Casey's Theorem and its Applications

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Abstract. We present a proof of the generalized Ptolemy's theorem, also known as Casey's theorem and its applications in the resolution of difficult geometry problems.

1 Casey's Theorem.

Theorem 1. Two circles $\Gamma_1(r_1)$ and $\Gamma_2(r_2)$ are internally/externally tangent to a circle $\Gamma(R)$ through A, B, respetively. The length δ_{12} of the common external tangent of Γ_1, Γ_2 is given by:

$$\delta_{12} = \frac{AB}{R}\sqrt{(R \pm r_1)(R \pm r_2)}$$

Proof. Without loss of generality assume that $r_1 \geq r_2$ and we suppose that Γ_1 and Γ_2 are internally tangent to Γ. The remaining case will be treated analogously. A common external tangent between Γ_1 and Γ_2 touches Γ_1, Γ_2 at A_1, B_1 and A_2 is the orthogonal projection of O_2 onto O_1A_1. (See Figure 1). By Pythagorean theorem for $\triangle O_1O_2A_2$, we obtain

$$\delta_{12}{}^2 = (A_1B_1)^2 = (O_1O_2)^2 - (r_1 - r_2)^2$$

Let $\angle O_1OO_2 = \lambda$. By cosine law for $\triangle OO_1O_2$, we get

$$(O_1O_2)^2 = (R - r_1)^2 + (R - r_2)^2 - 2(R - r_1)(R - r_2)\cos\lambda$$

By cosine law for the isosceles triangle $\triangle OAB$, we get

$$AB^2 = 2R^2(1 - \cos\lambda)$$

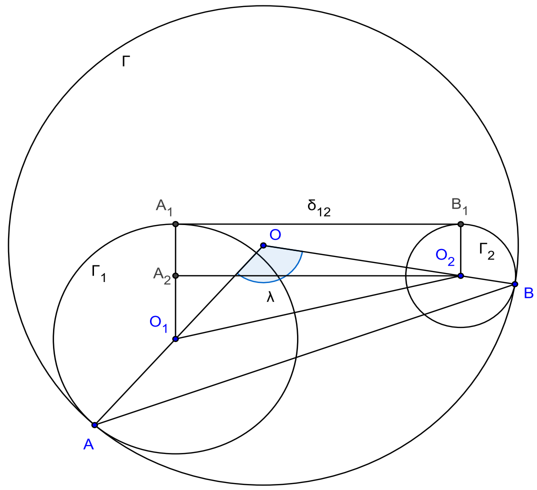

Figure 1: Theorem 1

Eliminating $\cos\lambda$ and O_1O_2 from the three previous expressions yields

$$\delta_{12}{}^2 = (R-r_1)^2 + (R-r_2)^2 - (r_1-r_2)^2 - 2(R-r_1)(R-r_2)\left(1-\frac{AB^2}{2R^2}\right)$$

Subsequent simplifications give

$$\delta_{12} = \frac{AB}{R}\sqrt{(R-r_1)(R-r_2)} \quad (1)$$

Analogously, if Γ_1, Γ_2 are externally tangent to Γ, then we will get

$$\delta_{12} = \frac{AB}{R}\sqrt{(R+r_1)(R+r_2)} \quad (2)$$

If Γ_1 is externally tangent to Γ and Γ_2 is internally tangent to Γ, then a similar reasoning gives that the length of the common internal tangent between Γ_1 and Γ_2 is given by

$$\delta_{12} = \frac{AB}{R}\sqrt{(R+r_1)(R-r_2)} \quad (3)$$

Theorem 2 (Casey). Given four circles $\Gamma_i, i = 1, 2, 3, 4$, let δ_{ij} denote the length of a common tangent (either internal or external) between Γ_i and Γ_j. The four circles are tangent to a fith circle Γ (or line) if and only if for appropriate choice of signs,

$$\delta_{12} \cdot \delta_{34} \pm \delta_{13} \cdot \delta_{42} \pm \delta_{14} \cdot \delta_{23} = 0$$

The proof of the direct theorem is straightforward using Ptolemy's theorem for the quadrilateral $ABCD$ whose vertices are the tangency points of $\Gamma_1(r_1), \Gamma_2(r_2), \Gamma_3(r_3), \Gamma_4(r_4)$ with $\Gamma(R)$. We susbtitute the lengths of its sides and digonals in terms of the lenghts of the tangents δ_{ij}, by using the formulas (1), (2) and (3). For instance, assuming that all tangencies are external, then using (1), we get

