V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (D.Prokopenko) (8) Points B_1 and B_2 lie on ray AM, and points C_1 and C_2 lie on ray AK. The circle with center O is inscribed into triangles AB_1C_1 and AB_2C_2. Prove that the angles B_1OB_2 and C_1OC_2 are equal.

   Solution. Let D be the common point of segments B_1C_1 and B_2C_2 (Fig.1). Then by theorem on exterior angles of a triangle, we have $\angle B_1OB_2 = \angle AOB_2 - \angle AOB_1 = \angle AB_1O - \angle AB_2O = (\angle AB_1C_1 - \angle AB_2C_2)/2 = \angle B_1DB_2/2$. Similarly $\angle C_1OC_2 = \angle C_1DC_2/2$, thus these angles are equal.

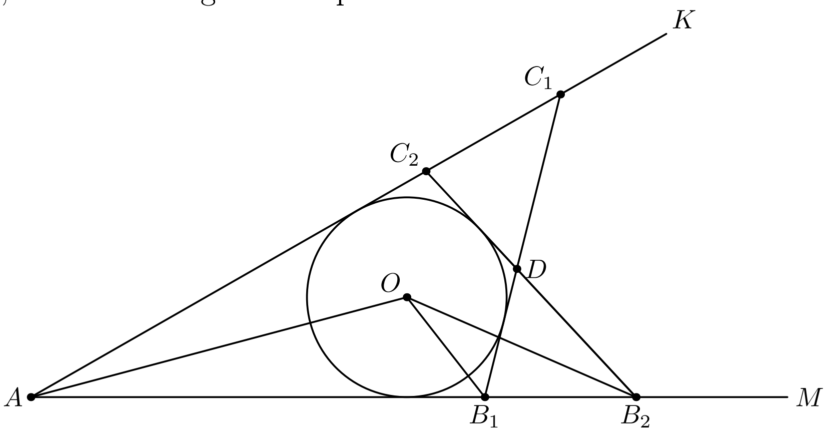

Fig.1

2. (B.Frenkin) (8) Given non-isosceles triangle ABC. Consider three segments passing through different vertices of this triangle and bisecting its perimeter. Are the lengths of these segments certainly different?

   Answer. Yes.

   Solution. Suppose for example that segments AA' and BB' are equal. Since the perimeters of triangles $AA'B$ and $AA'C$ are equal, we have $BA' = (AB + BC + CA)/2 - AB$. Similarly $AB' = (AB + BC + CA)/2 - AB$, and so triangles ABA' и BAB' are equal. Thus $\angle A = \angle B$, but this is impossible because triangle ABC is non-isosceles.

3. (D.Shnol) (8) The bisectors of trapezoid's angles form a quadrilateral with perpendicular diagonals. Prove that this trapezoid is isosceles.

   Solution. Let $KLMN$ be the quadrilateral formed by the bisectors (Fig. 3). Since AK, BK are the bisectors of adjacent trapezoid's angles, we have $\angle LKN = 90°$. Similarly $\angle LMN = 90°$. So $LK^2+KN^2 = LM^2+MN^2$. But by perpendicularity of the diagonals, $KL^2+MN^2 = KN^2+LM^2$. These two equalities yield that $KL = LM$ and $MN = NK$, thus $\angle NKM = \angle NMK$. But points K, M as common points of bisectors of adjacent angles, lie on the midline of the trapezoid, i.e. $KM \parallel AD$. So $\angle CAD = \angle BDA$ and the trapezoid is isosceles.

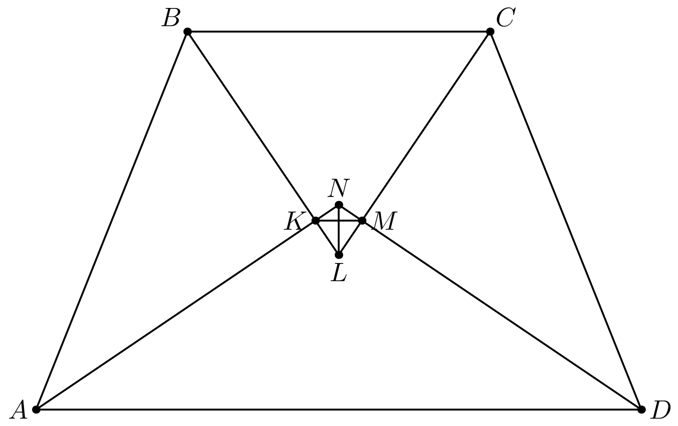

Fig.3

4. (D.Prokopenko) (8–9) Let P and Q be the common points of two circles. The ray with origin Q reflects from the first circle in points $A_1, A_2, \dots$ according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin Q reflects from the second circle in the points $B_1, B_2, \dots$ in the same manner. Points A_1, B_1 and P occurred to be collinear. Prove that all lines A_iB_i pass through P.

   Solution. When the rays reflect from the circles, we have $QA_1 = A_1A_2 = A_2A_3 = \cdots$ and $QB_1 = B_1B_2 = B_2B_3 = \cdots$. So $\angle(PQ, PA_1) = \angle(PA_1, PA_2) = \angle(PA_2, PA_3) = \cdots$ and $\angle(PQ, PB_1) = \angle(PB_1, PB_2) = \angle(PB_2, PB_3) = \cdots$ (for oriented angles). Also, since points A_1, B_1, P are collinear, we have $\angle(PQ, PA_1) = \angle(PQ, PB_1)$. Thus for any i we have $\angle(PA_{i-1}, PA_i) = \angle(PB_{i-1}, PB_i)$, and by induction A_i, B_i, P are collinear.

