# V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
# THE CORRESPONDENCE ROUND

Below are the conditions for the correspondence round of the V Geometrical Olympiad in honour of I.F.Sharygin.

Participation in the olympiad is possible for pupils of 8–11 forms (these are four elder forms in Russian school). In the list of tasks presented below, each task is indicated by the numbers of forms for which it is intended. However the participants may solve the tasks for elder forms as well (the solutions of problems for younger forms will not be considered).

Your work containing the solutions for the tasks (in Russian or English) must be sent, at the latest, April 1, 2009. We recommend to send the work by e-mail to geomolymp@mccme.ru in pdf, doc or jpg files. To avoid the loss of your work, please maintain the following rules.

*1. Each work must be sent by a separate message.*

*2. If your work is contained in several files send it as an archive.*

*3. In the subject of the message, write "The work for Sharygin olympiad", and present the following information in the text:*

*- last name, first name;*

*- post address, phone number, E-mail;*

*- the current number of your form;*

*- the number and the address of your school;*

*- full names of your teachers in maths and/or of instructors of your math circle.*

If you can't send your work by e-mail, please send it through the post to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of I.F.Sharygin.* In the title page present the information indicated in the item 3 above.

Please write down each task from a new page: first write down the condition, and next, the solution. Present your solution in detail, including all essential arguments and calculations, with exact figures. The solution of a computational task must be completed by a distinctly presented answer. Please be accurate: you are interested in good understanding and correct estimating of your work!

If your solution depends on some well-known theorem or a fact from a standard textbook, you may simply refer to it (to explain which theorem or fact is used). But if you use a fact not contained in the curriculum, you have to prove it (or to indicate the source).

Your work will be thoroughly examined, and you will receive the answer, at the latest, in the middle of May 2009. The winners of the correspondence round will be invited to the final round which will be held in summer 2009 in Dubna (near Moscow).

1. (8) Points $B_1$ and $B_2$ lie on ray $AM$, and points $C_1$ and $C_2$ lie on ray $AK$. The circle with center $O$ is inscribed into triangles $AB_1C_1$ and $AB_2C_2$. Prove that the angles $B_1OB_2$ and $C_1OC_2$ are equal.

2. (8) Given nonisosceles triangle $ABC$. Consider three segments passing through different vertices of this triangle and bisecting its perimeter. Are the lengths of these segments certainly different?

3. (8) The bisectors of trapezoid's angles form a quadrilateral with perpendicular diagonals. Prove that this trapezoid is isosceles.

4. (8–9) Let $P$ and $Q$ be the common points of two circles. The ray with origin $Q$ reflects from the first circle in points $A_1, A_2, \ldots$ according to the rule "the angle of incidence is equal to the angle of reflection". Another ray with origin $Q$ reflects from the second circle in the points $B_1, B_2, \ldots$ in the same manner. Points $A_1$, $B_1$ and $P$ occurred to be collinear. Prove that all lines $A_iB_i$ pass through $P$.

5. (8–9) Given triangle $ABC$. Point $O$ is the center of the excircle touching the side $BC$. Point $O_1$ is the reflection of $O$ in $BC$. Determine angle $A$ if $O_1$ lies on the circumcircle of $ABC$.

6. (8–9) Find the locus of excenters of right triangles with given hypotenuse.

7. (8–9) Given triangle $ABC$. Points $M$, $N$ are the projections of $B$ and $C$ to the bisectors of angles $C$ and $B$ respectively. Prove that line $MN$ intersects sides $AC$ and $AB$ in their points of contact with the incircle of $ABC$.

8. (8–10) Some polygon can be divided into two equal parts by three different ways. Is it certainly valid that this polygon has an axis or a center of symmetry?

9. (8–11) Given $n$ points on the plane, which are the vertices of a convex polygon, $n > 3$. There exists $k$ regular triangles with the side equal to 1 and the vertices at the given points.

   a) Prove that $k < \frac{2}{3}n$.

   b) Construct the configuration with $k > 0,666n$.

