IV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below are the conditions of tasks constituting the correspondence round of the IV geometrical olympiad in honour of I.F.Sharygin.

Participation in the olympiad is possible for pupils of 8–11 forms (these are four elder forms in Russian school). In the list of tasks presented below, each task is indicated by the numbers of forms for which it is intended. However the participants may solve the tasks for elder forms as well.

Your work in a school copy-book containing the solutions for the tasks must be sent, at the latest, April 1, 2008 to the following address: Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of I.F.Sharygin.

The work must be written in Russian or in English. On the cover of the copy-book, the following information is obedient: all names (underline the surname); post address including post index, phone number and/or E-mail; the number of your present form; the number and address of your school; full names of your teachers in maths and/or of instructors of your math circle.

Please write down each task from a new page: first write down the condition, and then the solution. Present your solution in detail, including all essential arguments and calculations, with exact figures. The solution of a computational task must be completed by a distinctly presented answer. Please be accurate: you are interested in good understanding and true estimating of your work!

If your solution depends on some well-known theorem or a fact from a standard textbook, you may simply refer to it (explaining which theorem or fact is used). But if you use a fact not contained in the curriculum, you have to prove it (or to indicate the source).

Your work will be thoroughly examined, and you will receive the answer, at the latest, in the middle of May 2008. The winners of the correspondence round will be invited to the final round which will be held in summer 2008 in Dubna (near Moscow).

1. (8) Does a regular polygon exist such that just half of its diagonals are parallel to its sides?

2. (8) For a given pair of circles, construct two concentric circles such that both are tangent to the given two. What is the number of solutions, depending on location of the circles?

3. (8) A triangle can be dissected into three equal triangles. Prove that some its angle is equal to 60°.

4. (8–9) The bisectors of two angles in an inscribed quadrangle are parallel. Prove that the sum of squares of some two sides in the quadrangle equals the sum of squares of two remaining sides.

5. (8–9) Reconstruct the square $ABCD$, given its vertex A and distances of vertices B and D from a fixed point O in the plane.

6. (8-9) In the plane, given two concentric circles with the center A. Let B be an arbitrary point on some of these circles, and C on the other one. For every triangle ABC, consider two equal circles mutually tangent at the point K, such that one of these circles is tangent to the line AB at point B and the other one is tangent to the line AC at point C. Determine the locus of points K.

7. (8–9) Given a circle and a point O on it. Another circle with center O meets the first one at points P and Q. The point C lies on the first circle, and the lines CP, CQ meet the second circle for the second time at points A and B. Prove that $AB = PQ$.

8. (8–11) a) Prove that for $n > 4$, any convex n-gon can be dissected into n obtuse triangles.

   b) Prove that for any n, there exists a convex n-gon which cannot be dissected into less than n obtuse triangles.

   c) In a dissection of a rectangle into obtuse triangles, what is the least possible number of triangles?

9. (9–10) The lines symmetrical to diagonal BD of a rectangle $ABCD$ relative to bisectors of angles B and D pass through the midpoint of diagonal AC. Prove that the lines symmetrical to diagonal AC relative to bisectors of angles A and C pass through the midpoint of diagonal BD.

10. (9–10) Quadrangle $ABCD$ is circumscribed around a circle with center I. Prove that the projections of points B and D to the lines IA and IC lie on a single circle.

11. (9–10) Given four points A, B, C, D. Any two circles such that one of them contains A and B, and the other one contains C and D, meet. Prove that common chords of all these pairs of circles pass through a common point.

12. (9–10) Given a triangle ABC. Point A_1 is chosen on the ray BA so that segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that lines A_1A_2, B_1B_2, C_1C_2 are parallel.

13. (9–10) Given triangle ABC. One of its excircles is tangent to the side BC at point A_1 and to the extensions of two other sides. Another excircle is tangent to side AC at point B_1. Segments AA_1 and BB_1 meet at point N. Point P is chosen on the ray AA_1 so that $AP = NA_1$. Prove that P lies on the incircle.

14. (9–10) The line connecting the incenter and the orthocenter of a non-isosceles triangle is parallel to the bisector of one of its angles. Determine this angle.

15. (9–11) Given two circles and point P not lying on them. Draw a line through P which cuts chords of equal length from these circles.

16. (9–11) Given two circles. Their common external tangent is tangent to them at points A and B. Points X, Y on these circles are such that some circle is tangent to the given two circles at these points, and in similar way (external or internal). Determine the locus of intersections of lines AX and BY.

17. (9–11) Given triangle ABC and a ruler with two marked intervals equal to AC and BC. By this ruler only, find the incenter of the triangle formed by midlines of triangle ABC.

18. (9–11) Prove that the triangle having sides a, b, c and area S satisfies the inequality

$$a^2 + b^2 + c^2 - \frac{1}{2}(|a-b| + |b-c| + |c-a|)^2 \geq 4\sqrt{3}S.$$

19. (10-11) Given parallelogram $ABCD$ such that $AB = a$, $AD = b$. The first circle has its center at vertex A and passes through D, and the second circle has its center at C and passes through D. A circle with center B meets the first circle at points M_1, N_1, and the second circle at points M_2, N_2. Determine the ratio M_1N_1/M_2N_2.

20. (10–11) a) Some polygon has the following property: if a line passes through any two points which bisect its perimeter then this line bisects the area of the polygon. Is it true that the polygon is central symmetrical?

    b) Is it true that any figure with the property from part a) is central symmetrical?

21. (10–11) In a triangle, one has drawn middle perpendiculars to its sides and has measured their segments lying inside the triangle.

    a) All three segments are equal. Is it true that the triangle is equilateral?

    b) Two segments are equal. Is it true that the triangle is isosceles?

    c) Can the segments have length 4, 4 and 3?

22. (10–11) a) All vertices of a pyramid lie on the facets of a cube but not on its edges, and each facet contains at least one vertex. What is the maximum possible number of the vertices of the pyramid?

    b) All vertices of a pyramid lie in the facet planes of a cube but not on the lines including its edges, and each facet plane contains at least one vertex. What is the maximum possible number of the vertices of the pyramid?

23. (10–11) In the space, given two intersecting spheres of different radii and a point A belonging to both spheres. Prove that there is a point B in the space with the following property: if an arbitrary circle passes through points A and B then the second points of its meet with the given spheres are equidistant from B.

24. (11) Let h be the least altitude of a tetrahedron, and d the least distance between its opposite edges. For what values of t the inequality $d > th$ is possible?

Feel free to ask any questions via Organizing committee official e-mail *geomolymp@mccme.ru*