5. (D.Shnol) (8–9) Given triangle ABC. Point O is the center of the excircle touching the side BC. Point O_1 is the reflection of O in BC. Determine angle A if O_1 lies on the circumcircle of ABC.

   Solution. The condition yields that $\angle BOC = \angle BO_1C = \angle A$. On the other hand, $\angle BOC = 180^\circ - (180^\circ - \angle B)/2 - (180^\circ - \angle C)/2 = (180^\circ - \angle A)/2$. So $\angle A = 60^\circ$.

6. (B.Frenkin) (8–9) Find the locus of excenters of right triangles with given hypotenuse.

   Solution. Let ABC be a right triangle with hypotenuse AB, and I_a, I_b, I_c be its excenters (Fig. 6). Then $\angle AI_cB = \angle AI_aB = \angle AI_bB = 45^\circ$, and points I_a, I_b lie on the same side from line AB, I_c on the other side. So these three points lie on two circles c_1, c_2 passing through A, B, such that their arc AB is equal to 90°. When C runs a semicircle with diameter AB then each excenter runs a quarter of the circle.

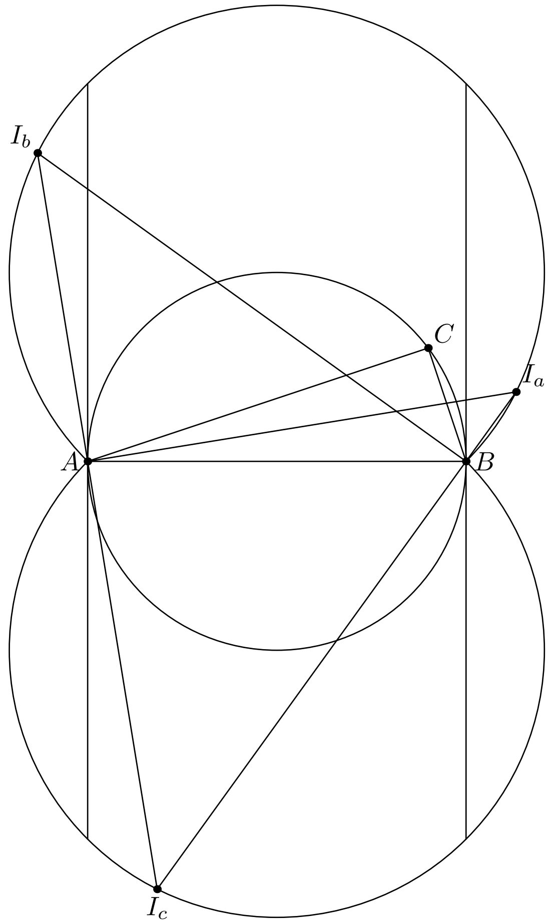

Fig.6

Namely, I_a runs the arc between B and the meet of the circle with l; I_b runs the arc between A and the meet of the circle with k; I_c runs the arc between the meets of the circle with k and l. When C runs the whole circle with diameter AB except points A, B, the excentres run the required locus, namely the arcs of c_1, c_2, lying outside the circle with diameter AB, except their ends A, B and their meets with k, l.

7. (V.Protasov) (8–9) Given triangle ABC. Points M, N are the projections of B and C to the bisectors of angles C and B respectively. Prove that line MN intersects sides AC and AB in their points of contact with the incircle of ABC.

   Solution. Let I be the incenter of ABC, P be the common point of MN and AC (Fig. 7). Points M, N lie on the circle with diameter BC, so $\angle MNB = \angle MCB = \angle ACI$. Hence C, I, P, N are concyclic and $\angle CPI = \angle CNI = 90°$. Thus, P is the touching point of AC with the incircle. For side AB the proof is similar.

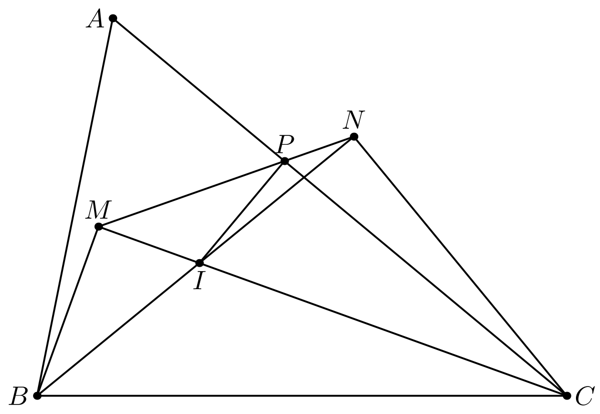

Fig.7

8. (S.Markelov) (8–10) Some polygon can be divided into two equal parts by three different ways. Is it certainly valid that this polygon has an axis or a center of symmetry?

   Answer. No, see Fig.8.

