A Short Proof of Lamoen’s Generalization of the Droz-Farny Line Theorem

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Abstract

We give a short proof of a slightly more general version of the Droz-Farny line theorem mentioned by Floor van Lamoen in [5].

1. The Droz-Farny line theorem and Lamoen’s generalization

In 1899, Arnold Droz-Fary discovered the following beautiful result, known nowadays as the Droz-Farny line theorem:

Theorem 1 (Droz-Farny). If two perpendicular straight lines are drawn through the orthocenter of a triangle, they intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.

As illustrated in Figure 1, we have denoted by $A_1, B_1, C_1$, and $A_2, B_2, C_2$ the intersections points of the two perpendicular lines $d_1, d_2$ with the sidelines $BC, CA$, and $AB$, respectively. The Droz-Farny line theorem states that the midpoints $A_3, B_3, C_3$ of the segments $A_1A_2, B_1B_2, C_1C_2$ are collinear. Despite of the simple configuration, the first known proof is the analytical one from [7]. Years later, on the Hyacinthos forum, several proofs were given by N. Reingold [6], D. Grinberg [2], [3], [4] and M. Stevanovic [8]. In 2004,
J. -L. Ayme ends this sequence of proofs by presenting a beautiful synthetic approach [1]. A month before the apparition of Ayme’s article, Lamoen [5] mentioned, without proof, the following generalization:

**Theorem 2** (Lamoen). If the midpoints of the intercepted segments are replaced by three points \(A_3, B_3, C_3\) dividing into the same ratio the corresponding segments \(A_1A_2, B_1B_2, \) and \(C_1C_2\), then \(A_3, B_3, C_3\) remain collinear.

2. **Proof of Theorem 2**

Denote by \(e, f\) the lines through the orthocenter \(H\) parallel to \(AB, AC\), respectively. Furthermore, denote by \(x, y\) the lines through the vertex \(A\) parallel to the lines \(d_1, d_2\), and let \(X, Y\) be the intersection points of the sideline \(BC\) with \(x, y\), respectively.

Since the pencil \((HC_1, HC_2, HB, e)\) is the image of \((HB_2, HB_1, f, HC)\) under the rotation \(\Psi(H, +\pi/2)\),

\[
\frac{BC_1}{BC_2} = \frac{CB_1}{CB_2}\quad \text{if and only if}\quad \frac{BC_1}{CB_1} = \frac{BC_2}{CB_2},
\]

and thus, by multiplying with \(AC/AB\),

\[
\frac{C_1B}{AB} \cdot \frac{AC}{B_1C} = \frac{C_2B}{AB} \cdot \frac{AC}{B_2C}.
\]

On other hand, since

\[
\frac{C_1B}{AB} = \frac{A_1B}{XB}, \quad \frac{AC}{B_1C} = \frac{XC}{A_1C}, \quad \frac{C_2B}{AB} = \frac{A_2B}{YB}, \quad \frac{AC}{B_2C} = \frac{YC}{A_2C}
\]
it follows that
\[
\frac{A_1B}{A_1C} \cdot \frac{XB}{XC} = \frac{A_2B}{A_2C} \cdot \frac{YB}{YC},
\]
which is equivalent with the congruence of the pencils \((B, C, A_1, X)\) and \((B, C, A_2, Y)\). By intersecting now \((AB, AC, AA_1, AX)\) with \(d_1\) and \((AB, AC, AA_2, AY)\) with \(d_2\), we deduce that
\[
\frac{C_1A_1}{C_1B_1} = \frac{C_2A_2}{C_2B_2},
\]
the two degenerated triangles \(A_1B_1C_1\) and \(A_2B_2C_2\) being similar.

For a point \(P\) denote by \(\overrightarrow{XP}\) the vector \(\overrightarrow{XP}\), where \(X\) is a fixed point in plane of triangle \(ABC\). Since \(C_1A_1/C_1B_1 = C_2A_2/C_2B_2\), there exist two real numbers \(k\) and \(l\), satisfying \(k + l = 1\), such that
\[
C_1 = kA_1 + lB_1, \quad C_2 = kA_2 + lB_2.
\]
On other hand, since \(A_3, B_3, C_3\) divide the segments \(A_1A_2, B_1B_2,\) and \(C_1C_2\), respectively, into the same ratio, there exist two real numbers \(u\) and \(v\), satisfying \(u + v = 1\), such that
\[
A_3 = uA_1 + vA_2, \quad B_3 = uB_1 + vB_2, \quad C_3 = uC_1 + vC_2.
\]
Therefore,
\[
C_3 = uC_1 + vC_2 = u(kA_1 + lB_1) + v(kA_2 + lB_2) = k(uA_1 + vA_2) + l(uB_1 + vB_2) = kA_3 + lB_3.
\]
According to the fact that \(k + l = 1\), this implies that the points \(A_3, B_3, C_3\) are collinear. This completes the proof of Theorem 2.

References