10. (9) Let $ABC$ be an acute triangle, $CC_1$ its bisector, $O$ its circumcenter. The perpendicular from $C$ to $AB$ meets line $OC_1$ in a point lying on the circumcircle of $AOB$. Determine angle $C$.

11. (9) Given quadrilateral $ABCD$. The circumcircle of $ABC$ is tangent to side $CD$, and the circumcircle of $ACD$ is tangent to side $AB$. Prove that the length of diagonal $AC$ is less than the distance between the midpoints of $AB$ and $CD$.

12. (9–10) Let $CL$ be a bisector of triangle $ABC$. Points $A_1$ and $B_1$ are the reflections of $A$ and $B$ in $CL$; points $A_2$ and $B_2$ are the reflections of $A$ and $B$ in $L$. Let $O_1$ and $O_2$ be the circumcenters of triangles $AB_1B_2$ and $BA_1A_2$ respectively. Prove that angles $O_1CA$ and $O_2CB$ are equal.

13. (9–10) In triangle $ABC$, one has marked the incenter, the foot of altitude from vertex $C$ and the center of the excircle tangent to side $AB$. After this, the triangle was erased. Restore it.

14. (9–10) Given triangle $ABC$ of area 1. Let $BM$ be the perpendicular from $B$ to the bisecor of angle $C$. Determine the area of triangle $AMC$.

15. (9–10) Given a circle and a point $C$ not lying on this circle. Consider all triangles $ABC$ such that points $A$ and $B$ lie on the given circle. Prove that the triangle of maximal area is isosceles.

16. (9–11) Three lines passing through point $O$ form equal angles by pairs. Points $A_1$, $A_2$ on the first line and $B_1$, $B_2$ on the second line are such that the common point $C_1$ of $A_1B_1$ and $A_2B_2$ lies on the third line. Let $C_2$ be the common point of $A_1B_2$ and $A_2B_1$. Prove that angle $C_1OC_2$ is right.

17. (9–11) Given triangle $ABC$ and two points $X$, $Y$ not lying on its circumcircle. Let $A_1$, $B_1$, $C_1$ be the projections of $X$ to $BC$, $CA$, $AB$, and $A_2$, $B_2$, $C_2$ be the projections of $Y$. Prove that the perpendiculars from $A_1$, $B_1$, $C_1$ to $B_2C_2$, $C_2A_2$, $A_2B_2$, respectively, concur iff line $XY$ passes through the circumcenter of $ABC$.

18. (9–11) Given three parallel lines on the plane. Find the locus of incenters of triangles with vertices lying on these lines (a single vertex on each line).

19. (10–11) Given convex $n$-gon $A_1 \dots A_n$. Let $P_i$ $(i = 1, \dots, n)$ be such points on its boundary that $A_iP_i$ bisects the area of polygon. All points $P_i$ don't coincide with any vertex and lie on $k$ sides of $n$-gon. What is the maximal and the minimal value of $k$ for each given $n$?

20. (10–11) Suppose $H$ and $O$ are the orthocenter and the circumcenter of acute triangle $ABC$; $AA_1$, $BB_1$ and $CC_1$ are the altitudes of the triangle. Point $C_2$ is the reflection of $C$ in $A_1B_1$. Prove that $H$, $O$, $C_1$ and $C_2$ are concyclic.

21. (10–11) The opposite sidelines of quadrilateral $ABCD$ intersect at points $P$ and $Q$. Two lines passing through these points meet the side of $ABCD$ in four points which are the vertices of a parallelogram. Prove that the center of this parallelogram lies on the line passing through the midpoints of diagonals of $ABCD$.

22. (10–11) Construct a quadrilateral which is inscribed and circumscribed, given the radii of the respective circles and the angle between the diagonals of quadrilateral.

23. (10–11) Is it true that for each $n$, the regular $2n$-gon is a projection of some polyhedron having not greater than $n + 2$ faces?

24. (11) A sphere is inscribed into a quadrangular pyramid. The point of contact of the sphere with the base of the pyramid is projected to the edges of the base. Prove that these projections are concyclic.