Casey’s Theorem and its Applications

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Abstract. We present a proof of the generalized Ptolemy’s theorem, also known as Casey’s theorem and its applications in the resolution of difficult geometry problems.

1 Casey’s Theorem.

Theorem 1. Two circles $\Gamma_1(r_1)$ and $\Gamma_2(r_2)$ are internally/externally tangent to a circle $\Gamma(R)$ through $A, B$, respectively. The length $\delta_{12}$ of the common external tangent of $\Gamma_1, \Gamma_2$ is given by:

$$\delta_{12} = \frac{AB}{R} \sqrt{(R \pm r_1)(R \pm r_2)}$$

Proof. Without loss of generality assume that $r_1 \geq r_2$ and we suppose that $\Gamma_1$ and $\Gamma_2$ are internally tangent to $\Gamma$. The remaining case will be treated analogously. A common external tangent between $\Gamma_1$ and $\Gamma_2$ touches $\Gamma_1, \Gamma_2$ at $A_1, B_1$ and $A_2$ is the orthogonal projection of $O_2$ onto $O_1A_1$. (See Figure 1). By Pythagorean theorem for $\triangle O_1O_2A_2$, we obtain

$$\delta_{12}^2 = (A_1B_1)^2 = (O_1O_2)^2 - (r_1 - r_2)^2$$

Let $\angle O_1OO_2 = \lambda$. By cosine law for $\triangle OO_1O_2$, we get

$$(O_1O_2)^2 = (R - r_1)^2 + (R - r_2)^2 - 2(R - r_1)(R - r_2) \cos \lambda$$

By cosine law for the isosceles triangle $\triangle OAB$, we get

$$AB^2 = 2R^2(1 - \cos \lambda)$$
Eliminating $\cos \lambda$ and $O_1O_2$ from the three previous expressions yields

$$\delta_{12}^2 = (R - r_1)^2 + (R - r_2)^2 - (r_1 - r_2)^2 - 2(R - r_1)(R - r_2) \left( 1 - \frac{AB^2}{2R^2} \right)$$

Subsequent simplifications give

$$\delta_{12} = \frac{AB}{R} \sqrt{(R - r_1)(R - r_2)} \quad (1)$$

Analogously, if $\Gamma_1, \Gamma_2$ are externally tangent to $\Gamma$, then we will get

$$\delta_{12} = \frac{AB}{R} \sqrt{(R + r_1)(R + r_2)} \quad (2)$$

If $\Gamma_1$ is externally tangent to $\Gamma$ and $\Gamma_2$ is internally tangent to $\Gamma$, then a similar reasoning gives that the length of the common internal tangent between $\Gamma_1$ and $\Gamma_2$ is given by

$$\delta_{12} = \frac{AB}{R} \sqrt{(R + r_1)(R - r_2)} \quad (3)$$
Theorem 2 (Casey). Given four circles $\Gamma_i, i = 1, 2, 3, 4$, let $\delta_{ij}$ denote the length of a common tangent (either internal or external) between $\Gamma_i$ and $\Gamma_j$. The four circles are tangent to a fifth circle $\Gamma$ (or line) if and only if for appropriate choice of signs,

$$\delta_{12} \cdot \delta_{34} \pm \delta_{13} \cdot \delta_{42} \pm \delta_{14} \cdot \delta_{23} = 0$$

The proof of the direct theorem is straightforward using Ptolemy’s theorem for the quadrilateral $ABCD$ whose vertices are the tangency points of $\Gamma_1(r_1), \Gamma_2(r_2), \Gamma_3(r_3), \Gamma_4(r_4)$ with $\Gamma(R)$. We substitute the lengths of its sides and digonals in terms of the lengths of the tangents $\delta_{ij}$, by using the formulas (1), (2) and (3). For instance, assuming that all tangencies are external, then using (1), we get

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \frac{(AB + CD + AD - BC)}{R^2} \sqrt{(R - r_1)(R - r_2)(R - r_3)(R - r_4)}$$

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \frac{(AC - BD)}{R^2} \sqrt{(R - r_1)(R - r_3) \cdot \sqrt{(R - r_2)(R - r_4)}}$$

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \delta_{13} \cdot \delta_{42}.$$

Casey established that this latter relation is sufficient condition for the existence of a fifth circle $\Gamma(R)$ tangent to $\Gamma_1(r_1), \Gamma_2(r_2), \Gamma_3(r_3), \Gamma_4(r_4)$. Interestingly, the proof of this converse is a much tougher exercise. For a proof you may see [1].