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \left(\tfrac{AB \cdot CD + AD \cdot BC}{R^2}\right) \sqrt{(R - r_1)(R - r_2)(R - r_3)(R - r_4)}$$

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \left(\tfrac{AC \cdot BD}{R^2}\right) \sqrt{(R - r_1)(R - r_3)} \cdot \sqrt{(R - r_2)(R - r_4)}$$

$$\delta_{12} \cdot \delta_{34} + \delta_{14} \cdot \delta_{23} = \delta_{13} \cdot \delta_{42}.$$

Casey established that this latter relation is sufficient condition for the existence of a fith circle $\Gamma(R)$ tangent to $\Gamma_1(r_1), \Gamma_2(r_2), \Gamma_3(r_3), \Gamma_4(r_4)$. Interestingly, the proof of this converse is a much tougher exercise. For a proof you may see [1].

2 Some Applications.

I) $\triangle ABC$ is isosceles with legs $AB = AC = L$. A circle ω is tangent to $\overline{BC}$ and the arc BC of the circumcircle of $\triangle ABC$. A tangent line from A to ω touches ω at P. Describe the locus of P as ω varies.

Solution. We use Casey's theorem for the circles $(A), (B), (C)$ (with zero radii) and ω, all internally tangent to the circumcircle of $\triangle ABC$. Thus, if ω touches $\overline{BC}$ at Q, we have:

$$L \cdot CQ + L \cdot BQ = AP \cdot BC \Longrightarrow AP = \frac{L(BQ + CQ)}{BC} = L$$

The length AP is constant, i.e. Locus of P is the circle with center A and radius $AB = AC = L$.

II) (O) is a circle with diameter $\overline{AB}$ and P, Q are two points on (O) lying on different sides of $\overline{AB}$. T is the orthogonal projection of Q onto $\overline{AB}$. Let $(O_1), (O_2)$ be the circles with diameters TA, TB and PC, PD are the tangent segments from P to $(O_1), (O_2)$, respectively. Show that $PC + PD = PQ$. [2].

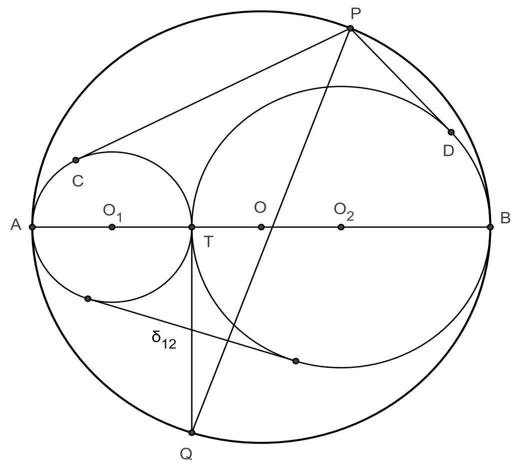

Figure 2: Application II

Solution. Let δ_{12} denote the length of the common external tangent of $(O_1), (O_2)$. We use Casey's theorem for the circles $(O_1), (O_2), (P), (Q)$, all internally tangent to (O).

$$PC \cdot QT + PD \cdot QT = PQ \cdot \delta_{12} \Longrightarrow PC + PD = PQ \cdot \frac{\delta_{12}}{QT} = PQ \cdot \frac{\sqrt{TA \cdot TB}}{TQ} = PQ.$$

III) In $\triangle ABC$, let $\omega_A, \omega_B, \omega_C$ be the circles tangent to BC, CA, AB through their midpoints and the arcs BC, CA, AB of its circumcircle (not containing A, B, C). If $\delta_{BC}, \delta_{CA}, \delta_{AB}$ denote the lengths of the common external tangents between $(\omega_B, \omega_C), (\omega_C, \omega_A)$ and (ω_A, ω_B), respectively, then prove that

$$\delta_{BC} = \delta_{CA} = \delta_{AB} = \frac{a+b+c}{4}$$

Solution. Let $\delta_A, \delta_B, \delta_C$ denote the lengths of the tangents from A, B, C to $\omega_A, \omega_B, \omega_C$, respectively. By Casey's theorem for the circles $(A), (B), (C), \omega_B$, all tangent to the circumcircle of $\triangle ABC$, we get

$$\delta_B \cdot b = a \cdot AE + c \cdot CE \Longrightarrow \delta_B = \tfrac{1}{2}(a+c)$$

