V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 8 form. Solutions.

1. (A.Blinkov, Y.Blinkov) Minor base BC of trapezoid $ABCD$ is equal to side AB, and diagonal AC is equal to base AD. The line passing through B and parallel to AC intersects line DC in point M. Prove that AM is the bisector of angle BAC.

   First solution. We have $\angle BMC = \angle ACD = \angle CDA = \angle BCM$ (first and third equality follow from parallelism of BM and AC, BC and AD; second equality follows from $AC = AD$). Thus, $BM = BC = AB$, and $\angle BAM = \angle BMA = \angle MAC$ (fig.8.1).

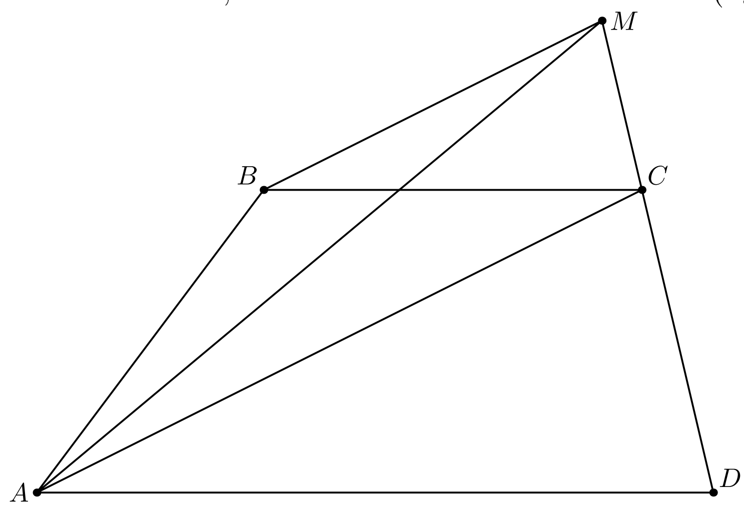

Fig.8.1

   Second solution. Let point P lie on the extension of side AB (beyond point B), and point K lie on the extension of diagonal AC (beyond point C). Then $\angle MCK = \angle ACD = \angle ADC = \angle BCM$, i.e CM is the bisector of angle BCK. Since AC bisects angle BAD and $BM \parallel AC$, then BM is the bisector of angle PBC. Thus M is the common point of two external bisectors of triangle ABC, therefore AM is the bisector of angle BAC.

2. (A.Blinkov) A cyclic quadrilateral is divided into four quadrilaterals by two lines passing through its inner point. Three of these quadrilaterals are cyclic with equal circumradii. Prove that the fourth part also is cyclic quadrilateral and its circumradius is the same.

   Solution. Let the parts adjacent to vertices A, B, C of cyclic quadrilateral $ABCD$ be cyclic quadrilaterals. Since angles A and C are opposite to equal angles in point of division L we have $\angle A = \angle C = 90°$. So two dividing lines are perpendicular. Thus angle B is also right and $ABCD$ is a rectangle. So the fourth quadrilateral is cyclic. Now the angles corresponding to arcs AL, BL, CL are equal, and since the radii of these circles also are equal, we have $AL = BL = CL$. So L is the center of the rectangle and the fourth circle has the same radius.

3. (A.Akopjan, K.Savenkov) Let AH_a and BH_b be the altitudes of triangle ABC. Points P and Q are the projections of H_a to AB and AC. Prove that line PQ bisects segment H_aH_b.

   Solution. Let CH_c be the third altitude of ABC. Then $\angle H_aH_cB = \angle H_bH_cA = \angle C$ because quadrilaterals CBH_cH_b and CAH_cH_a are cyclic. So the reflection of H_a in AB

lies on H_bH_c. Similarly the reflection of H_a in AC also lies on this line. Thus P and Q lie on the medial line of triangle $H_aH_bH_c$ (fig.8.3).

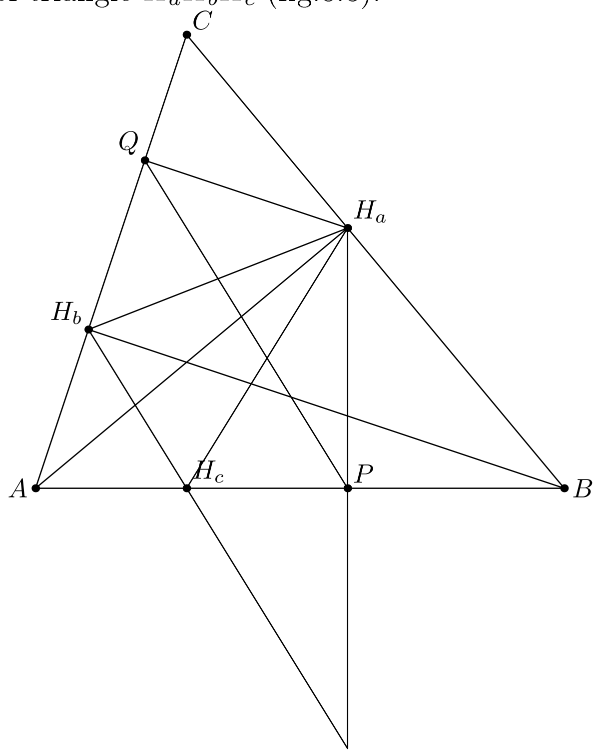

Fig.8.3

4. (N.Beluhov) Given is $\triangle ABC$ such that $\angle A = 57°$, $\angle B = 61°$ and $\angle C = 62°$. Which segment is longer: the angle bisector through A or the median through B?

   First solution. Let K be the midpoint of arc ABC in the circumcircle of ABC. Let also the circumcenter of the triangle be O, and AL and BM be the angle bisector and the median. Define $AL \cap CK = N$ and let AH be an altitude in $\triangle AKC$. Since $\angle A < \angle C$, B lies inside arc KC, therefore N lies inside segment AL and $AL > AN > AH$. Moreover $AH > KM$ as altitudes from a smaller and a greater angle in $\triangle AKC$. Finally, $KM = MO + OK = MO + OB > MB$, and the problem is solved: the angle bisector is longer (fig.8.4).