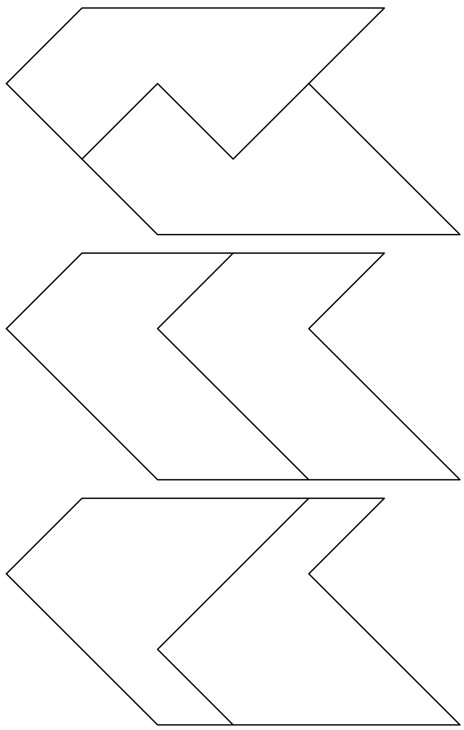

Fig.8

9. (V.A.Yasinsky) (8–11) Given n points on the plane, which are the vertices of a convex polygon, $n > 3$. There exist k regular triangles with the side equal to 1 and the vertices at the given points.

   a) Prove that $k < \frac{2}{3}n$.

   b) Construct the configuration with $k > 0,666n$.

   Solution. a) For any given point there exists a line passing through this point, such that all other given points lie on the same side from this line. This enables us to choose among all triangles having this point as a vertex, two "extreme"triangles (maybe coinciding), "left"and 'right'. We will call these two triangles "attached"to the given vertex.

   Lemma. Each triangle is attached at least three times.

Proof. Suppose that triangle ABC isn't "extremely left"for vertex C and isn't "extremely right"for vertex B (Fig.9).

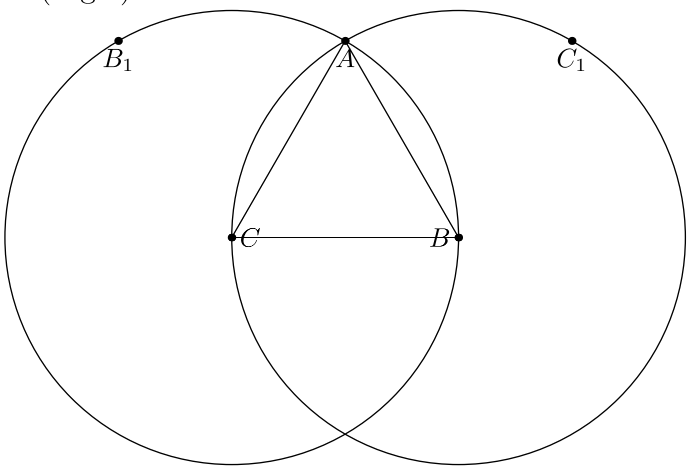

Fig.9

Then arcs AB_1 and AC_1 contain some of given points. But these points and A, B, C can't be vertices of a convex polygon. So ABC is attached by one of two indicated ways, i.e., it is "extremely left"for C or "extremely right"for B. Similarly it is "extremely left"for A or "extremely right"for C. Also it is "extremely left"for B or "extremely right"for A. Hence ABC is attached at least three times as required.

Suppose now that for n given points there exist k unit triangles. Since for any point there exist at most two attached triangles, $2n$ is the maximum number of attachments. Since each unit triangle is attached at least three times, $3k$ is the minimum number of attachments. Thus $3k \le 2n$ and $k \le \frac{2}{3}n$.

b) Consider the rhombus formed by two triangles. Rotating it around its obtuse-angled vertex we obtain m rhombuses.

If all rotation angles are less than $\pi/3$, then all vertices of obtained rhombuses form a convex polygon. Also we have $n = 3m + 1$, $k = 2m$, and for m sufficiently great we have $k > 0,666n$.

10. (F.Ivlev) (9) Let ABC be an acute triangle, CC_1 its bisector, O its circumcenter. The perpendicular from C to AB meets line OC_1 in a point lying on the circumcircle of AOB. Determine angle C.

    Solution. Let D be the common point of OC_1 and the perpendicular from C to AB. Since D lies on circle AOB and $AO = OB$, we have $\angle ADC_1 = \angle BDC_1$. So $AD/BD = AC_1/BC_1 = AC/BC$. On the other hand, $CD \perp AB$ implies $AC^2 + BD^2 = AD^2 + BC^2$. From these two relations $AC = AD$, i.e., D is the reflection of C in AB. But then CC_1 intersects the medial perpendicular of AB in the point symmetric to O (Fig.10). Since the medial perpendicular and the bisector meet on the circumcircle, AB bisects perpendicular radius. So $\angle C = 60°$.

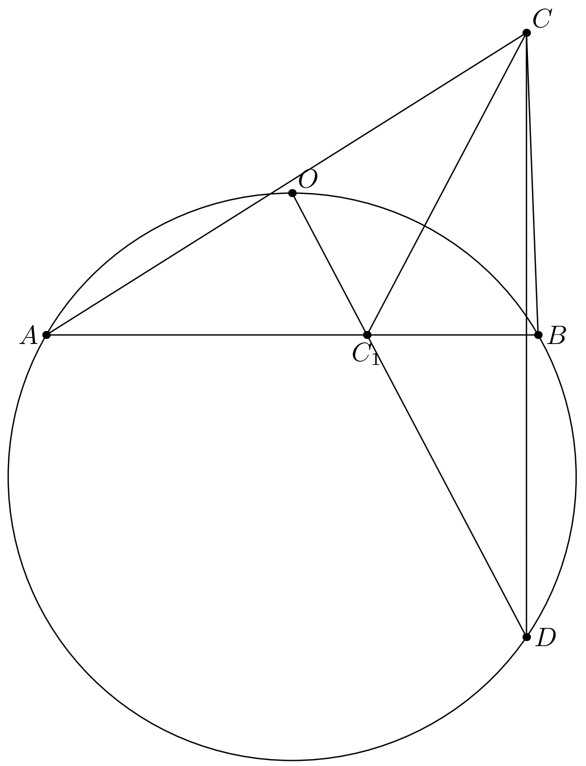

Fig.10

11. (A.Blinkov) (9) Given quadrilateral $ABCD$. The circumcircle of ABC is tangent to side CD, and the circumcircle of ACD is tangent to side AB. Prove that the length of diagonal AC is less than the distance between the midpoints of AB and CD.