Similarly, by Casey's theorem for $(A), (B), (C), \omega_C$ we'll get $\delta_C = \frac{1}{2}(a+b)$

Now, by Casey's theorem for $(B), (C), \omega_B, \omega_C$, we get $\delta_B \cdot \delta_C = \delta_{BC} \cdot a + BF \cdot BE \Longrightarrow$

$$\delta_{BC} = \frac{\delta_B \cdot \delta_C - BF \cdot BE}{a} = \frac{(a+c)(a+b) - bc}{4a} = \frac{a+b+c}{4}$$

By similar reasoning, we'll have $\delta_{CA} = \delta_{AB} = \frac{1}{4}(a+b+c)$.

IV) A circle $\mathcal{K}$ passes through the vertices B, C of $\triangle ABC$ and another circle ω touches $AB, AC, \mathcal{K}$ at P, Q, T, respectively. If M is the midpoint of the arc BTC of $\mathcal{K}$, show that BC, PQ, MT concur. [3]

Solution. Let R, ϱ be the radii of $\mathcal{K}$ and ω, respectively. Using formula (1) of Theorem 1 for $\omega, (B)$ and $\omega, (C)$. Both $(B), (C)$ with zero radii and tangent to $\mathcal{K}$ through B, C, we obtain:

$$TC^2 = \frac{CQ^2 \cdot R^2}{(R-\varrho)(R-0)} = \frac{CQ^2 \cdot R}{R-\varrho}\ ,\ TB^2 = \frac{BP^2 \cdot R^2}{(R-\varrho)(R-0)} = \frac{BP^2 \cdot R}{R-\varrho} \Longrightarrow \frac{TB}{TC} = \frac{BP}{CQ}$$

Let PQ cut BC at U. By Menelaus' theorem for $\triangle ABC$ cut by $\overline{UPQ}$ we have

$$\frac{UB}{UC} = \frac{BP}{AP} \cdot \frac{AQ}{CQ} = \frac{BP}{CQ} = \frac{TB}{TC}$$

Thus, by angle bisector theorem, U is the foot of the T-external bisector TM of $\triangle BTC$.

V) If D, E, F denote the midpoints of the sides BC, CA, AB of $\triangle ABC$. Show that the incircle (I) of $\triangle ABC$ is tangent to $\odot(DEF)$. (Feuerbach theorem).

Solution. We consider the circles $(D), (E), (F)$ with zero radii and (I). The notation δ_{XY} stands for the length of the external tangent between the circles $(X), (Y)$, then

$$\delta_{DE} = \frac{c}{2}\ ,\ \delta_{EF} = \frac{a}{2}\ ,\ \delta_{FD} = \frac{b}{2}\ ,\ \delta_{DI} = \left|\frac{b-c}{2}\right|\ ,\ \delta_{EI} = \left|\frac{a-c}{2}\right|\ ,\ \delta_{FI} = \left|\frac{b-a}{2}\right|$$

For the sake of applying the converse of Casey's theorem, we shall verify if, for some combination of signs $+$ and $-$, we get $\pm c(b-a) \pm a(b-c) \pm b(a-c) = 0$, which is trivial. Therefore, there exists a circle tangent to $(D), (E), (F)$ and (I), i.e. (I) is internally tangent to $\odot(DEF)$. We use the same reasoning to show that $\odot(DEF)$ is tangent to the three excircles of $\triangle ABC$.

VI) $\triangle ABC$ is scalene and D, E, F are the midpoints of BC, CA, AB. The incircle (I) and 9 point circle $\odot(DEF)$ of $\triangle ABC$ are internally tangent through the Feuerbach point F_e. Show that one of the segments $\overline{F_eD}, \overline{F_eE}, \overline{F_eF}$ equals the sum of the other two. [4]

Solution. WLOG assume that $b \geq a \geq c$. Incircle (I, r) touches BC at M. Using formula (1) of Theorem 1 for (I) and (D) (with zero radius) tangent to the 9-point circle $(N, \frac{R}{2})$, we have:

$$F_eD^2 = \frac{DM^2 \cdot (\frac{R}{2})^2}{(\frac{R}{2} - r)(\frac{R}{2} - 0)} \implies F_eD = \sqrt{\frac{R}{R-2r}} \cdot \frac{(b-c)}{2}$$