2 Some Applications.

I) $\triangle ABC$ is isosceles with legs $AB = AC = L$. A circle $\omega$ is tangent to $BC$ and the arc $BC$ of the circumcircle of $\triangle ABC$. A tangent line from $A$ to $\omega$ touches $\omega$ at $P$. Describe the locus of $P$ as $\omega$ varies.

Solution. We use Casey’s theorem for the circles $(A), (B), (C)$ (with zero radii) and $\omega$, all internally tangent to the circumcircle of $\triangle ABC$. Thus, if $\omega$ touches $BC$ at $Q$, we have:

$$L \cdot CQ + L \cdot BQ = AP \cdot BC \implies AP = \frac{L(BQ + CQ)}{BC} = L$$

The length $AP$ is constant, i.e. Locus of $P$ is the circle with center $A$ and radius $AB = AC = L$.

II) $(O)$ is a circle with diameter $AB$ and $P, Q$ are two points on $(O)$ lying on different sides of $AB$. $T$ is the orthogonal projection of $Q$ onto $AB$. Let $(O_1), (O_2)$ be the circles with diameters $TA, TB$ and $PC, PD$ are the tangent segments from $P$ to $(O_1), (O_2)$, respectively. Show that $PC + PD = PQ$. [2].
Solution. Let $\delta_{12}$ denote the length of the common external tangent of $(O_1), (O_2)$. We use Casey’s theorem for the circles $(O_1), (O_2), (P), (Q)$, all internally tangent to $(O)$.

$$PC \cdot QT + PD \cdot QT = PQ \cdot \delta_{12} \implies PC + PD = PQ \cdot \frac{\delta_{12}}{QT} = PQ \cdot \frac{\sqrt{TA \cdot TB}}{TQ} = PQ.$$  

III) In $\triangle ABC$, let $\omega_A, \omega_B, \omega_C$ be the circles tangent to $BC, CA, AB$ through their midpoints and the arcs $BC, CA, AB$ of its circumcircle (not containing $A, B, C$). If $\delta_{BC}, \delta_{CA}, \delta_{AB}$ denote the lengths of the common external tangents between $(\omega_B, \omega_C), (\omega_C, \omega_A)$ and $(\omega_A, \omega_B)$, respectively, then prove that

$$\delta_{BC} = \delta_{CA} = \delta_{AB} = \frac{a + b + c}{4}.$$  

Solution. Let $\delta_A, \delta_B, \delta_C$ denote the lengths of the tangents from $A, B, C$ to $\omega_A, \omega_B, \omega_C$, respectively. By Casey’s theorem for the circles $(A), (B), (C), \omega_B$, all tangent to the circumcircle of $\triangle ABC$, we get

$$\delta_B \cdot b = a \cdot AE + c \cdot CE \implies \delta_B = \frac{1}{2}(a + c)$$

Similarly, by Casey’s theorem for $(A), (B), (C), \omega_C$ we’ll get $\delta_C = \frac{1}{2}(a + b)$. 

Figure 2: Application II
Now, by Casey’s theorem for \((B), (C), \omega_B, \omega_C\), we get
\[
\delta_{BC} = \frac{\delta_B \cdot \delta_C - BF \cdot BE}{a} = \frac{(a + c)(a + b) - bc}{4a} = \frac{a + b + c}{4}
\]
By similar reasoning, we’ll have \(\delta_{CA} = \delta_{AB} = \frac{1}{4}(a + b + c)\).