    Solution. The condition implies $\angle BAC + \angle BCD = \angle ACD + \angle BAD = 180°$. Thus $\angle BCA = \angle CAD$, i.e. $AD \parallel BC$ and the segment between the midpoints of AB and CD is the medial line of the trapezoid and equals $(AD + BC)/2$. Also $\angle ACD = \angle ABC$ and $\angle BAC = \angle CDA$, so that triangles ABC and DCA are similar. Thus $AC^2 = AD \cdot BC$ and the assertion of the problem follows from Cauchi inequality.

12. (D.Prokopenko) (9–10) Let CL be a bisector of triangle ABC. Points A_1 and B_1 are the reflections of A and B in CL; points A_2 and B_2 are the reflections of A and B in L. Let O_1 and O_2 be the circumcenters of triangles AB_1B_2 and BA_1A_2 respectively. Prove that angles O_1CA and O_2CB are equal.

    Solution. The condition implies $CB_1/CA = CB/CA = BL/LA = B_2L/AL$, i.e., $B_1B_2 \parallel CL$. Similarly $A_1A_2 \parallel CL$. So $\angle AB_1B_2 = \angle BA_1A_2 = \angle C/2$. The reflection in CL transforms points B and A_1 to B_1 and A. Also it transforms A_2 to some point A'. We obtain $\angle A'AB_2 + \angle A'B_1B_2 = \angle A + \angle B + 2\angle C/2 = 180°$. Thus quadrilateral $AA'B_1B_2$ is cyclic and points O_1, O_2 are symmetric wrt CL (Fig.12).

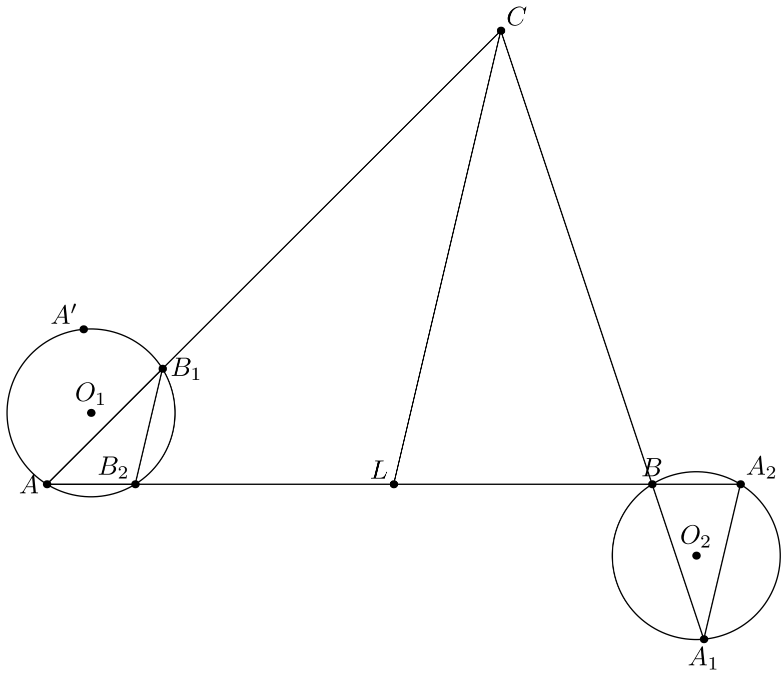

Fig.12

13. (A.Zaslavsky) (9–10) In triangle ABC, one has marked the incenter, the foot of altitude from vertex C and the center of the excircle tangent to side AB. After this, the triangle was erased. Restore it.

Solution. Incenter I and excenter I_c lie on the bisector CC' of angle C. If r and r_c are the inradius and the exradius then $CI/CI_c = C'I/C'I_c = r/r_c$. So for any point X lying on the circle with diameter CC', the ratio XI/XI_c is the same. As the foot H of altitude to AB lies on this circle, we have $HI/HI_c = CI/C/I_c = C'I/C'I_c$, i.e., HC' and HC are the bisectors of angle IHI_c (Fig.13). Constructing these bisectors, we restore point C and line AB. As $\angle IAI_c = \angle IBI_c = 90°$, points A, B lie on the circle with diameter II_c. Constructing this circle and its common points with line AB, we restore the triangle.