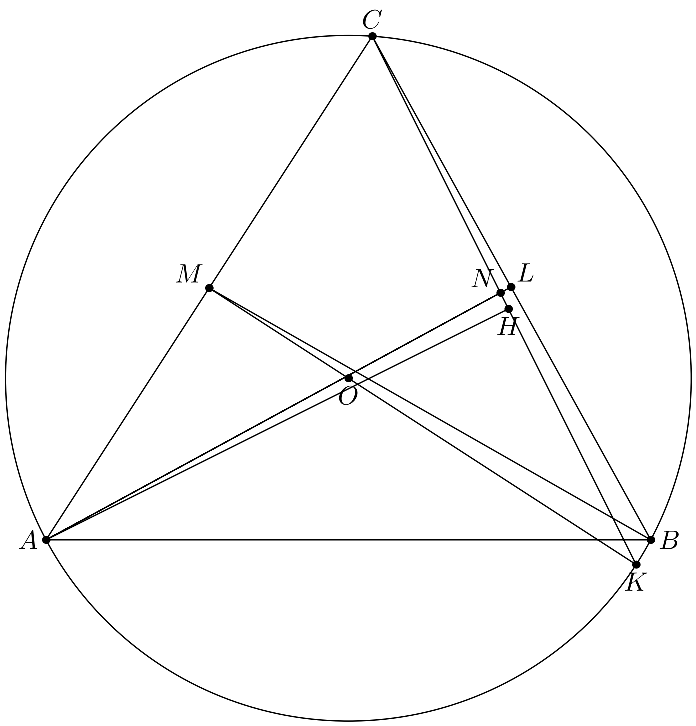

Fig.8.4

Second solution. Since $AB > BC$, we have $\angle MBC > 30°$. Construct the altitude AH and the perpendicular MK from M to BC. We have $AL > AH = 2MK > BM$, because $\sin \angle BMK = \frac{MK}{BM} > \frac{1}{2}$.

Third solution. (K.Ivanov, Moscow). Consider regular triangle ABC'. Since ray BC' lies inside angle ABC, we have that the bisector of angle A is longer that the altitude of regular triangle. In the other hand let M, N be the midpoints of AC and AC' respectively. Since ray AC lies inside angle $C'AB$, we have $\angle BMN > \angle BMA$. But $\angle BMA > 90°$ because $AB > BC$. Thus $BN > BM$ and the bisector of angle A is longer that the median from B.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 8 form. Solutions.

5. (V.Protasov) Given triangle ABC. Point M is the projection of vertex B to bisector of angle C. K is the touching point of the incircle with side BC. Find angle MKB if $\angle BAC = \alpha$

   Solution. Let I be the incenter of ABC. Then quadrilateral $BMIK$ is cyclic because $\angle BMI = \angle BKI = 90^\circ$ (fig.8.5). Thus $\angle MKB = \angle MIB = \angle IBC + \angle ICB = \frac{\angle B + \angle C}{2} = 90^\circ - \frac{\alpha}{2}$.

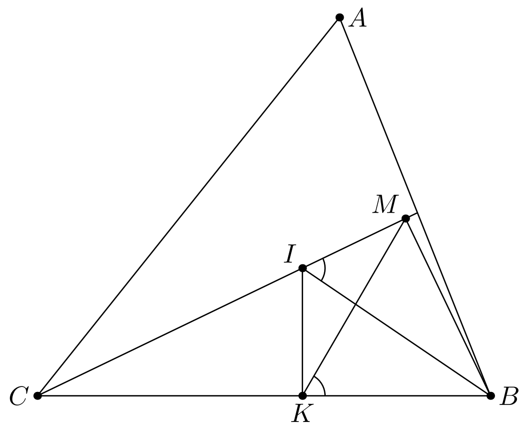

Fig.8.5

6. (S.Markelov) Can four equal polygons be placed on the plane in such a way that any two of them don't have common interior points, but have a common boundary segment?

   Solution. Yes, see fig.8.6.

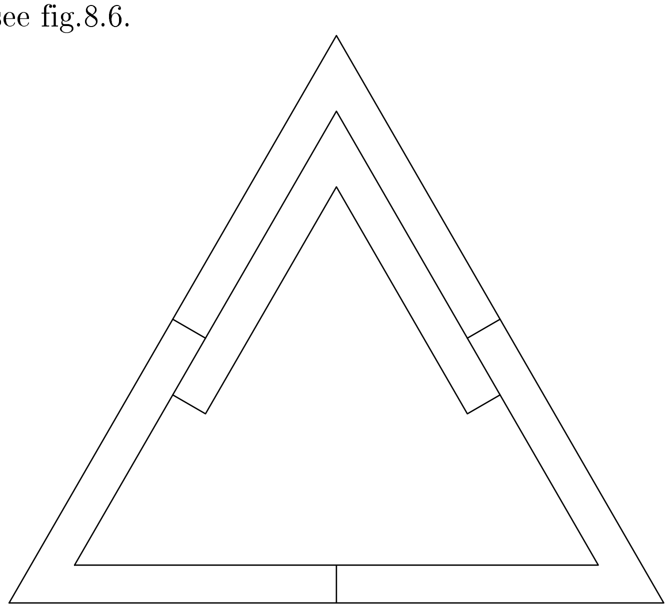

Fig.8.6

7. (D.Prokopenko) Let s be the circumcircle of triangle ABC, L and W be common points of angle's A bisector with side BC and s respectively, O be the circumcenter of triangle ACL. Restore triangle ABC, if circle s and points W and O are given.

Solution. Let O' be the circumcenter of ABC. Then lines $O'O$ and $O'W$ are perpendicular to sides AC and BC, so the directions of these sides are known. Also $\angle COL = 2\angle CAL = 2\angle LCW$, thus $\angle OCW = 90°$ (fig.8.7). Therefore C is the common point of s and the circle with diameter OW.

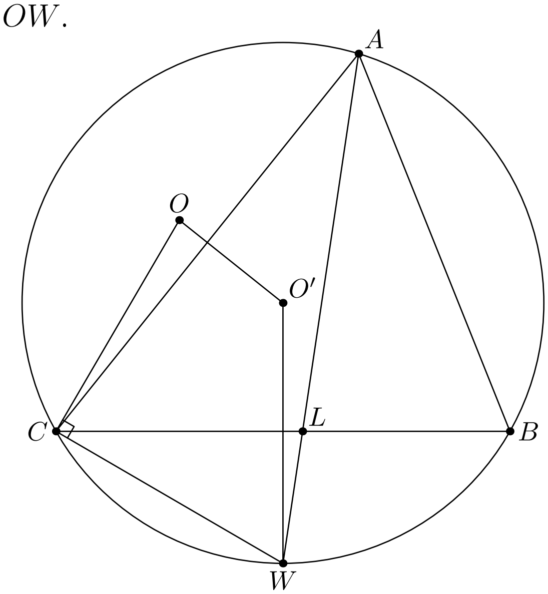

Fig.8.7

8. (N.Beluhov) A triangle ABC is given, in which the segment BC touches the incircle and the corresponding excircle in points M and N. If $\angle BAC = 2\angle MAN$, show that $BC = 2MN$.