**IV) A circle \(K\) passes through the vertices \(B, C\) of \(\triangle ABC\) and another circle \(\omega\) touches \(AB, AC, K\) at \(P, Q, T\), respectively. If \(M\) is the midpoint of the arc \(BTC\) of \(K\), show that \(BC, PQ, MT\) concur. [3]**

**Solution.** Let \(R, \varrho\) be the radii of \(K\) and \(\omega\), respectively. Using formula (1) of Theorem 1 for \(\omega, (B)\) and \(\omega, (C)\). Both \((B), (C)\) with zero radii and tangent to \(K\) through \(B, C\), we obtain:
\[
TC^2 = \frac{CQ^2 \cdot R^2}{(R - \varrho)(R - 0)} = \frac{CQ^2 \cdot R}{R - \varrho}, \quad TB^2 = \frac{BP^2 \cdot R^2}{(R - \varrho)(R - 0)} = \frac{BP^2 \cdot R}{R - \varrho} \implies \frac{TB}{TC} = \frac{BP}{CQ}
\]
Let \(PQ\) cut \(BC\) at \(U\). By Menelaus’ theorem for \(\triangle ABC\) cut by \(U PQ\) we have
\[
\frac{UB}{UC} = \frac{BP}{AP} \cdot \frac{AQ}{CQ} = \frac{BP}{CQ} = \frac{TB}{TC}
\]
Thus, by angle bisector theorem, \(U\) is the foot of the \(T\)-external bisector \(TM\) of \(\triangle BTC\).

**V) If \(D, E, F\) denote the midpoints of the sides \(BC, CA, AB\) of \(\triangle ABC\). Show that the incircle \((I)\) of \(\triangle ABC\) is tangent to \(\odot(DEF)\). (Feuerbach theorem).**

**Solution.** We consider the circles \((D), (E), (F)\) with zero radii and \((I)\). The notation \(\delta_{XY}\) stands for the length of the external tangent between the circles \((X), (Y)\), then
\[
\delta_{DE} = \frac{c}{2}, \quad \delta_{EF} = \frac{a}{2}, \quad \delta_{FD} = \frac{b}{2}, \quad \delta_{DI} = \left| \frac{b - c}{2} \right|, \quad \delta_{EI} = \left| \frac{a - c}{2} \right|, \quad \delta_{FI} = \left| \frac{b - a}{2} \right|
\]
For the sake of applying the converse of Casey’s theorem, we shall verify if, for some combination of signs + and −, we get \(\pm c(b - a) \pm a(b - c) \pm b(a - c) = 0\), which is trivial. Therefore, there exists a circle tangent to \((D), (E), (F)\) and \((I)\), i.e. \((I)\) is internally tangent to \(\odot(DEF)\). We use the same reasoning to show that \(\odot(DEF)\) is tangent to the three excircles of \(\triangle ABC\).

**VI) \(\triangle ABC\) is scalene and \(D, E, F\) are the midpoints of \(BC, CA, AB\). The incircle \((I)\) and 9 point circle \(\odot(DEF)\) of \(\triangle ABC\) are internally tangent through the Feuerbach point \(F_e\). Show that one of the segments \(F_eD, F_eE, F_eF\) equals the sum of the other two. [4]**
Solution. WLOG assume that $b \geq a \geq c$. Incircle $(I, r)$ touches $BC$ at $M$. Using formula (1) of Theorem 1 for $(I)$ and $(D)$ (with zero radius) tangent to the 9-point circle $(N, R/2)$, we have:

$$F_e D^2 = DM^2 \cdot \left(\frac{R}{2} \right)^2 \quad \Rightarrow \quad F_e D = \sqrt{\frac{R}{R-2r} \cdot \frac{(b-c)}{2}}$$

By similar reasoning, we have the expressions

$$F_e E = \sqrt{\frac{R}{R-2r} \cdot \frac{(a-c)}{2}} \quad , \quad F_e F = \sqrt{\frac{R}{R-2r} \cdot \frac{(b-a)}{2}}$$