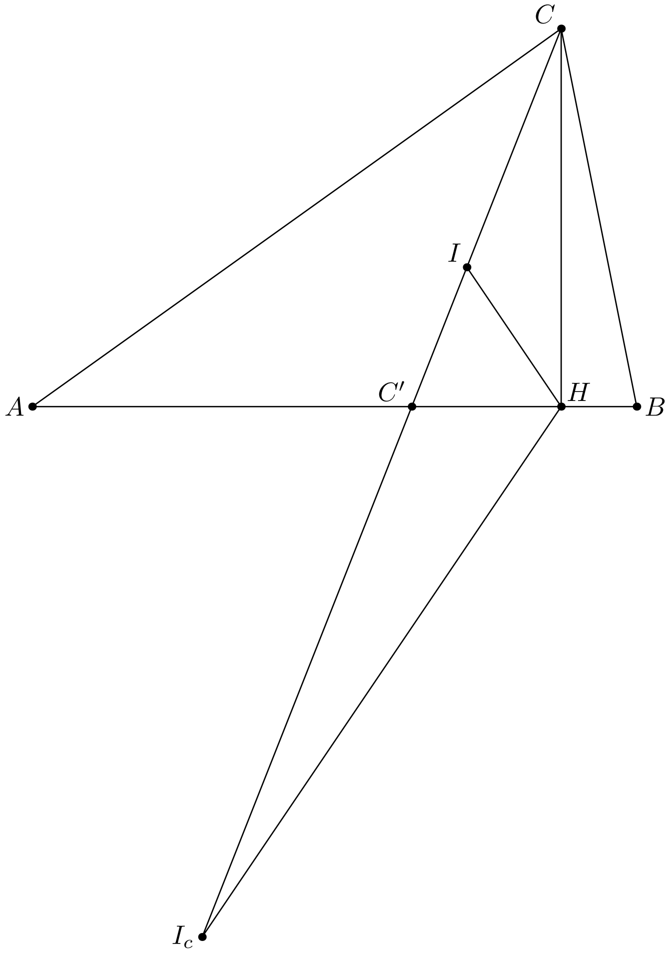

Fig.13

14. (V.Protasov) (9–10) Given triangle ABC of area 1. Let BM be the perpendicular from B to the bisector of angle C. Determine the area of triangle AMC.

    First solution. Let the line passing through B and parallel to AC meet the bisector of angle C in point N (Fig.14). Since $\angle BNC = \angle ACN = \angle BCN$, triangle BCN is isosceles and BM is its median. Thus $S_{AMC} = \frac{1}{2}S_{ANC} = \frac{1}{2}S_{ABC} = \frac{1}{2}$.

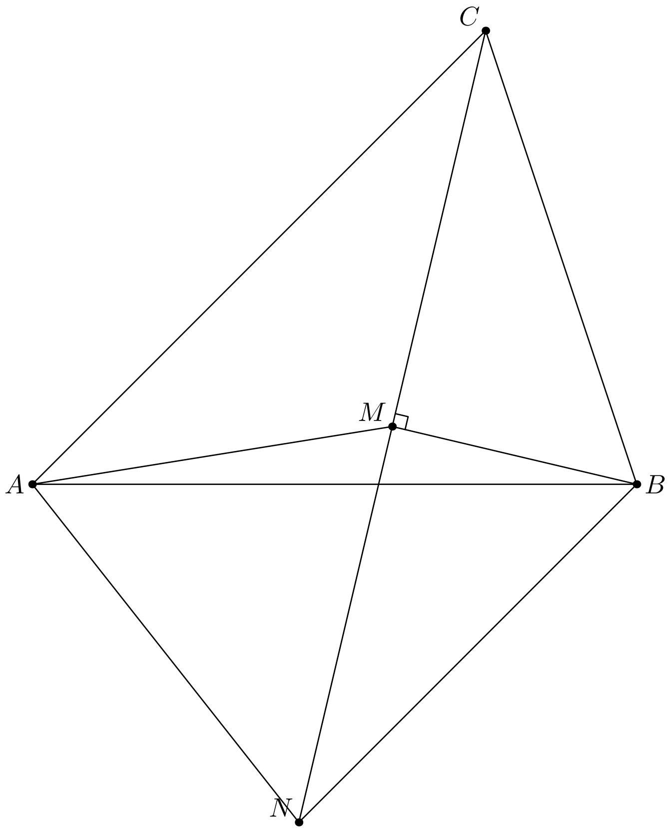

Fig.14

Second solution. Since $S_{AMC} = \frac{1}{2}AC \cdot CM \sin\frac{C}{2}$ and $CM = BC\cos\frac{C}{2}$, we have $S_{AMC} = \frac{1}{4}AC \cdot BC \sin C = \frac{S_{ABC}}{2} = \frac{1}{2}$.

15. (B.Frenkin) (9–10) Given a circle and a point C not lying on this circle. Consider all triangles ABC such that points A and B lie on the given circle. Prove that the triangle of maximal area is isosceles.

    Solution. Let C be the given point and A, B lie on the circle. If the tangent to the circle in A isn't parallel to CB, then moving point A, we can increase the distance from A to BC and the area of the triangle. Similarly the tangent at B is parallel to CA. So lines AC and BC are symmetric wrt the medial perpendicular to AB, and $AC = BC$ (Fig.15).