   Solution. We may assume that $AB > AC$, and therefore the points B, N, M, C lie on the line in this order. We will use the following well-known

   Lemma. Let K be the midpoint of AB, and I and J be the incenter and the excenter opposite to A. Then $AN \parallel IK$ and $AM \parallel JK$.

   Now the lemma shows that the original condition is equivalent to $\angle IKJ = 180 - \alpha/2$. We will show first that if $BC = 2MN$ then this is true. In this case, since the midpoints of BC and MN coincide, we have that M and N are midpoints of KC and KB, and therefore, IM and NJ are perpendicular bisectors of KC and KB. Thus triangles IKC and JKB are isosceles, and $\angle JKB = 90 - \beta/2, \angle IKC = \gamma/2$, yielding the claim (fig.8.8).

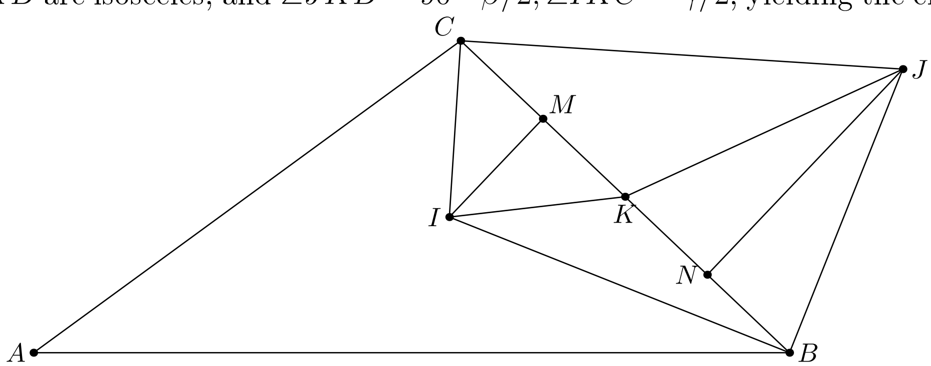

Fig.8.8

Now, consider the circle $(BICJ)$. Given α, we see that IJ is determined as a diameter, and BC as an arc constituting angle $90 + \alpha/2$. When the chord BC runs along the circle, its midpoint K runs along a smaller circle. In the same time the locus of the points K'

such that $\angle IKJ = 180 - \alpha/2$, consists of two arcs of circles with endpoints I and J. Obviously, these loci intersect in four points, symmetric to each other with respect to IJ and its perpendicular bisector, thus corresponding to four equal quadrilaterals $BICJ$. So this quadrilateral is completely determined by the condition $\angle IKJ = 180 - \alpha/2$. But the one obtained when $BC = 2MN$ satisfies this condition, hence the claim.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 9 form. Solutions.

1. (A.Blinkov, Y.Blinkov) The midpoint of triangle's side and the base of the altitude to this side are symmetric wrt the touching point of this side with the incircle. Prove that this side equals one third of triangle's perimeter

   First solution. Let a, b be the lengths of two sides, and the altitude divide the third side into segments with lengths x, y (if the base of the altitude lies out of the side then one of these lengths is negative). By the Pythagorean theorem $x^2 - y^2 = a^2 - b^2$. But the touching point divides the side into segments with lengths $p - a$ and $p - b$. So the condition of the problem is equivalent to $x - y = 2(a - b)$. Dividing the first equality by the second one we obtain that $x + y = (a + b)/2 = 2p/3$.

   Second solution. Let c be the side in question, then $r/r_c = (p - c)/p$. Let K and P be the touching points of this side with the incircle and the excircle, I and Q be the centers of these circles. It is known that the midpoint of altitude CH lies on line IP. Using similarity of two pairs of triangles we obtain that $r = h/3$, $r_c = h$ (fig.9.1). From the first equality we obtain the assertion of the problem.

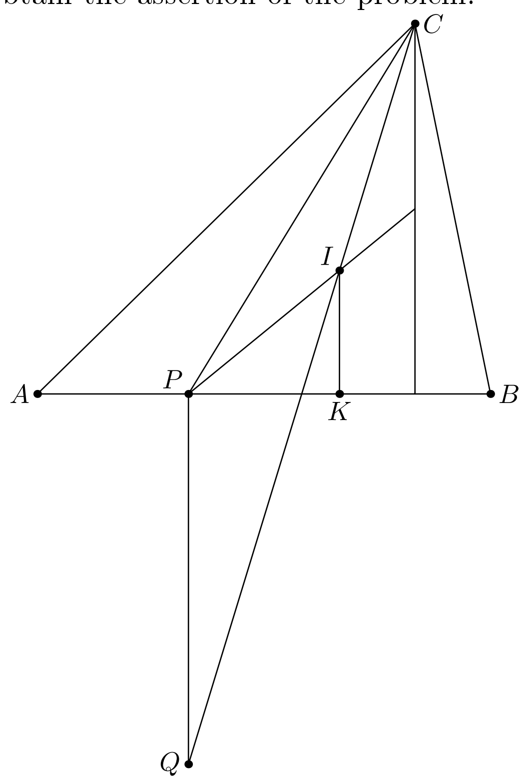

Fig.9.1

2. (O.Musin) Given a convex quadrilateral $ABCD$. Let R_a, R_b, R_c and R_d be the circumradii of triangles DAB, ABC, BCD, CDA. Prove that inequality $R_a < R_b < R_c < R_d$ is equivalent to

$$180^\circ - \angle CDB < \angle CAB < \angle CDB.$$

Solution. Let the angles of the quadrilateral satisfy the given inequality. Then $\sin\angle CAB > \sin\angle CDB$ and so $R_b < R_c$. Since angle CDB is obtuse, this implies that point A lies out of the circle CDB, thus $\angle CAD < \angle CBD$. As these angles are both acute, we have $\sin\angle CAD < \sin\angle CBD$ and $R_c < R_d$. Moreover $\angle ACB < \angle ADB < 90^\circ$, so $R_a < R_b$.

Conversely, from $R_b < R_c$ it follows that angle CAB lies between angles CDB and $180^\circ - \angle CDB$. If angle CDB is acute, we have $\angle ABD < \angle ACD$, and since $R_a < R_d$ then $\angle ABD > 180^\circ - \angle ACD$. But in this case we obtain by repeating previous argument that $R_b < R_a < R_d < R_c$.

3. (I.Bogdanov) Quadrilateral $ABCD$ is circumscribed, rays BA and CD intersect in point E, rays BC and AD intersect in point F. The incircle of the triangle formed by lines AB, CD and the bisector of angle B, touches AB in point K, and the incircle of the triangle formed by lines AD, BC and the bisector of angle B, touches BC in point L. Prove that lines KL, AC and EF concur.