By similar reasoning, we have the expressions

$$F_eE = \sqrt{\frac{R}{R-2r}} \cdot \frac{(a-c)}{2}\ , \ F_eF = \sqrt{\frac{R}{R-2r}} \cdot \frac{(b-a)}{2}$$

Therefore, the addition of the latter expressions gives

$$F_eE + F_eF = \sqrt{\frac{R}{R-2r}} \cdot \frac{b-c}{2} = F_eD$$

VII) $\triangle ABC$ is a triangle with $AC > AB$. A circle ω_A is internally tangent to its circumcircle ω and AB, AC. S is the midpoint of the arc BC of ω, which does not contain A and ST is the tangent segment from S to ω_A. Prove that

$$\frac{ST}{SA} = \frac{AC - AB}{AC + AB} \quad [5]$$

Solution. Let M, N be the tangency points of ω_A with AC, AB. By Casey's theorem for $\omega_A, (B), (C), (S)$, all tangent to the circumcircle ω, we get

$$ST \cdot BC + CS \cdot BN = CM \cdot BS \implies ST \cdot BC = CS(CM - BN)$$

If U is the reflection of B across AS, then $CM - BN = UC = AC - AB$. Hence

$$ST \cdot BC = CS(AC - AB)\ (\star)$$

By Ptolemy's theorem for $ABSC$, we get $SA \cdot BC = CS(AB + AC)$. Together with $(\star)$, we obtain

$$\frac{ST}{SA} = \frac{AC - AB}{AC + AB}$$

VIII) Two congruent circles $(S_1), (S_2)$ meet at two points. A line ℓ cuts (S_2) at A, C and (S_1) at B, D (A, B, C, D are collinear in this order). Two distinct circles ω_1, ω_2 touch the line ℓ and the circles $(S_1), (S_2)$ externally and internally respectively. If ω_1, ω_2 are externally tangent, show that $AB = CD$. [6]

Solution. Let $P \equiv \omega_1 \cap \omega_2$ and M, N be the tangency points of ω_1 and ω_2 with an external tangent. Inversion with center P and power $PB \cdot PD$ takes (S_1) and the line ℓ into themselves. The circles ω_1 and ω_2 go to two parallel lines k_1 and k_2 tangent to (S_1) and the circle (S_2) goes to another circle $(S_2{}')$ tangent to k_1, k_2. Hence, (S_2) is congruent to its inverse $(S_2{}')$. Further, $(S_2), (S_2{}')$ are symmetrical about $P \Longrightarrow PC \cdot PA = PB \cdot PD$.

By Casey's theorem for $\omega_1, \omega_2, (D), (B), (S_1)$ and $\omega_1, \omega_2, (A), (C), (S_2)$ we get:

$$DB = \frac{2PB \cdot PD}{MN} \ , \ AC = \frac{2PA \cdot PC}{MN}$$

Since $PC \cdot PA = PB \cdot PD \Longrightarrow \ AC = BD \ \Longrightarrow AB = CD$.

IX) $\triangle ABC$ is equilateral with side length L. Let (O, r) and (O, R) be the incircle and circumcircle of $\triangle ABC$. P is a point on (O, r) and P_1, P_2, P_3 are the projections of P onto BC, CA, AB. Circles $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ touch BC, CA, AB through P_1, P_2, P_2 and (O, R) (internally), their centers lie on different sides of BC, CA, AB with respect to A, B, C. Prove that the sum of the lengths of the common external tangents of $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ is a constant value.

Solution. Let δ_1 denote the tangent segment from A to $\mathcal{T}_1$. By Casey's theorem for $(A), (B), (C), \mathcal{T}_1$, all tangent to (O, R), we have $L \cdot BP_1 + L \cdot CP_1 = \delta_1 \cdot L \Longrightarrow \ \delta_1 = L$. Similarly, we have $\delta_2 = \delta_3 = L$. By Euler's theorem for the pedal triangle $\triangle P_1P_2P_3$ of P, we get:

$$[P_1P_2P_3] = \frac{p(P, (O))}{4R^2}[ABC] = \frac{R^2 - r^2}{4R^2}[ABC] = \frac{3}{16}[ABC]$$

Therefore, we obtain

$$AP_2 \cdot AP_3 + BP_3 \cdot BP_1 + CP_1 \cdot CP_2 = \frac{2}{\sin 60^\circ}\left([ABC] - [P_1P_2P_3]\right) = \tfrac{13}{16}L^2. \ (\star)$$