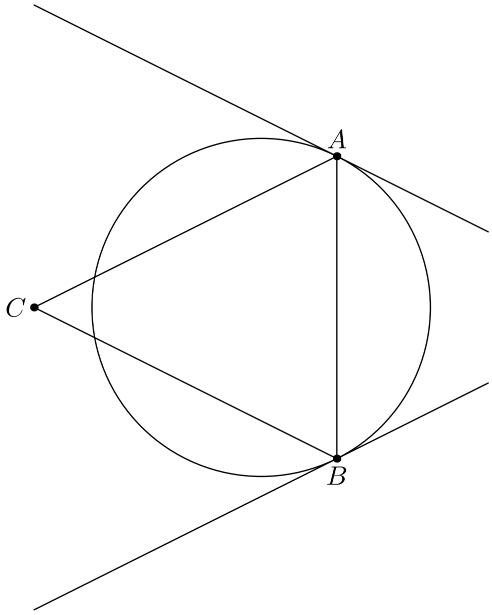

Fig.15

Note that the above argument does not depend on the location of the given point inside or outside the circle.

16. (A.Zaslavsky) (9–11) Three lines passing through point O form equal angles by pairs. Points A_1, A_2 on the first line and B_1, B_2 on the second line are such that the common point C_1 of A_1B_1 and A_2B_2 lies on the third line. Let C_2 be the common point of A_1B_2 and A_2B_1. Prove that angle C_1OC_2 is right.

    Solution. Let C_3 be the common point of lines OC_1 and A_2B_1 (Fig.16). Applying the Ceva and Menelaes theorems to triangle OA_2B_1 we obtain $C_2A_2/C_2B_1 = C_3A_2/C_3B_1 = OA_2OB_1$. So OC_2 is the external bisector of angle A_2OB_1, and $OC_2 \perp OC_1$.

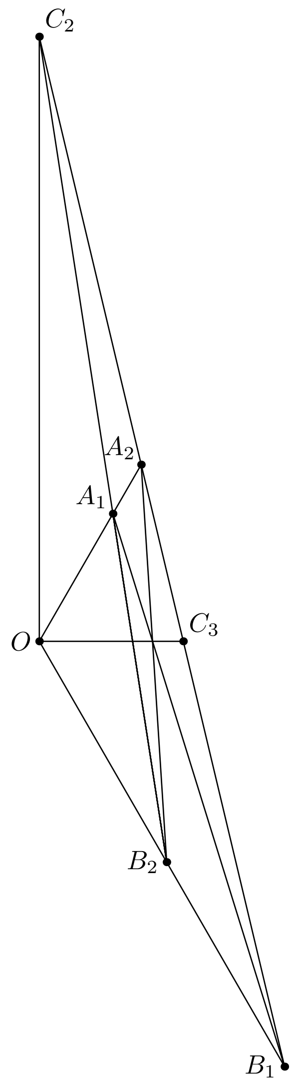

Fig.16

17. (A.Zaslavsky) (9–11) Given triangle ABC and two points X, Y not lying on its circumcircle. Let A_1, B_1, C_1 be the projections of X to BC, CA, AB, and A_2, B_2, C_2 be the projections of Y. Prove that the perpendiculars from A_1, B_1, C_1 to B_2C_2, C_2A_2, A_2B_2, respectively, concur iff line XY passes through the circumcenter of ABC.

    Solution. Let line XY pass through the circumcenter O. Fix point Y and move X along this line. The perpendiculars from A_1, B_1, C_1 to sidelines of $A_2B_2C_2$ move uniformly and remain self-parallel, so their common points move along some lines. When X coincides with O or Y, the three perpendiculars concur. Hence this is correct for all X.

    The above argument yields that for point Y fixed, the locus of points X such that the perpendiculars concur is line OY or the whole plane. Supposing the second case, take point C for X. Then A_1, B_1 coincide with C, and C_1 is the foot of the altitude from C. Since the three perpendiculars concur, we have $A_2B_2 \parallel AB$, and so Y lies on OC. Taking now another vertex for X, we obtain that Y coincides with O.

18. (B.Frenkin) (9–11) Given three parallel lines on the plane. Find the locus of incenters of triangles with vertices lying on these lines (a single vertex on each line).

Answer. The stripe whose bounds do not belong to the locus, are parallel to the given lines and are equidistant from the medium line and one of the extreme lines.

Solution. If we shift an arbitrary triangle whose vertices lie on the given lines by a vector parallel to these lines, its incenter is shifted by the same vector. So the desired locus is a stripe. Let us find its bounds.

Let b be the medium line and a, c the extreme lines. Let the vertex A, B, C lie on a, b, c respectively. Consider the diameter of the incircle, perpendicular to the given lines (Fig.18), and its endpoint nearest to a. This point lies nearer to a than the touching point of the incircle with AB, and thus it is nearer to a than b. Since another extreme point of the diameter lies nearer to a than c, the midpoint I of the diameter lies nearer to a than the line equidistant from b and c. Interchanging a and c in this argument, we obtain that I lies inside the stripe indicated in the answer.

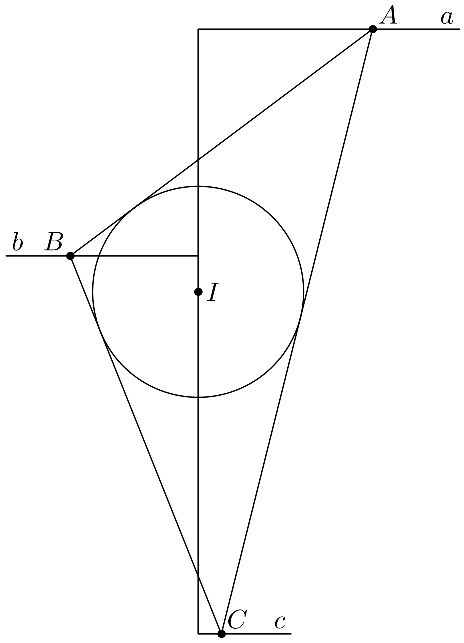

Fig.18

Now consider some triangle ABC with vertices on given lines. Move vertex B to make side AB perpendicular to the given lines. Now let point C tend to infinity. Then angles A and B of the triangle tend to right angles and point I tends to a vertex of the right isosceles triangle with hypotenuse AB. So I tends to the line equidistant from a and b. Similarly, starting from the same triangle, we can tend I to the line equidistant from b and c. Thus the locus of points I is the whole indicated stripe.