Solution. Let the incircle of $ABCD$ touch sides AB and BC in points U and V. Then we have

$$(EB; KU) = \frac{EK}{BK} : \frac{EU}{BU} = \frac{\operatorname{ctg}\frac{\angle BEC}{2}}{\operatorname{ctg}\frac{\angle B}{4}} : \frac{\operatorname{ctg}\frac{\angle BEC}{2}}{\operatorname{ctg}\frac{\angle B}{2}} = (FB; LV).$$

This means that lines KL, EF, UV concur. Similarly lines AC, EF, UV concur (fig.9.3).

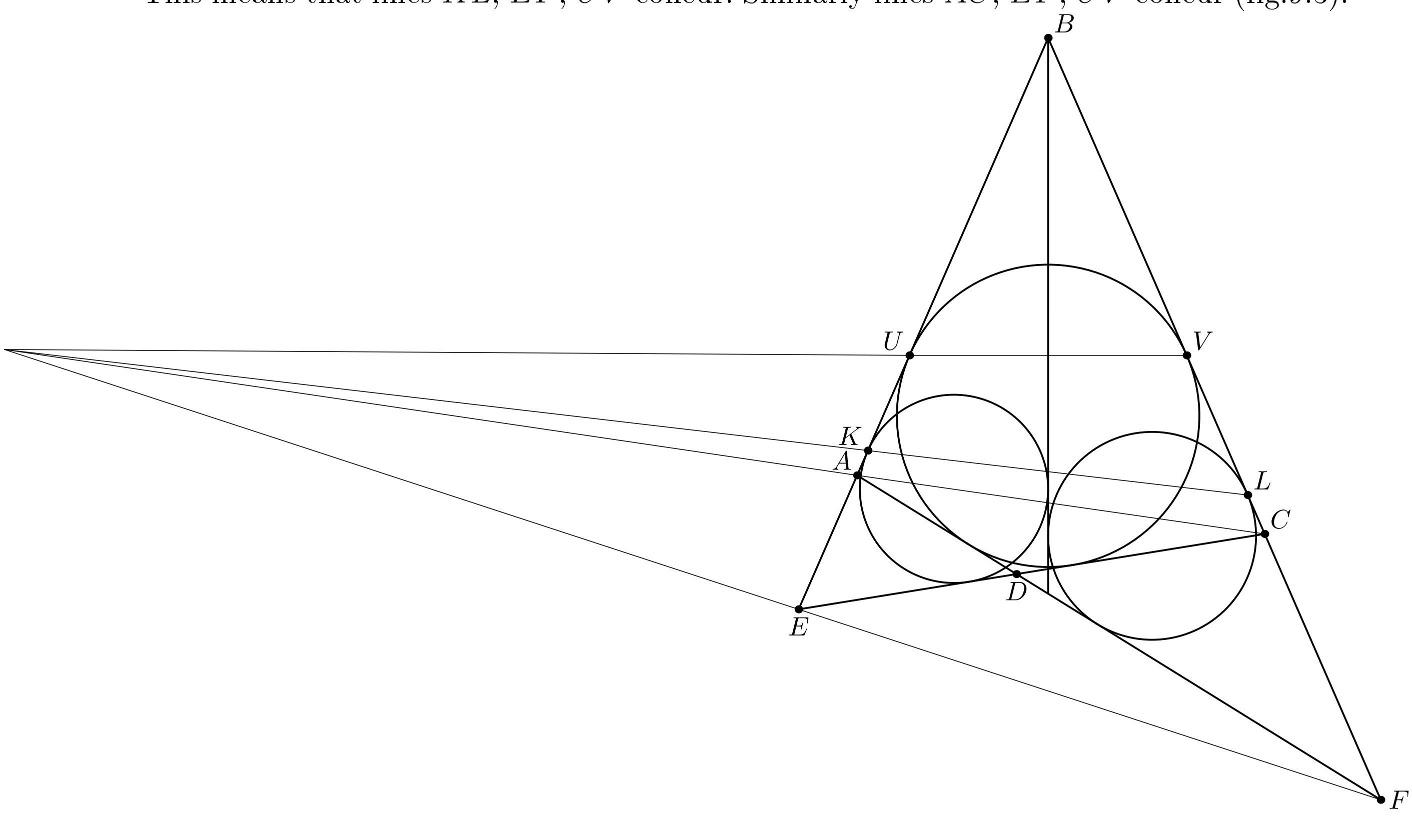

Fig.9.3

4. (N.Beluhov) Given regular 17-gon $A_1 \dots A_{17}$. Prove that two triangles formed by lines A_1A_4, A_2A_{10}, $A_{13}A_{14}$ and A_2A_3, A_4A_6 $A_{14}A_{15}$ are equal.

Solution. Firstly note that $A_1A_4 \parallel A_2A_3$, $A_2A_{10} \parallel A_{14}A_{15}$, $A_{13}A_{14} \parallel A_4A_6$. So we have to prove that given triangles are central symmetric.

Let A, B, C, D, E, F be the midpoints of A_1A_2, A_3A_4, A_4A_{13}, A_6A_{14}, $A_{10}A_{14}$, $A_{15}A_2$ respectively. Lines BC, DE, FA as medial lines of three triangles are parallel to $A_3A_{13} \parallel A_6A_{10} \parallel A_1A_{15}$. Lines AD, BE, CF as axes of three isosceles trapezoids concur at the center of 17-gon. By dual Pappus theorem AB, CD, EF concur at some point P (fig.9.4). But these lines are the medial lines of three strips formed by parallel sidelines of given triangles. Therefore these triangles are symmetric wrt P.

