectively, and line FO meets AD and BD in points U and V. Prove that $XV = YU$.

Solution. It is sufficient to prove that $XO = OY$. Indeed, we then similarly have $UO = OV$ and so $XUYV$ is a parallelogram.

Let EO meet the circle in points P and Q (Fig. 19). The sought equality is equivalent to $(PX; OY) = (QY; OX)$. Projecting line EO to the circle from point C we obtain an equivalent equality $(PA; C'B) = (QB; C'A)$. It is correct because PQ, AB and the tangent in C' concur.

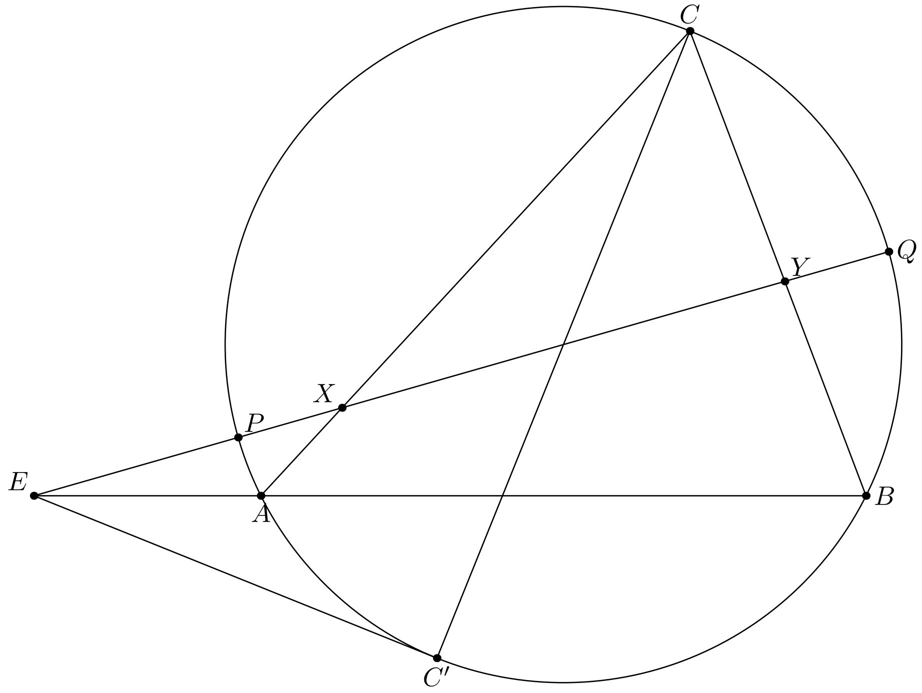

Fig. 19

20. (F.Ivlev) (10) The incircle of an acute-angled triangle ABC touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, B_2 are the midpoints of segments B_1C_1, A_1C_1 respectively. Let P be a common point of the incircle and the line CO, where O is the circumcenter of ABC. Let also A' and B' be the second common points of PA_2 and PB_2 with the incircle. Prove that a common point of AA' and BB' lies on the altitude of the triangle dropped from the vertex C.

Solution. It is sufficient to prove that $\angle CAP = \angle A'AB$. Indeed, from this we obtain that line AA' is the reflection of AP in the bisector of angle A. Similarly line BB' is the reflection of BP in the bisector of angle B, and so the common point of these two lines lies on the reflection of line CP in the bisector of angle C, i.e. on the altitude.

Let Q be the common point of line AP with the incircle and S be the midpoint of arc B_1C_1 (fig. 20). Consider the composition f of the projections of incircle to itself from A and A_2. We have $f(B_1) = C_1$, $f(C_1) = B_1$, $f(Q) = A'$ and $f(S) = S$. Thus $(B_1Q; SC_1) = (C_1A'; SB_1)$, i.e. A' is the reflection of Q in AA_2, which proves the sought equality.