19. (B.Frenkin) (10–11) Given convex n-gon $A_1 \dots A_n$. Let $P_i\,(i = 1, \dots, n)$ be points on its boundary such that A_iP_i bisects the area of the polygon. All points P_i don't coincide with any vertex and lie on k sides of the n-gon. What is the maximal and the minimal value of k for each given n?

    Answer. The minimal value is 3, the maximal value is equal to $n - 1$ for n even and equal to n for n odd.

    Solution. Since segments A_iP_i bisect the area of the polygon, any two of them intersect. Let point P_i lie on side A_jA_{j+1}. Then points P_j and P_{j+1} lie on the boundary of the polygon in opposite directions from A_i, i.e., some of given points lie on three distinct sides of the polygon. Now let two vertices of the polygon be the vertices of a regular triangle and all other vertices lie near the third vertex of this triangle. Then all points P_i lie on three sides of the polygon.

If n is odd then it is evident that for a regular n-gon all P_i lie on different sides. Let $n = 2m$. Since segments A_mP_m and $A_{2m}P_{2m}$ intersect, points P_m and P_{2m} lie on the same side from diagonal A_mA_{2m}. There exist m sides of polygon lying on the other side from this diagonal, and point P_i can lie on one of these sides only if the corresponding vertex A_i lies between P_m and P_{2m}. But there exist at most $m - 1$ such vertices, and so some side doesn't contain points P_i.

Consider now the n-gon such that vertices $A_1, \dots, A_{n-2}$ coincide with the vertices of a regular $(n-1)$-gon, and vertices A_{n-1}, A_n lie near the remaining vertex of this $(n-1)$-gon. Points P_i lie on all sides of this polygon except $A_{n-1}A_n$.

20. (D.Prokopenko) (10–11) Suppose H and O are the orthocenter and the circumcenter of acute triangle ABC; AA_1, BB_1 and CC_1 are the altitudes of the triangle. Point C_2 is the reflection of C in A_1B_1. Prove that H, O, C_1 and C_2 are concyclic.

    Solution. As $CA_1/CA = CB_1/CB = \cos C$, triangles ABC and A_1B_1C are similar. So, since $\angle ACO = \angle BCC_1 = 90° - \angle B$, line CO contains an altitude of A_1B_1C, thus points C, O, C_2 are collinear (Fig.20). From similarity of ABC and A_1B_1C it follows as well that $CC_2/CC_1 = 2\cos C$. But it is known that $CH = 2CO\cos C$. Thus $CO \cdot CC_2 = CH \cdot CC_1$, which is equivalent to the assertion of the problem.

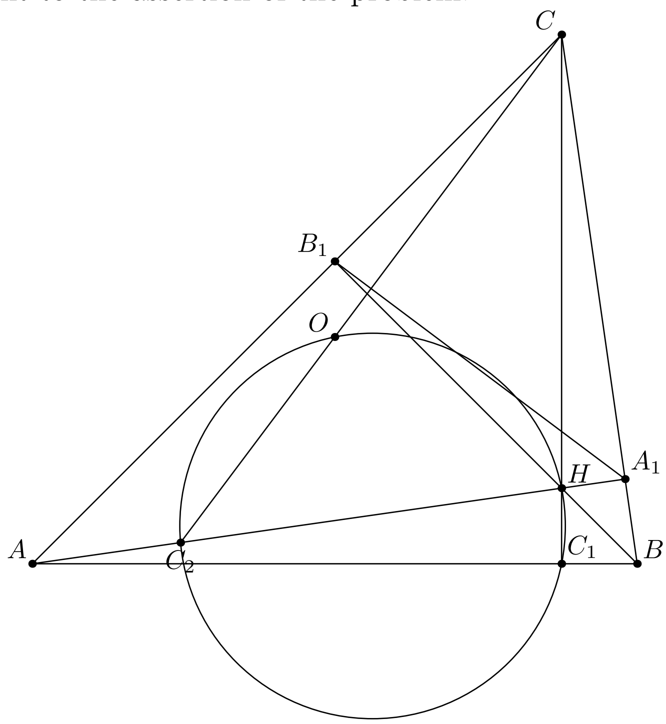

Fig.20

21. (F.Nilov) (10–11) The opposite sidelines of quadrilateral $ABCD$ intersect at points P and Q. Two lines passing through these points meet sides of $ABCD$ in four points which are vertices of a parallelogram. Prove that the center of this parallelogram lies on the line passing through the midpoints of diagonals of $ABCD$.