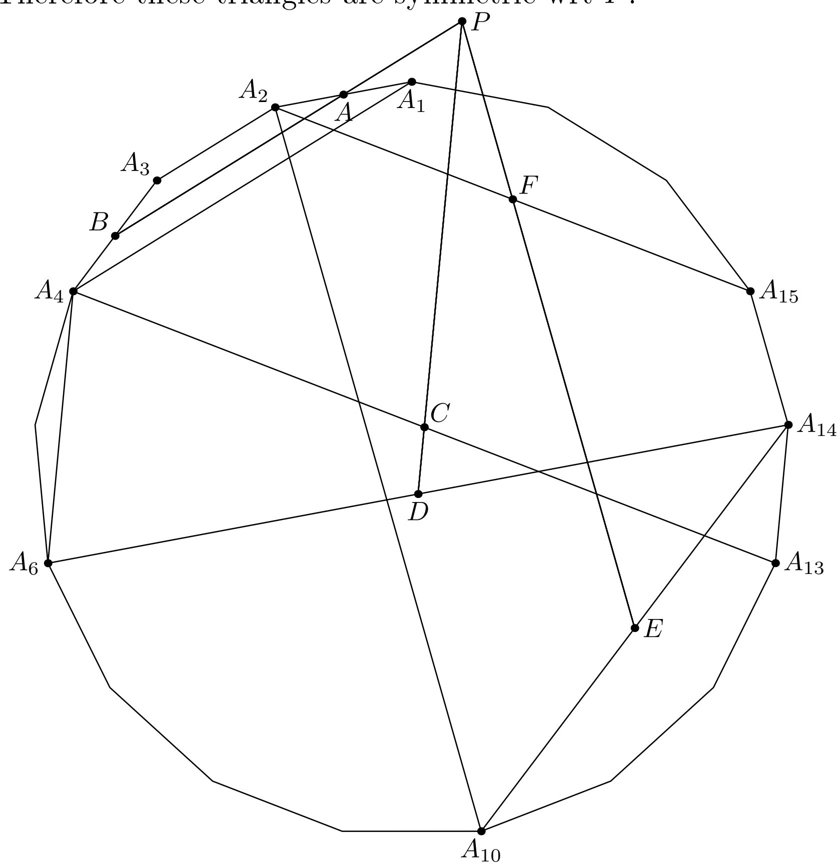

Fig.9.4

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 9 form. Solutions.

5. (B.Frenkin) Let n points lie on the circle. Exactly half of triangles formed by these points are acute-angled. Find all possible n.

   Answer. $n = 4$ or $n = 5$.

   Solution. It is evident that $n > 3$. Consider any quadrilateral formed by marked points. If the center of the circle lies inside this quadrilateral but not on its diagonal (call such quadrilateral "good"), then exactly two of four triangles formed by the vertices of the quadrilateral are acute-angled. In other cases less than two triangles are acute-angled. Therefore the condition of the problem is true only when all quadrilaterals are good. If $n = 4$ or $n = 5$ this is possible (consider for example the vertices of a regular pentagon).

   Now let $n > 5$. Consider one of marked points A and the diameter AA'. If point A' also is marked then the quadrilateral formed by A, A' and any two of remaining points isn't good. Otherwise there exist three marked points lying on the same side from AA'. The quadrilateral formed by these points and A isn't good.

6. (A.Akopjan) Given triangle ABC such that $AB - BC = \frac{AC}{\sqrt{2}}$. Let M be the midpoint of AC, and N be the base of the bisector from B. Prove that

$$\angle BMC + \angle BNC = 90^\circ.$$

   Solution. Let C' be the reflection of C in BN. Then $AC' = AB - BC$ and by condition $AM/AC' = AC'/AC$. Thus triangles $AC'M$ and ACC' are similar and $\angle AC'M = \angle C'CA = 90^\circ - \angle BNC$. Furthermore using the formula for a median we obtain that $BM^2 = AB \cdot BC$, so $BC'/BM = BM/BA$. Therefore triangles $BC'M$ and BMA are also similar and $\angle BMC' = \angle BAM$. Finally $\angle BMC = 180^\circ - \angle BMC - \angle C'MA = \angle MC'A$ q.e.d. (fig.9.6).

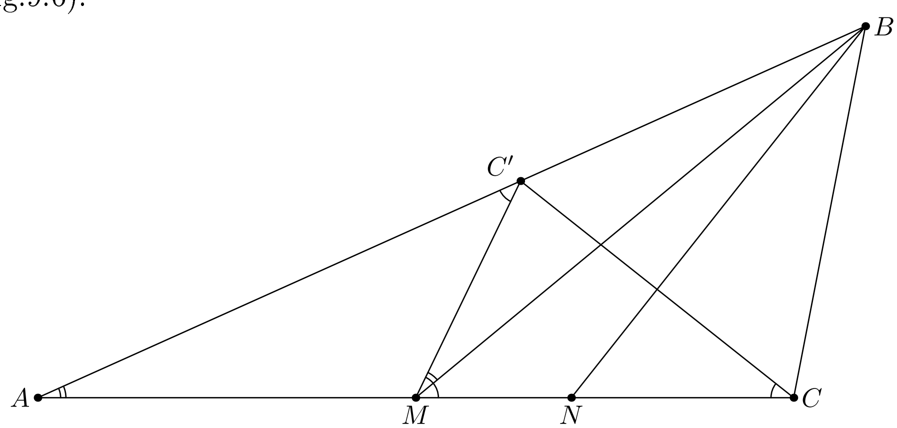

Fig.9.6

7. (M.Volchkevich) Given two intersecting circles with centers O_1, O_2. Construct the circle touching one of them externally and the second one internally such that the distance from its center to O_1O_2 is maximal.

   Solution. Let O, r be the center and the radius of some circle touching the two given; r_1, r_2 be the radii of the given circles. Then $OO_1 = r_1 - r$, $OO_2 = r_2 + r$, or $OO_1 = r_1 + r$,

$OO_2 = r_2 - r$, and in both cases $OO_1 + OO_2 = r_1 + r_2$. Therefore we must find the point satisfying this condition with maximal distance from line O_1O_2. It is known that the isosceles triangle has the minimal perimeter among all triangles with given base and altitude. Therefore the isosceles triangle has also the maximal altitude among all triangles with given side and the sum of two other sides. From this we obtain that the center of the required circle lies on equal distances $(r_1 + r_2)/2$ from points O_1 and O_2, and its radius is equal to $|r_1 - r_2|/2$.

8. (C.Pohoata, A.Zaslavsky) Given cyclic quadrilateral $ABCD$. Four circles each touching its diagonals and the circumcircle internally are equal. Is $ABCD$ a square?

Answer. Yes.

First solution. Let $AC \cap BD = P$, and let the incircles of the circular triangles ABP, BCP, CDP, DAP touch the circumcircle of $ABCD$ in K, L, M, N.

Consider the segment ABC. When a variable point X moves along the arc ABC from A to C, the radius of the circle inscribed in the segment and touching the arc in X changes as follows: it increases until X becomes the midpoint of the arc, and then decreases. Therefore, each value of radius is reached in exactly two, symmetrically situated positions of X.

Therefore $\smile AK = \smile LC$. Analogously $\smile AN = \smile MC$. So $\smile NK = \smile LM$. Analogously $\smile KL = \smile MN$. Now $\smile NL = \smile NK + \smile KL = 180^\circ$ i.e. NL is a diameter. Analogously KM is also a diameter.