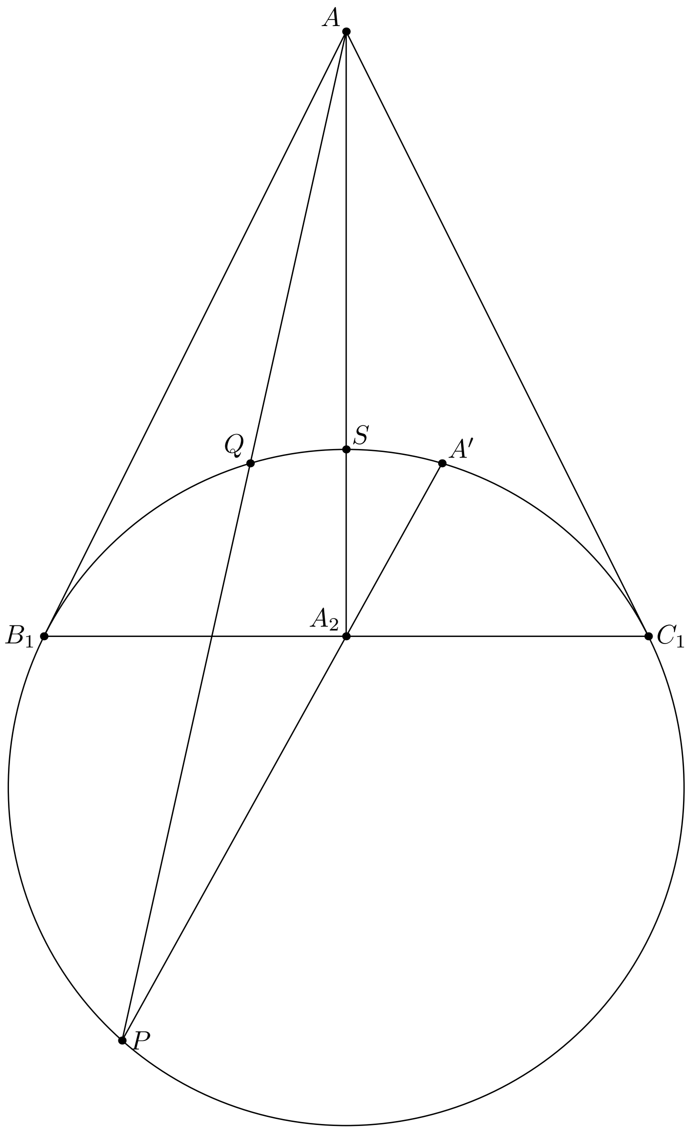

Fig. 20

21. (A.Akopjan) (10–11) A given convex quadrilateral $ABCD$ is such that $\angle ABD + \angle ACD > \angle BAC + \angle BDC$. Prove that $S_{ABD} + S_{ACD} > S_{BAC} + S_{BDC}$.

    Solution. If $AB \parallel CD$ then $\angle ABD = \angle BDC$ and $\angle ACD = \angle BAC$. Thus the given equality is equivalent to the fact that rays AB and DC intersect, i.e. the distance from C to line AB is less than the distance from D, and the distance from B to line CD is less than the distance from A. So $S_{ABD} > S_{ABC}$ and $S_{ACD} > S_{BCD}$.

22. (A.Zaslavsky) (10–11) A circle centered at a point F and a parabola with focus F have two common points. Prove that there exist four points A, B, C, D on the circle such that lines AB, BC, CD and DA touch the parabola.

    Solution. Take an arbitrary point A lying on the circle and outside the parabola. Line AF and the line passing through A and parallel to the axis of parabola intersect the circle in points symmetric wrt the axis. Thus the tangents from A to the parabola also intersect the circle in symmetric points B and D. Similarly the second tangents from B and D intersect the circle in point C symmetric to A. Thus A, B, C, D are the sought points.

23. (N.Beluhov, Bulgaria) (10–11) A cyclic hexagon $ABCDEF$ is such that $AB \cdot CF = 2BC \cdot FA$, $CD \cdot EB = 2DE \cdot BC$, and $EF \cdot AD = 2FA \cdot DE$. Prove that lines AD, BE and CF concur.

    Solution. Let P be the common point of FC and AD and G be the second common point of BP with the circumcircle of the hexagon. Then $2 = (AC; BF) = (DF; GB) = (DF; EB)$, and so points G and F coincide.

24. (A.Akopjan) (10–11) Given a line l in the space and a point A not lying on l. For an arbitrary line l' passing through A, XY (Y is on l') is the common perpendicular to the lines l and l'. Find the locus of points Y.

    Solution. Let the plane passing through A and perpendicular to l intersect l in point B. Let C be the projection of Y to this plane. Hence $BC \parallel XY$, thus $BC \perp AY$ and by three perpendiculars theorem $BC \perp AC$. So C lies on the circle with diameter AB, and Y lies on the cylinder constructed over this circle. It is clear that all points of the cylinder are on the sought locus.

25. (N.Beluhov, Bulgaria) (11) It is known for two different regular icosahedrons that some six of their vertices are vertices of a regular octahedron. Find the ratio of the edges of these icosahedrons.

    Solution. Note that no icosahedron can contain four vertices of octahedron. Indeed, between four vertices of octahedron there exist two opposite vertices that together with each of remaining vertices form an isosceles right-angled triangle . But no three vertices of icosahedron form such triangle.

    Thus one of given icosahedrons contains three vertices lying on one face of octahedron, and the other icosahedron contains three vertices lying on the opposite face. Now note that there exist only three different distances between the vertices of icosahedron: one is equal to the edge, the second is equal to the diagonal of a regular pentagon with the side equal to the edge, and the third is the distance between two opposite vertices. If three vertices form a regular triangle then the distance between them is one of two first types. Since two icosahedrons aren't equal, then some face of the octahedron is a face for one of them and a triangle formed by diagonals for the other. Thus the ratio of edges is the ratio of the diagonal and the side of a regular pentagon, i.e. $\dfrac{\sqrt{5}+1}{2}$.