Solution. Using affine map, transform the parallelogram to a square and consider the coordinate system having the diagonals of this square for axes. Let the sides of the given quadrilateral intersect the coordinate axes in points $(\pm 1, 0)$, $(0, \pm 1)$, and the coordinates of P, Q be $(p, 0)$ and $(0, q)$ respectively. Then the equations of sidelines are $\frac{x}{p} \pm y = 1$, $\pm x + \frac{y}{q} = 1$; the vertices have coordinates $(\frac{p(q-1)}{pq-1}, \frac{q(p-1)}{pq-1})$, $(-\frac{p(q-1)}{pq+1}, \frac{q(p+1)}{pq+1})$, $(-\frac{p(q+1)}{pq-1}, -\frac{q(p+1)}{pq-1})$, $(\frac{p(q+1)}{pq+1}, -\frac{q(p-1)}{pq+1})$, and the midpoints of diagonals are collinear with the origin of the coordinate system.

22. (A.Zaslavsky) (10–11) Construct a quadrilateral which is inscribed and circumscribed, given the radii of the respective circles and the angle between the diagonals of the quadrilateral.

    Solution. If R, r are the radii of the circumcircle and the incircle and d is the distance between their centers O and I, then it is known that

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}.$$

    So we can define d and construct these circles. The diagonals of all quadrilaterals with the given circumcircle and incircle intersect at the same point L lying on line OI, and their midpoints lie on the circle with diameter OL. Furthermore the segment between midpoints of the diagonals passes through I, and its length is equal to $OL \sin\phi$, where ϕ is the given angle. Constructing the chord of this length passing through I, we find the midpoints of the diagonals and so the vertices of the quadrilateral.

23. (V.Protasov) (10–11) Is it true that for each n, the regular $2n$-gon is a projection of some polyhedron having not greater than $n+2$ faces?

    Answer. Yes.

    Solution. Apply to regular $2n$-gon $A_1 \dots A_{2n}$ the dilation wrt $A_n A_{2n}$ with coefficient $k > 1$ (Fig.23). Now bend the obtained polygon along line $A_n A_{2n}$ to project its vertices $B_1, \dots, B_{n-1}, B_{n+1}, \dots, B_{2n-1}$ to the vertices of the original regular polygon. Then all lines $B_i B_{2n-i}$ are parallel, and the polyhedron bounded by triangles $B_{n-1} B_n B_{n+1}$, $B_{2n-1} B_{2n} B_1$, trapezoids $B_i B_{i+1} B_{2n-i-1} B_{2n-i}$ and two halves of the $2n$-gon is the desired one.

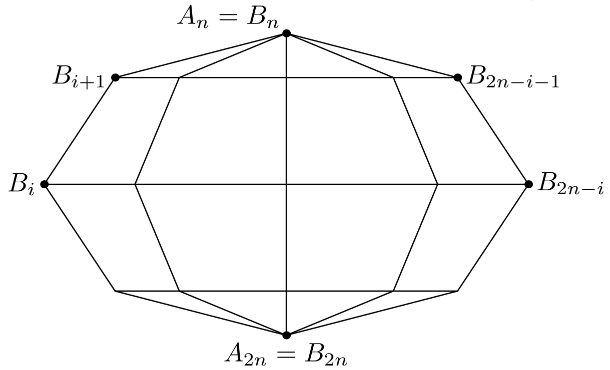

Fig.23

24. (F.Nilov) (11) A sphere is inscribed into a quadrangular pyramid. The point of contact of the sphere with the base of the pyramid is projected to the edges of the base. Prove that these projections are concyclic.

Solution. Let $ABCD$ be the base of the pyramid, P its touching point with the insphere, P' its touching point with the exsphere touching the base and the extensions of lateral faces. Then the ratio of distances from P to sidelines of the base is the same as for cotangents of halves of dihedral angles at the corresponding edges, and the ratio of distances from P' to sidelines is the same as for tangents of these half angles. Thus the lines joining each vertex of $ABCD$ with P and P' are symmetric wrt the bisector of the respective angle.

Now let K, L, M, N be the reflections of P wrt AB, BC, CD, DA. Since, for example, $BK = BP = BL$, the medial perpendicular to KL coincides with the bisector of angle KBL, i.e., line BP' (Fig.24). So P' is the circumcenter of $KLMN$. Using the homothety with center P and coefficient $1/2$ we obtain that the midpoint of PP' is the center of the circle passing through the projections of P, to the edges of the base.

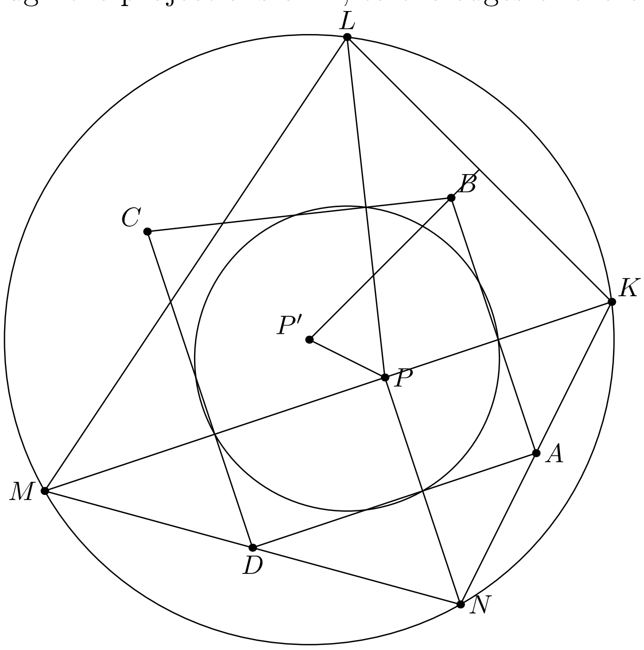

Fig.24