Now symmetry with respect to O sends the pair of circles touching the circumcircle in M and N in the analogous pair touching it in K and L. So the same symmetry sends the common external tangent of the first pair in that of the second namely it sends AC in CA. Therefore AC is a diameter and similarly, BD is a diameter.

So $ABCD$ is a rectangle. Its diagonals divide the circumcircle into four sectors with equal radii of incircles. Therefore these sectors are also equal and $ABCD$ is square.

Second solution. Use **the Thebault theorem**: let point M lie on side AC of triangle ABC and two circles touch ray MB, line AC and internally the circumcircle of ABC. Then two centers of these circles and the incenter of ABC are collinear.

Applying the Thebault theorem to triangles ABC, BCD, CDA, DAB and the common point of diagonals we obtain that the inradii of these four triangles are equal. Calculating the areas of triangles as product of semiperimeter by inradius and finding the area of quadrilateral in two ways we obtain that $AC = BD$, so $ABCD$ is an isosceles trapezoid. Suppose that AD, BC are its bases and $AD > BC$. Then $S_{ABD}/S_{ABC} = AD/BC > (AD + BD + AB)/(BC + AB + AC)$, and the inradii of these triangles can't be equal. Thus $ABCD$ is a rectangle. As in the first solution $ABCD$ must be a square.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. First day. 10 form. Solutions.

1. (D.Shvetsov) Let a, b, c be the lengths of some triangle's sides; p, r be the semiperimeter and the inradius of triangle. Prove an inequality

$$\sqrt{\frac{ab(p-c)}{p}}+\sqrt{\frac{ca(p-b)}{p}}+\sqrt{\frac{bc(p-a)}{p}}\geq 6r.$$

Solution. By Cauchi inequality the left part isn't less than

$$3\sqrt[3]{\frac{abc}{p}\sqrt{\frac{(p-a)(p-b)(p-c)}{p}}}=3\sqrt[3]{4r^2R}.$$

As $R\geq 2r$ we obtain the demanded inequality.

2. (F.Nilov) Given quadrilateral $ABCD$. Its sidelines AB and CD intersect in point K. Its diagonals intersect in point L. It is known that line KL pass through the centroid of $ABCD$. Prove that $ABCD$ is trapezoid.

Solution. Suppose that lines AD and BC intersect in point M. Let X, Y be the common points of these lines with line KL. Then $(AD;MX)=(BC;MY)=1$. Therefore relations AX/XD and BY/YC are both greater or are both less than 1, and segment XY doesn't intersect the segment between the midpoints of AD and BC. As this last segment contains the centroid of $ABCD$, the condition of problem is true only when $AD\parallel BC$.

3. (A.Zaslavsky, A.Akopjan) The cirumradius and the inradius of triangle ABC are equal to R and r; O, I are the centers of respective circles. External bisector of angle C intersect AB in point P. Point Q is the projection of P to line OI. Find distance OQ.

Solution. Let A', B', C' be the excenters of ABC. Then I is the orthocenter of triangle $A'B'C'$, A, B, C are the bases of its altitudes and so the circumcircle of ABC is the Euler circle of $A'B'C'$. Thus the circumradius of $A'B'C'$ is $2R$, and its circumcenter O' is the reflection of I in O. Furthermore points A, B, A', B' lie on the circle. Line AB is the common chord of this circle and the circumcircle of ABC, and the external bisector of C is the common chord of this circle and the circumcircle of $A'B'C'$. So P is the radical center of three circles, and line PQ is the radical axis of circles ABC and $A'B'C'$ (fig.10.3). Therefore $OQ^2-R^2=(OQ+OO')^2-4R^2$. As $OO'=OI=\sqrt{R^2-2Rr}$, we have $OQ=R(R+r)/\sqrt{R^2-2Rr}$.