Therefore, the addition of the latter expressions gives

$$F_e E + F_e F = \sqrt{\frac{R}{R-2r} \cdot \frac{b-c}{2}} = F_e D$$

VII) $\triangle ABC$ is a triangle with $AC > AB$. A circle $\omega_A$ is internally tangent to its circumcircle $\omega$ and $AB, AC$. $S$ is the midpoint of the arc $BC$ of $\omega$, which does not contain $A$ and $ST$ is the tangent segment from $S$ to $\omega_A$. Prove that

$$\frac{ST}{SA} = \frac{AC - AB}{AC + AB} \quad [5]$$

Solution. Let $M, N$ be the tangency points of $\omega_A$ with $AC, AB$. By Casey’s theorem for $\omega_A, (B), (C), (S)$, all tangent to the circumcircle $\omega$, we get

$$ST \cdot BC + CS \cdot BN = CM \cdot BS \quad \Rightarrow \quad ST \cdot BC = CS(CM - BN)$$

If $U$ is the reflection of $B$ across $AS$, then $CM - BN = UC = AC - AB$. Hence

$$ST \cdot BC = CS(AC - AB) \quad (\ast)$$

By Ptolemy’s theorem for $ABSC$, we get $SA \cdot BC = CS(AB + AC)$. Together with (\ast), we obtain

$$\frac{ST}{SA} = \frac{AC - AB}{AC + AB}$$
VIII) Two congruent circles \((S_1), (S_2)\) meet at two points. A line \(\ell\) cuts \((S_2)\) at \(A, C\) and \((S_1)\) at \(B, D\) \((A, B, C, D\) are collinear in this order). Two distinct circles \(\omega_1, \omega_2\) touch the line \(\ell\) and the circles \((S_1), (S_2)\) externally and internally respectively. If \(\omega_1, \omega_2\) are externally tangent, show that \(AB = CD\). [6]

Solution. Let \(P \equiv \omega_1 \cap \omega_2\) and \(M, N\) be the tangency points of \(\omega_1\) and \(\omega_2\) with an external tangent. Inversion with center \(P\) and power \(PB \cdot PD\) takes \((S_1)\) and the line \(\ell\) into themselves. The circles \(\omega_1\) and \(\omega_2\) go to two parallel lines \(k_1\) and \(k_2\) tangent to \((S_1)\) and the circle \((S_2)\) goes to another circle \((S'_2)\) tangent to \(k_1, k_2\). Hence, \((S_2)\) is congruent to its inverse \((S'_2)\). Further, \((S_2), (S'_2)\) are symmetrical about \(P \implies PC \cdot PA = PB \cdot PD\).

By Casey’s theorem for \(\omega_1, \omega_2, (D), (B), (S_1)\) and \(\omega_1, \omega_2, (A), (C), (S_2)\) we get:

\[
DB = \frac{2PB \cdot PD}{MN} , \ AC = \frac{2PA \cdot PC}{MN}
\]

Since \(PC \cdot PA = PB \cdot PD \implies AC = BD \implies AB = CD\).

IX) \(\triangle ABC\) is equilateral with side length \(L\). Let \((O, r)\) and \((O, R)\) be the incircle and circumcircle of \(\triangle ABC\). \(P\) is a point on \((O, r)\) and \(P_1, P_2, P_3\) are the projections of \(P\) onto \(BC, CA, AB\). Circles \(T_1, T_2\) and \(T_3\) touch \(BC, CA, AB\) through \(P_1, P_2, P_2\) and \((O, R)\) (internally), their centers lie on different sides of \(BC, CA, AB\) with respect to \(A, B, C\). Prove that the sum of the lengths of the common external tangents of \(T_1, T_2\) and \(T_3\) is a constant value.