By Casey's theorem for $(B), (C), \mathcal{T}_2, \mathcal{T}_3$, all tangent to (O, R), we get

$$\delta_2 \cdot \delta_3 = L^2 = BC \cdot \delta_{23} + CP_2 \cdot BP_3 = L \cdot \delta_{23} + (L - AP_1)(L - AP_2)$$

By cyclic exchange, we have the expressions:

$$L^2 = L \cdot \delta_{31} + (L - BP_3)(L - BP_1) \ , \ L^2 = L \cdot \delta_{12} + (L - CP_1)(L - CP_2)$$

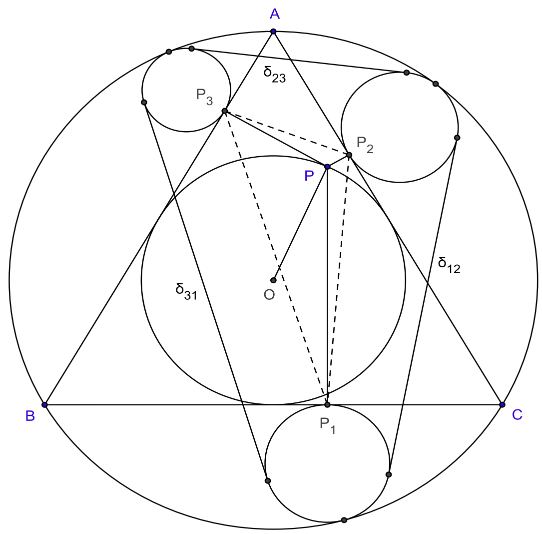

Figure 3: Application VII

Adding the three latter equations yields

$$3L^2 = L(\delta_{23} + \delta_{31} + \delta_{12}) + 3L^2 - 3L^2 + AP_3 \cdot AP_2 + BP_3 \cdot BP_1 + CP_1 \cdot CP_2$$

Hence, combining with $(\star)$ gives

$$\delta_{23} + \delta_{31} + \delta_{12} = 3L - \tfrac{13}{16}L = \tfrac{35}{16}L$$

3 Proposed Problems.

1) *Purser's theorem*: $\triangle ABC$ is a triangle with circumcircle (O) and ω is a circle in its plane. AX, BY, CZ are the tangent segments from A, B, C to ω. Show that ω is tangent to (O), if and only if

$$\pm AX \cdot BC \pm BY \cdot CA \pm CZ \cdot AB = 0$$

2) Circle ω touches the sides AB, AC of $\triangle ABC$ at P, Q and its circumcircle (O). Show that the midpoint of $\overline{PQ}$ is either the incenter of $\triangle ABC$ or the A-excenter of $\triangle ABC$, according to whether $(O), \omega$ are internally tangent or externally tangent.

3) $\triangle ABC$ is A-right with circumcircle (O). Circle Ω_B is tangent to the segments $\overline{OB}, \overline{OA}$ and the arc AB of (O). Circle Ω_C is tangent to the segments $\overline{OC}, \overline{OA}$ and the arc AC of (O). Ω_B, Ω_C touch $\overline{OA}$ at P, Q, respectively. Show that:

$$\frac{AB}{AC} = \frac{AP}{AQ}$$

4) *Gumma, 1874.* We are given a cirle (O, r) in the interior of a square $ABCD$ with side length L. Let (O_i, r_i) $i = 1, 2, 3, 4$ be the circles tangent to two sides of the square and (O, r) (externally). Find L as a fuction of r_1, r_2, r_3, r_4.

5) Two parallel lines τ_1, τ_2 touch a circle $\Gamma(R)$. Circle $k_1(r_1)$ touches Γ, τ_1 and a third circle $k_2(r_2)$ touches Γ, τ_2, k_1. We assume that all tangencies are external. Prove that $R = 2\sqrt{r_1 \cdot r_2}$.

6)*Victor Thébault. 1938.* $\triangle ABC$ has incircle (I, r) and circumcircle (O). D is a point on $\overline{AB}$. Circle $\Gamma_1(r_1)$ touches the segments $\overline{DA}$, $\overline{DC}$ and the arc CA of (O). Circle $\Gamma_2(r_2)$ touches the segments $\overline{DB}, \overline{DC}$ and the arc CB of (O). If $\angle ADC = \varphi$, show that:

$$r_1 \cdot \cos^2 \frac{\varphi}{2} + r_2 \cdot \sin^2 \frac{\varphi}{2} = r$$

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