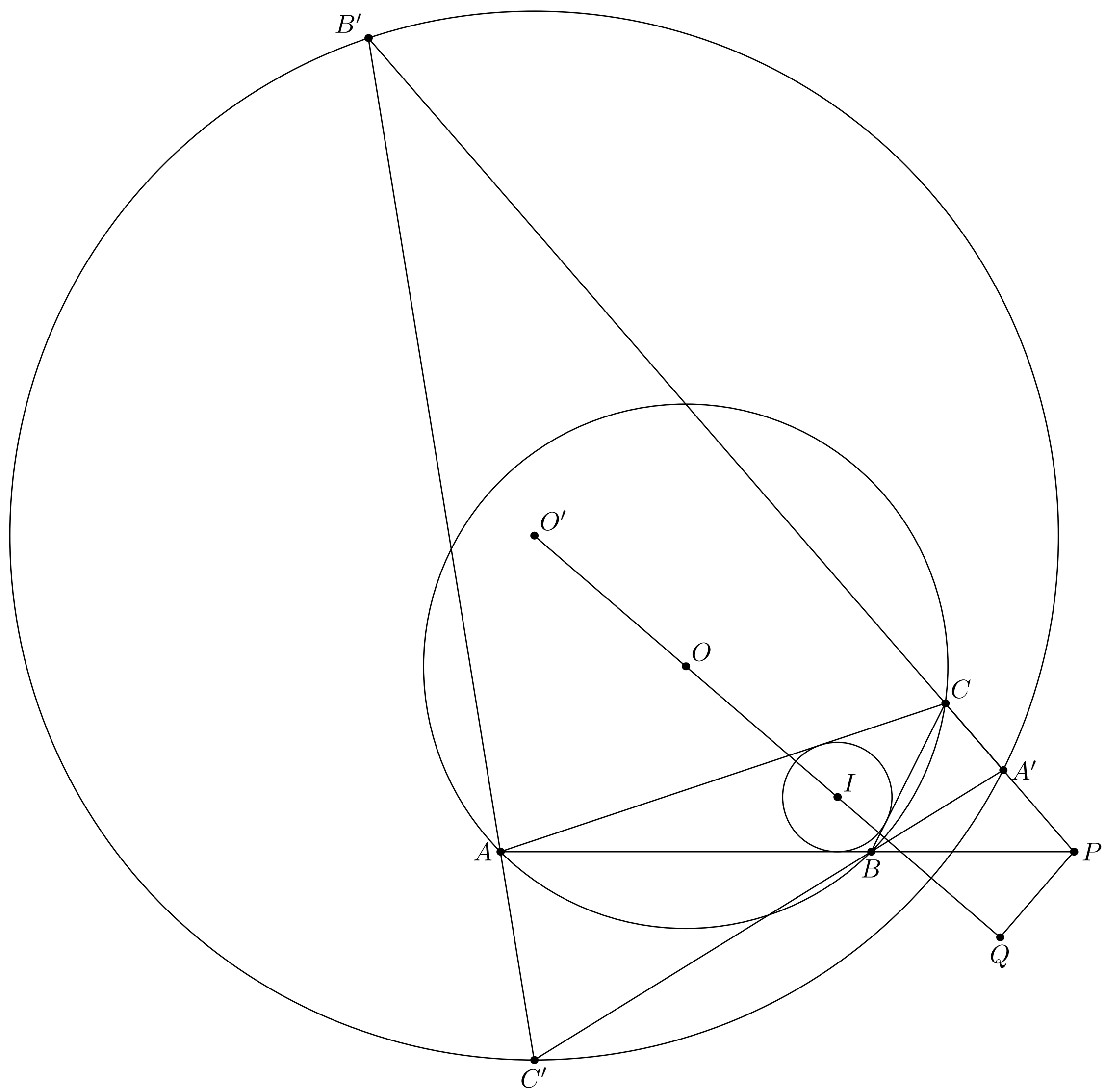

Fig.10.3

4. (C.Pohoata) Three parallel lines d_a, d_b, d_c pass through the vertex of triangle ABC. The reflections of d_a, d_b, d_c in BC, CA, AB respectively form triangle XYZ. Find the locus of incenters of such triangles.

   First solution. When d_a, d_b, d_c rotate around the vertices the symmetric lines rotate with the same velocity around the reflections of the vertices in opposite sidelines. Thus, firstly, the angles of XYZ don't depend on d_a, d_b, d_c, so all these triangles are similar, and secondly, points X, Y, Z move with equal angle velocity along three circles. Therefore the incenter also moves along some circles and it is sufficient to find three points of this circle.

   Take d_a, d_b coinciding with line AB. Let A', B' be the reflections of A, B in opposite sidelines. Then Z is the common point of lines AB' and BA', Y and X are the common points of these lines with the line parallel to AB and lying twice as far from C. Note that C and circumcenter O of ABC lie on equal distances from AB' and BA', so the bisector of angle XZY coincides with line CO. Also it is easy to see that the bisectors of angles ZXY and ZYX are perpendicular to AC and BC respectively.

   Consider the projections of O and of the incenter of XYZ to line AC. The projection of O is the midpoint of AC. Also it is the projection of the common point of AB' and d_c, because these two lines form equal angles with AC. Thus the projection of X and the incenter of XYZ is symmetric to the midpoint of AC wrt A (fig.10.4). Therefore the distance from the incenter to O is twice as large as the circumradius of ABC. When d_a, d_b, d_c are parallel to other sidelines of ABC, we obtain the same result. So the demanded locus is the circle with center O and radius twice as large as the circumradius of ABC .

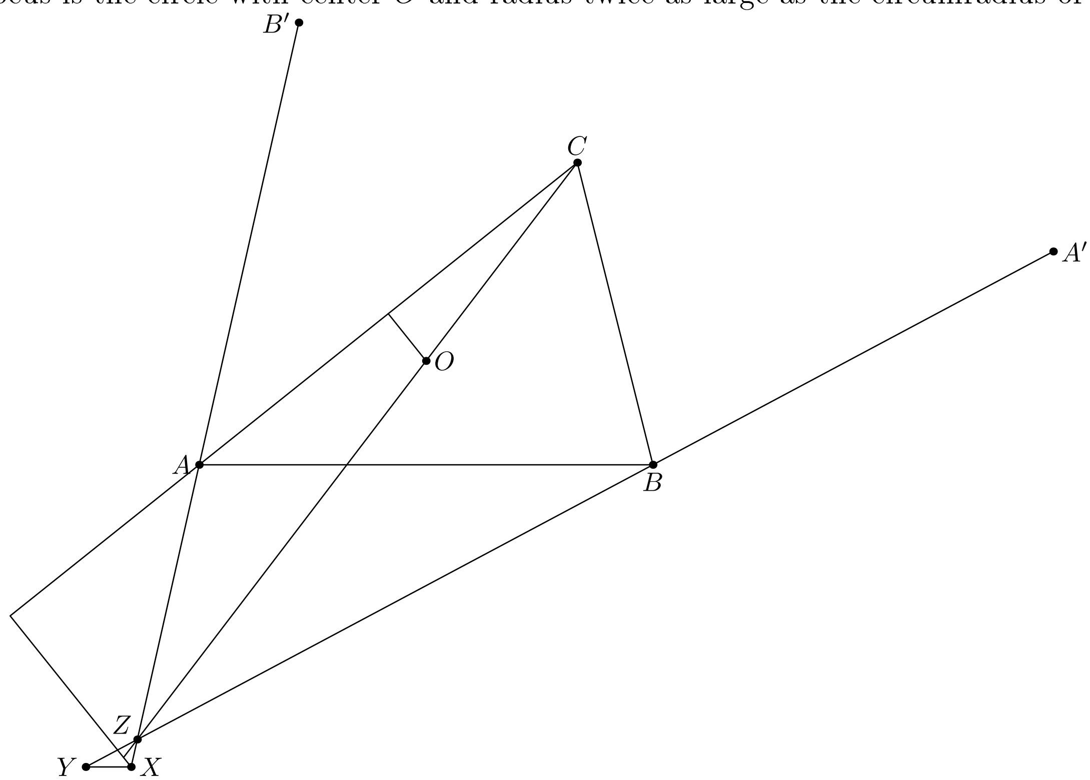

Fig.10.4

Second solution. As in the previous solution we obtain that when the direction of the lines d changes with a constant angular speed, so do the directions of XY, YZ, ZX. Therefore, the vertex X of triangle XYZ traces a circle with chord $B'C'$, and the angle bisector of $\angle YXZ$ rotates around the midpoint W_a of the arc $\smile B'C'$ with constant angular speed, too. So do the angle bisectors of $\angle Y$ and $\angle Z$ around the midpoints W_b, W_c of the corresponding arcs $\smile A'C'$ and $\smile A'B'$.

Therefore, their intersection I traces in the same time the circumcircles of triangles IW_aW_b, IW_bW_c and IW_cW_a. So, these three circumcircles do in fact coincide, and we are left to describe the circumcircle of triangle $W_aW_bW_c$.

We will show that all the points W_a, W_b, W_c are of distance $2R$ from O. Indeed, take W_a. Let BH_b, CH_c be the altitudes in triangle ABC, O_a be the circumcenter of triangle AH_bH_c, O' be the reflection of O in BC, and M_a be the midpoint of BC. The figures BCO', $H_bH_cO_a$ and $B'C'W_a$ are similar, and the figures BH_bB_1 and CH_cC_1 are also similar, therefore they are similar to $O'O_aW_a$, and M_aO_a is a mid-segment in triangle $O'OW_a$. Since M_aO_a is a diameter of the Euler circle, and thus equals R, the claim follows.

V GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10 form. Solutions.

5. (D.Prokopenko) Rhombus $CKLN$ is inscribed into triangle ABC in such way that point L lies on side AB, point N lies on side AC, point K lies on side BC. O_1, O_2 and O are the circumcenters of triangles ACL, BCL and ABC respectively. Let P be the common point of circles ANL and BKL, distinct from L. Prove that points O_1, O_2, O and P are concyclic.

   Solution. It is evident that L is the base of the bisector of angle C, and lines LN, LK are parallel to sides BC, AC. Thus $\angle AO_1L = 2\angle ACL = \angle C = \angle ANL$, so point O_1 lies on the circumcircle of triangle ANL and coincides with the midpoint of arc ANL. Thus, $\angle O_1PL = \angle APL + \angle O_1PA = \angle C + \frac{\angle A + \angle B}{2} = \frac{\pi + \angle C}{2}$. Similarly $\angle O_2Pl = \frac{\pi + \angle C}{2}$. Therefore $\angle O_1PO_2 = \pi - \angle C$. But angle O_1OO_2 is also equal to $\pi - \angle C$, because lines OO_1, OO_2 are medial perpendiculars to AC and BC (fig.10.5).

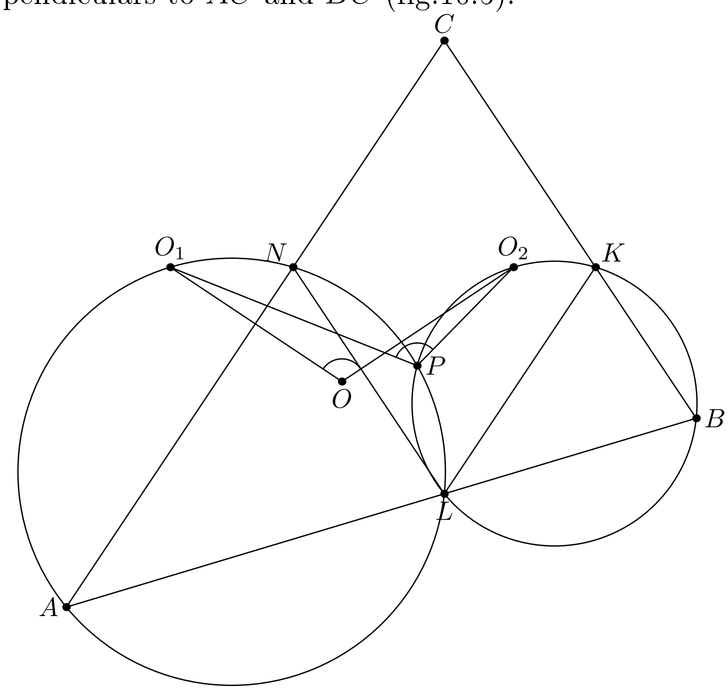

Fig.10.5

6. (A.Zaslavsky) Let M, I be the centroid and the incenter of triangle ABC, A_1 and B_1 be the touching points of the incircle with sides BC and AC, G be the common point of lines AA_1 and BB_1. Prove that angle CGI is right if and only if $GM \parallel AB$.

   First solution. Let C_1 be the touching point of incircle with side AB, C_2 be the second common point of incircle with CC_1. Then G lies on segment CC_1. As there exists central projection transforming the incircle to some circle and G to the center of this circle, then the cross-ratio $(CG; C_1C_2)$ is the same for any triangle and regular triangle. So this cross-ratio is equal to 3. Therefore we have the chain of equivalent assertions:

   - $\angle CGI = 90^\circ$;

   - G is the midpoint of C_1C_2;

   - $CC_1 = 3CC_2$;

- $CC_1 = 3GC_1$;

- $GM \parallel AB$.

Second solution. Let $AC_1 = x, BA_1 = y, CB_1 = z$. By Menelaus' theorem,

$$\frac{y+x}{x} . \frac{GC_1}{GC} . \frac{z}{y} = 1 \Rightarrow \frac{GC_1}{GC} = k = \frac{xy}{z(x+y)} = \frac{m}{z},$$

where $m = \frac{xy}{x+y}$.

Now,

$$\angle IGC = 90^\circ \Leftrightarrow CI^2 - r^2 = GC^2 - GC_1^2 \Leftrightarrow z^2 =$$

$$= CC_1^2 (\frac{1}{(1+k)^2} - \frac{k^2}{(1+k)^2}) = CC_1^2 (\frac{1-k}{1+k}) = CC_1^2 (\frac{z-m}{z+m}).$$

But, by Stewart's theorem,

$$CC_1^2 = \frac{x}{x+y}(z+y)^2 + \frac{y}{x+y}(z+x)^2 - xy = z(z+4m).$$

Then, these two equations yield

$$z^2 = z(z+4m)(\frac{z-m}{z+m}) \Leftrightarrow z(z+m) = (z+4m)(z-m) \Leftrightarrow$$

$$\Leftrightarrow 2zm = 4m^2 \Leftrightarrow z = 2m \Leftrightarrow k = \frac{1}{2},$$

as needed.

7. (A.Glazyrin) Given points O, A_1, A_2 ... A_n on the plane. For any two of these points the square of distance between them is natural number. Prove that there exist two vectors $\vec{x}$ and $\vec{y}$, such that for any point A_i $\vec{OA_i} = k\vec{x} + l\vec{y}$, where k and l are some integer numbers.

   Solution. By condition we obtain that for all i, j the product $(\vec{OA_i}, \vec{OA_j})$ is a half of an integer number. Thus for any integer $m_1, \ldots, m_n$ the square of vector $m_1\vec{OA_1} + \cdots + m_n\vec{OA_n}$ is a natural number. Consider all points which are the ends of such vectors. Let X be the nearest to O of these points, Y be the nearest to O of considered points not lying on line OX. Divide the plane into parallelograms formed by vectors $\vec{x} = \vec{OX}$ and $\vec{y} = \vec{OY}$. By definition of points X, Y all marked points are vertices of parallelograms, therefore $\vec{x}$, $\vec{y}$ are demanded vectors.

8. (B.Frenkin) Can the regular octahedron be inscribed into regular dodecahedron in such way that all vertices of octahedron be the vertices of dodecahedron?

   Answer. No.

   Solution. If an octahedron is inscribed into a dodecahedron then their circumspheres coincide. Therefore two opposite vertices of the octahedron are opposite vertices of the dodecahedron, and all other vertices of the octahedrons are equidistant from these two vertices. But the dodecahedron has no vertices equidistant from two opposite vertices.