Solution. Let \(\delta_1\) denote the tangent segment from \(A\) to \(T_1\). By Casey’s theorem for \((A), (B), (C), T_1\), all tangent to \((O, R)\), we have \(L \cdot BP_1 + L \cdot CP_1 = \delta_1 \cdot L \implies \delta_1 = L\). Similarly, we have \(\delta_2 = \delta_3 = L\). By Euler’s theorem for the pedal triangle \(\triangle P_1P_2P_3\) of \(P\), we get:

\[
[P_1P_2P_3] = \frac{p(P, (O))}{4R^2}[ABC] = \frac{R^2 - r^2}{4R^2}[ABC] = \frac{3}{16}[ABC]
\]

Therefore, we obtain

\[
AP_2 \cdot AP_3 + BP_3 \cdot BP_1 + CP_1 \cdot CP_2 = \frac{2}{\sin 60^\circ} ( [ABC] - [P_1P_2P_3] ) = \frac{13}{16}L^2. \, (\ast)
\]

By Casey’s theorem for \((B), (C), T_2, T_3\), all tangent to \((O, R)\), we get

\[
\delta_2 \cdot \delta_3 = L^2 = BC \cdot \delta_{23} + CP_2 \cdot BP_3 = L \cdot \delta_{23} + (L - AP_1)(L - AP_2)
\]

By cyclic exchange, we have the expressions:

\[
L^2 = L \cdot \delta_{31} + (L - BP_3)(L - BP_1) , \ L^2 = L \cdot \delta_{12} + (L - CP_1)(L - CP_2)
\]
Adding the three latter equations yields

\[ 3L^2 = L(\delta_{23} + \delta_{31} + \delta_{12}) + 3L^2 - 3L^2 + AP_3 \cdot AP_2 + BP_3 \cdot BP_1 + CP_1 \cdot CP_2 \]

Hence, combining with (⋆) gives

\[ \delta_{23} + \delta_{31} + \delta_{12} = 3L - \frac{13}{16} L = \frac{35}{16} L \]

3 Proposed Problems.

1) Purser’s theorem: \( \triangle ABC \) is a triangle with circumcircle \((O)\) and \( \omega \) is a circle in its plane. \( AX, BY, CZ \) are the tangent segments from \( A, B, C \) to \( \omega \). Show that \( \omega \) is tangent to \((O)\), if and only if

\[ \pm AX \cdot BC \pm BY \cdot CA \pm CZ \cdot AB = 0 \]
2) Circle $\omega$ touches the sides $AB, AC$ of $\triangle ABC$ at $P, Q$ and its circumcircle ($O$). Show that the midpoint of $PQ$ is either the incenter of $\triangle ABC$ or the A-excenter of $\triangle ABC$, according to whether ($O$), $\omega$ are internally tangent or externally tangent.

3) $\triangle ABC$ is A-right with circumcircle ($O$). Circle $\Omega_B$ is tangent to the segments $OB, OA$ and the arc $AB$ of ($O$). Circle $\Omega_C$ is tangent to the segments $OC, OA$ and the arc $AC$ of ($O$). $\Omega_B, \Omega_C$ touch $OA$ at $P, Q$, respectively. Show that:

$$\frac{AB}{AC} = \frac{AP}{AQ}$$

4) Gumma, 1874. We are given a circle ($O, r$) in the interior of a square $ABCD$ with side length $L$. Let ($O_i, r_i$) $i = 1, 2, 3, 4$ be the circles tangent to two sides of the square and ($O, r$) (externally). Find $L$ as a function of $r_1, r_2, r_3, r_4$.

5) Two parallel lines $\tau_1, \tau_2$ touch a circle $\Gamma(R)$. Circle $k_1(r_1)$ touches $\Gamma, \tau_1$ and a third circle $k_2(r_2)$ touches $\Gamma, \tau_2, k_1$. We assume that all tangencies are external. Prove that $R = 2\sqrt{r_1 \cdot r_2}$.

6) Victor Thébault. 1938. $\triangle ABC$ has incircle ($I, r$) and circumcircle ($O$). $D$ is a point on $\overline{AB}$. Circle $\Gamma_1(r_1)$ touches the segments $\overline{DA}, \overline{DC}$ and the arc $CA$ of ($O$). Circle $\Gamma_2(r_2)$ touches the segments $\overline{DB}, \overline{DC}$ and the arc $CB$ of ($O$). If $\angle ADC = \varphi$, show that:

$$r_1 \cdot \cos^2 \frac{\varphi}{2} + r_2 \cdot \sin^2 \frac{\varphi}{2} = r$$

References